What are we trying to do?

Define a non-commutative generalization of Gromov-Witten theory $\rightsquigarrow$ will call this theory Categorical Enumerative Invariants (CEIs).

What does NC mean? Kontsevich approach

Instead of starting with a space (variety, DM stack) start with an (enhanced) triangulated category $\mathcal{C}$

E.g. $D^b(X)$, $\mathrm{Fuk}(X)$, $MF(\times)$, $\mathrm{WFuk}(X, L)$

$D^b(X) \rightsquigarrow$ BCOV theory

$\mathrm{Fuk}(X) \rightsquigarrow$ GW invariants

$MF(\times) \rightsquigarrow$ B-model FJRW.

$\mathrm{WFuk}(X, L) \rightsquigarrow$ FJRW invariants

Mirror symmetry: $\mathrm{Fuk}(X) \simeq D^b(\check{X})$

$\Rightarrow GW(X) = BCOV(\check{X})$ ~ known for $g=0$
unknown for $g>0$.

Big picture idea:

Step 1: $\mathcal{C} \rightsquigarrow (1+1)$-enhanced TFT

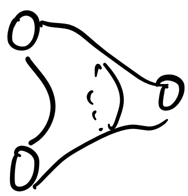
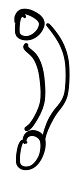
Explicitly computable!!

Polishchuk, ...

Tu cyclic A_∞ algebra Costello, Kontsevich–Soibelman

What is this enhanced TFT?

Classical result: 2d purely topological TFT $\Updownarrow$ Frobenius algebra

 = multiplication  = pairing

Conformal FT: put complex structure on surfaces. (hard to classify).

$\hookrightarrow$ would need to get an operation in the field theory for each point in $\mathcal{M}_{g,n}$.

Extended FT: get action of $C_*(\mathcal{M}_{g,n})$.

Specifically, an extended TFT constructs a dg-vector space L and maps

$$C_*(\mathcal{M}^{fr}_{g,k,\ell}) \longrightarrow \mathrm{Hom}(L^{\otimes k}, L^{\otimes \ell})$$

compatible w/ sewing framed punctures on Riemann surfaces

For a given CY[d] A_∞-algebra,

$$L := \left(C_*(A)[d],\ b\right) \text{ Hochschild differential}$$

$$C_k(A) = A^{\otimes k+1}$$

We think of the TFT maps as a chain level version of the Kontsevich-Manin CohFT where

$$(\alpha_1, \dots, \alpha_n) \longmapsto \Omega_{g,n}(\alpha_1, \dots \alpha_n) \longmapsto \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(\alpha_1, \dots \alpha_n)$$

now can integrate on a chain in $C_*(\mathcal{M}_{g,n})$.

Step 2: We cannot, in this 2d TFT approach, deal with

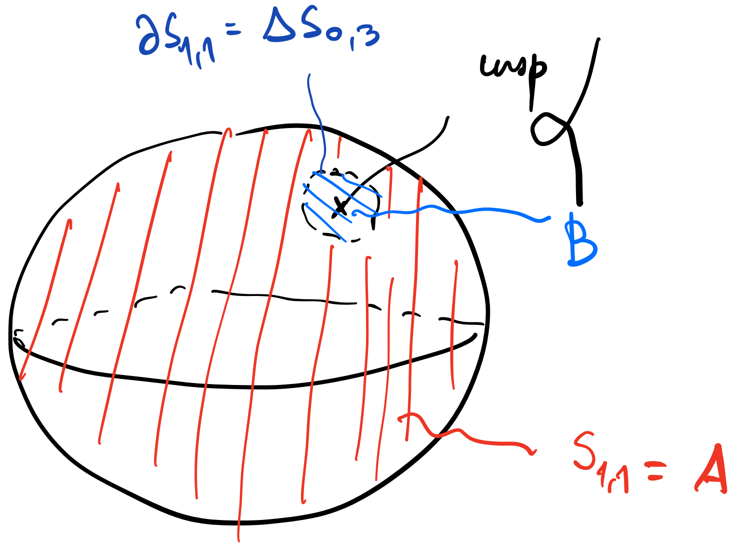
- $C_*(\overline{\mathcal{M}}_{g,n})$ only $C_*(\mathcal{M}^{fr}_{g,n})$
- cannot have 0 inputs

Idea: integrate on "most" of $\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \partial \mathcal{M}_{g,n}$

and use recursion on boundary strata that come from earlier $\mathcal{M}_{g,n}$'s.

Example:

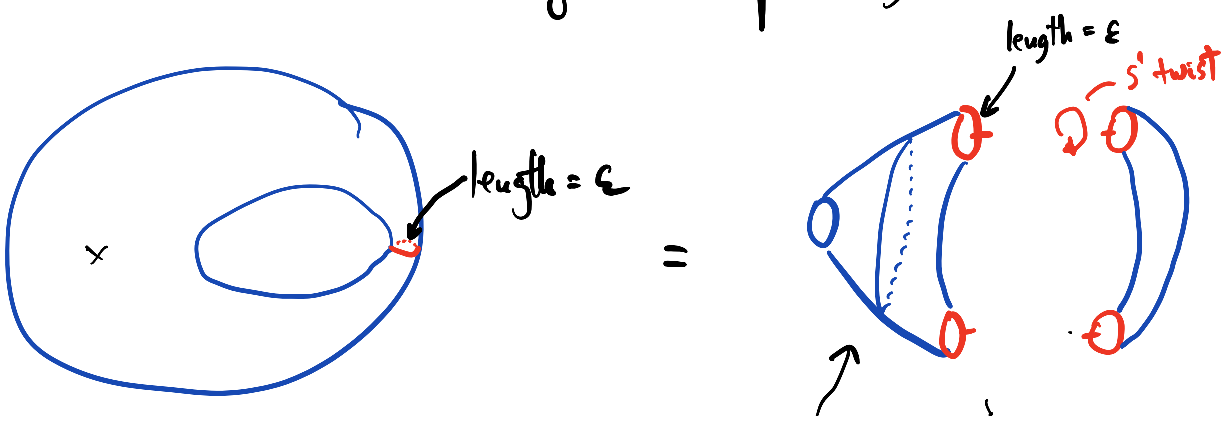
$\overline{\mathcal{M}}_{1,1}$:



$S_{1,1} = \overline{\mathcal{M}}_{1,1} \setminus$ tubular neighborhood of $\partial\overline{\mathcal{M}}_{g,n}$

$\rightsquigarrow$ can compute $\int_A \Omega_{1,1}(\alpha)$ using the TFT

$\partial S_{1,1}$ = 1-punctured elliptic curves where in hyperbolic metric have geodesic of length ε



$$\downarrow \qquad\qquad\qquad\qquad S_{0,3}' \qquad \downarrow$$

$$\partial S_{1,1} \qquad\qquad\qquad\qquad\qquad \Delta S_{0,3}.$$

Still need to compute $\int_B \Omega_{1,1}(\alpha)$

If we know that $\Omega_{1,1}(\alpha)\big|_B = d\,\Omega'(\alpha)$ for some special $\overline{\Omega}$ we'd have

$$\int_B \Omega_{1,1}(\alpha) = \int_B d\Omega'(\alpha) \overset{\text{Stokes}}{=} \int_{\partial B} \Omega'(\alpha) = \int_{\Delta S_{0,3}} \Omega'(\alpha)$$

This can again be computed from TFT.

Conclusion: We need more data than just the TFT (or A_∞-algebra, or CY category $\mathcal{C}$).

Namely we need to know how to find the special form $\Omega'(\alpha)$.

This data turns out to be exactly a choice of splitting of the non-commutative Hodge filtration coming from the spectral sequence

$$HH_*(\mathcal{C})[[u]] \Rightarrow HC^-_*(\mathcal{C}).$$

Examples of nice splittings:

$\mathcal{C} = D^b(X)$ $\quad X = X_\tau = \mathbb{C}/_{\mathbb{Z}\oplus\mathbb{Z}\tau}$ = elliptic curve

$$\begin{array}{ccccccccc}
0 \to & H^0(X, \Omega^1_X) & \to & H^1_{dR}(X) & \to & H^1(X, \mathcal{O}_X) & \to 0 \\
& \| & & \| & & \| & \\
& \mathbb{C}^1 & & \mathbb{C}^2 & & \mathbb{C}^1 & \\
& dz & \longmapsto & dz & & & \\
& & & d\bar{z} & \longleftarrow & (\text{generator}) & \\
& & & & & \text{or} & \\
& & & \frac{dz - d\bar{z}}{\tau - \bar{\tau}} & \longleftarrow & (\text{generator}) &
\end{array}$$

correct for computing B-model GW inv'ts.

Relationship between these two splittings gives us a proof that CEIs satisfy Holomorphic Anomaly Eqn.

How do we define the invariants in

general ?

Costello (non-constructive ~2005)
—, Costello, Tu (constructive ~2020)

The combinatorial information on how things should be glued up is encoded in a BV algebra, where the QME has a unique solution ... then use homological perturbation to get rid of anything defined in terms of the circle action.

I want to switch gears and talk about some explicit computations.

Main conjecture: $\mathrm{CEIs}(\mathrm{Fuk}(X)) = \mathrm{GW}(X)$.

Try to prove this for $X = T^2$ (two-torus)

RHS = known (Dijkgraaf, Okounkov-Pandharipande)

LHS = hard to work with $\mathrm{Fuk}(X)$
but Polishchuk-Zaslow proved HMS for elliptic curves $\rightsquigarrow (\mathrm{Fuk}(T^2), q) \simeq D^b(X_z)$

$q = \exp(2\pi i z)$.

So we can try to understand CEIs of $D^b(X_\tau)$.

Polishchuk constructs a cyclic A_∞ algebra A_τ s.t

$$D^b(X_\tau) = D^b(A_\tau).$$

Can do computer calculations for $g = 1, 2$. ✓

What do we actually know:

1. $CEI_{g,n}(X_\tau)\langle \dots \rangle$ is quasi-modular, holomorphic in τ.

$\rightsquigarrow$ a polynomial combination of E_2, E_4, E_6.

2. CEIs satisfy Holomorphic Anomaly Equation (Oberdieck - Pixton for GW for curves)

$\rightsquigarrow$ invariants in genus g are determined from lower genera, modulo holomorphic information

So if $CEI_{g,n} = \underbrace{E_2 \cdot P(E_2, E_4, E_6)}_{\text{determined by HAE.}} + Q(E_4, E_6)$

(3) $CEI_{g,n+1}$ are more or less determined by $CEI_{g,n}$ i.e., string, divisor axioms hold (recent work of Shen, Tu)

$\rightsquigarrow$ only need to compute $CEI_{g,1}(X_\tau)$

Only interesting invariant is

$$\langle [\xi] u^{2g-2} \rangle^{X_\tau}_{g,1} \overset{?}{=} \langle \tau_{2g-2}([pt]) \rangle^{T^2}_{g,1}$$

(1) + (2) $\Rightarrow$ For $g \le 5$ this invariant is completely determined by value at $q = 0 \Leftrightarrow \tau = i\infty$.

E.g. For $g = 2$, invariant is supposed to be

$$\underbrace{\frac{1}{2} C_2^2}_{\text{know from HAE}} + a C_4 \qquad \text{where} \qquad C_2 = -\frac{1}{24} E_2, \quad C_4 = \frac{1}{2880} E_4.$$

Value at $q = 0$ is $\frac{1}{1152} + \frac{1}{2880} \Rightarrow \boxed{a = 1}$

What does it mean to compute at $q = 0$?

In GW theory, this is the contribution of constant maps!!

$$\boxed{\begin{aligned}\langle \tau_{2g-2}([pt])\rangle^{T^2,0}_{g,1} &= \langle \tau_{2g-2}(\lambda_g)\rangle^{pt}_{g,1} \\ &= \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi^{2g-2} = \frac{2^{2g-1}-1}{2^{2g-1}} \cdot \frac{|B_{2g}|}{(2g)!}\end{aligned}}$$

(Getzler - Pandharipande, Faber - Pandharipande)

Can we compute this in CEI world?

Maybe. Three conjectures:

1. The CEIs have NO contributions from the higher multiplication of A_∞

2. The higher order terms of the lift

$$\tilde{\zeta} = \zeta + u\alpha_1 + \dots$$

DO NOT contribute to the computation, for the correct choice of string vertices.

3. The individual terms in the computation of the $a=0$ CEI invariant match the

ones in the computation of the double ramification cycle.