

On Complements of $\mathfrak{F}$ -Residuals in Finite Solvable Groups *)

By

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Introduction. A formation is a class $\mathfrak{F}$ of finite solvable groups which is closed under homomorphic images and is such that each finite group G has a unique smallest normal subgroup H with factor group G/H in $\mathfrak{F}$. This uniquely determined normal subgroup of G is called the $\mathfrak{F}$ -residual subgroup of G and will be denoted here by $G^{\mathfrak{F}}$. SHULT [13] and CARTER and HAWKES [2] have shown that if $\mathfrak{F}$ is saturated and contains all finite nilpotent groups and if $G^{\mathfrak{F}}$ is abelian then $G^{\mathfrak{F}}$ is complemented in G and all complements are conjugate. In this paper we show that $G^{\mathfrak{F}}$ is complemented under much weaker restrictions on $G^{\mathfrak{F}}$, and we obtain some sufficient conditions for conjugacy of the complements.

In order to describe the results, as well as to provide proofs, we need to develop a somewhat more general notion of $\mathfrak{F}$ -normalizers than that introduced in [2] by CARTER and HAWKES. Section 1 is devoted to such a development and to showing (for use in section 3) that the resulting definition also generalizes the definition of $\mathfrak{F}$ -normalizers given by FISCHER in [4]. The second section gives a condition under which $G^{\mathfrak{F}}$ is complemented by an $\mathfrak{F}$ -normalizer of G .

The question of conjugacy of complements of $G^{\mathfrak{F}}$ seems to be quite delicate. In section 3 we show that a sort of Frobenius action of an $\mathfrak{F}$ -normalizer on $G^{\mathfrak{F}}$ guarantees conjugacy. The last section gives an account of a special class of groups in which the $\mathfrak{F}$ -normalizer has such an action.

All groups considered will be finite and solvable. Notation is intended to be consistent with that of [2]. If Σ is a Sylow system of G we denote by Σ_{π} the π -Hall subgroup in Σ and also sometimes denote the p -complement Σ_p' by Σ^p . We use the notation $N_G(H \cap \Sigma)$ for the intersection, over all π , of the set of subgroups $N_G(H \cap \Sigma_{\pi})$. The formation of nilpotent groups will be denoted by $\mathfrak{N}$, and for each set of primes π the formation of π -groups will be called $\mathfrak{G}_{\pi}$.

1. $\mathfrak{F}$ -normalizers for general $\mathfrak{F}$. Let $\mathfrak{F}$ be a non-empty saturated formation. Then $\mathfrak{F}$ is locally induced by a set $\{\mathfrak{F}(p)\}$ of formations contained in $\mathfrak{F}$ and indexed by the set of primes. (This fact and other results which we shall need about saturated

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formations are contained in [2] and the last chapter of [7].) The set of primes for which $\mathfrak{F}(p)$ is non-empty will be called the *support* of $\mathfrak{F}$. Thus $\mathfrak{F} \supseteq \mathfrak{N}$ exactly when the support of $\mathfrak{F}$ is the set of all primes.

If $\mathfrak{F} \supseteq \mathfrak{N}$, then in view of Lemma 3.1 of [2] the $\mathfrak{F}$ -normalizers of CARTER and HAWKES are the subgroups of form $\bigcap_{p \text{ prime}} N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p)$, where Σ is a Sylow system of G . In general, however, if $\mathfrak{F}$ has support π it makes no sense to consider $G^{\mathfrak{F}(p)}$ for p not in π . We extend the definition as follows.

Definition. Let $\mathfrak{F}$ be a saturated formation with support π locally induced by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for each p . An $\mathfrak{F}$ -normalizer of the group G is a π -Hall subgroup of a subgroup of form $\bigcap_{p \in \pi} N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p)$, where Σ is a Sylow system of G .

We shall see in Theorem 1.6 that the definition of $\mathfrak{F}$ -normalizers is independent of which set $\{\mathfrak{F}(p)\}$ is used to define $\mathfrak{F}$, provided that $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for each p . Our first result shows that each Sylow system of G determines an $\mathfrak{F}$ -normalizer of G in a natural way.

Proposition 1.1. *Let $\{\mathfrak{F}(p) \mid p \in \pi\}$ be a set of formations and let G be a group with Sylow system Σ . Let*

$$N = \bigcap_{p \in \pi} N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p).$$

Then for each p in π ,

$$N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p) \cap \Sigma_p$$

is a p -Sylow subgroup of N , and if $\tau \subseteq \pi$ then $N \cap \Sigma_\tau$ is a τ -Hall subgroup of N .

Proof. For p in π let $N_p = N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p) \cap \Sigma_p$. If $q \in \pi$ but $q \neq p$, then $N_p \leq \Sigma^q \leq N_G(G^{\mathfrak{F}(q)} \cap \Sigma^q)$. Thus $N_p \leq N$. By the Frattini argument

$$G = N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p) \cdot (G^{\mathfrak{F}(p)} \cap \Sigma_p),$$

so that

$$|N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p) : N_p| = |G : \Sigma_p|,$$

a number prime to p . Thus N_p is a p -Sylow subgroup of N . It follows that $N_p = N \cap \Sigma_p$. If $\tau \subseteq \pi$, then $N \cap \Sigma_\tau = \langle N \cap \Sigma_p \mid p \in \tau \rangle$, a τ -Hall subgroup of N .

Corollary 1.2. *Let τ and π be sets of primes with $\tau \subseteq \pi$. Let $\mathfrak{F}$ be a formation with support τ locally induced by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for all p , and let $\mathfrak{H}$ be a formation with support π locally induced by $\{\mathfrak{H}(p)\}$ with $\mathfrak{H}(p) \subseteq \mathfrak{H}$ for all p . Suppose that $\mathfrak{F}(p) = \mathfrak{H}(p)$ for all p in τ . Then a τ -Hall subgroup of an $\mathfrak{H}$ -normalizer is an $\mathfrak{F}$ -normalizer.*

Proof. Let G be a group with Sylow system Σ . For p in $\pi - \tau$,

$$\Sigma_\tau \leq \Sigma^p \leq N_G(G^{\mathfrak{H}(p)} \cap \Sigma^p).$$

Hence

$$\Sigma_\tau \cap \bigcap_{p \in \pi} N_G(G^{\mathfrak{H}(p)} \cap \Sigma^p) = \Sigma_\tau \cap \bigcap_{p \in \tau} N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p).$$

The result now follows from the definition of $\mathfrak{H}$ - and $\mathfrak{F}$ -normalizers of G .

Corollary 1.2 is especially useful in the case in which π is the set of all primes and $\mathfrak{F}(p) = \{1\}$ for p not in τ . For example, we conclude that $(\mathfrak{N} \cap \mathfrak{G}_\pi)$ -normalizers are simply π -Hall subgroups of system normalizers.

Corollary 1.2 also allows us to make wholesale use of the results of [2]. Define a p -chief factor H/K of G to be $\mathfrak{F}$ -central in case $\Lambda_G(H/K) \in \mathfrak{F}(p)$ and to be $\mathfrak{F}$ -eccentric otherwise. Then Theorem 4.1 of [2] and Corollary 1.2 together show that $\mathfrak{F}$ -normalizers cover $\mathfrak{F}$ -central chief factors and avoid $\mathfrak{F}$ -eccentric chief factors. Moreover, the proof of Theorem 4.1 of [2] shows that $\mathfrak{F}$ -normalizers are in $\mathfrak{F}$.

FISCHER ([4], p. 63) gives a definition of $\mathfrak{F}$ -normalizer for $\mathfrak{F} \supseteq \mathfrak{N}$ which makes no mention of the formations $\mathfrak{F}(p)$ and makes sense in case $\mathfrak{F} \not\supseteq \mathfrak{N}$. The next major result, Theorem 1.6, shows that this definition leads to an alternate characterization of the $\mathfrak{F}$ -normalizers defined above, even when $\mathfrak{F} \not\supseteq \mathfrak{N}$.

We first extend Proposition 8.6 of [4] and Theorem 5.8 of [2].

Lemma 1.3. *Let $\mathfrak{F}$ be a saturated formation with support π . Let E be a subgroup of G such that E is in $\mathfrak{F}$ and $E \cdot F(G) \cdot \mathbf{O}_{\pi'}(G) = G$. Then the π -Hall subgroups of $N_G(E)$ are contained in $\mathfrak{F}$ -covering subgroups of G .*

Proof. Suppose the result false. Let G be a minimal counterexample and E a subgroup of G satisfying the hypothesis but not the conclusion of the lemma. Let A be a minimal normal subgroup of G and let S be a π -Hall subgroup of $N_G(E)$ with $E \leq S$. Then G/A and its subgroup EA/A satisfy the hypotheses, so that $SA \leq HA$ for some $\mathfrak{F}$ -covering subgroup H of G . Now

$$\mathbf{O}_{\pi'}(HA) \cdot F(HA) \geq [\mathbf{O}_{\pi'}(G) \cdot F(G)] \cap HA,$$

so that

$$E \cdot \mathbf{O}_{\pi'}(HA) \cdot F(HA) \geq [E \cdot \mathbf{O}_{\pi'}(G) \cdot F(G)] \cap HA = HA.$$

If $HA < G$, then by minimality of G , S is conjugate to a subgroup of the $\mathfrak{F}$ -covering subgroup H of HA , a contradiction. Thus $HA = G$ for every minimal normal subgroup A . Hence, H is maximal, $\text{core}(H) = 1$, G has a unique minimal normal subgroup, A , and $A = C_G(A)$. It follows that $A = F(G)$.

Since H is a π -group, $\mathbf{O}_{\pi'}(G) \leq A$. Thus $E \cdot A = G$ and E is conjugate to H . Since E is then an $\mathfrak{F}$ -covering subgroup of G and a π -group, $E < N_G(E)$, so that $E \triangleleft G$. Since E is coreless, $E = 1$ and G has prime order, p . Since $G \notin \mathfrak{F}$, $p \notin \pi$. But then 1 is a π -Hall subgroup of G , which is absurd.

Corollary 1.4. *If $\mathfrak{F}$ is a saturated formation with support π , and if E is a subgroup of G maximal subject to the conditions $E \in \mathfrak{F}$ and $E \cdot F(G) \cdot \mathbf{O}_{\pi'}(G) = G$, then E is an $\mathfrak{F}$ -covering subgroup of G and is a π -Hall subgroup of its normalizer.*

Lemma 1.5. *Let $\mathfrak{F}$ be a saturated formation locally induced by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for every prime p . Let Σ be a Sylow system for G and let E be an $\mathfrak{F}$ -covering subgroup of $N_G(G^\Sigma \cap \Sigma)$. Then E avoids every $\mathfrak{F}$ -eccentric chief factor of G .*

Proof. This is essentially contained in Theorem 8.5 (3) of [4].

Theorem 1.6. Let $\mathfrak{F}$ be a saturated formation with support π locally induced by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for each prime p . Let Σ be a Sylow system of the group G and let $D = \Sigma_\pi \cap \bigcap_{p \in \pi} N_G(G^{\mathfrak{F}(p)} \cap \Sigma_p)$. Then D is an $\mathfrak{F}$ -covering subgroup of $N_G(G^{\mathfrak{F}} \cap \Sigma)$.

Proof. Let $N = N_G(G^{\mathfrak{F}} \cap \Sigma)$. For $p \in \pi$, $D \leq N_G(G^{\mathfrak{F}} \cap \Sigma_p)$ (since $G^{\mathfrak{F}(p)} \geq G^{\mathfrak{F}}$), while for $p \notin \pi$, $D \leq \Sigma_p \leq N_G(G^{\mathfrak{F}} \cap \Sigma_p)$. Thus $D \leq N$. Since D covers $G/G^{\mathfrak{F}}$, $D \cdot (G^{\mathfrak{F}} \cap N) = N$. Since $N \cap G^{\mathfrak{F}}$ is nilpotent and normal in N , $N \cap G^{\mathfrak{F}} \leq F(N)$ and $D \cdot F(N) = N$. As we noted after Corollary 1.2, $D \in \mathfrak{F}$. Hence, by Lemma 1.3, D is contained in an $\mathfrak{F}$ -covering subgroup E of N . But by Lemma 1.5 E avoids every chief factor of G that D does, so that $D = E$, as desired.

Theorem 1.6 shows that we could have defined the $\mathfrak{F}$ -normalizers to be $\mathfrak{F}$ -covering subgroups of relative system normalizers of $G^{\mathfrak{F}}$. Such an approach avoids consideration of the formations $\mathfrak{F}(p)$ and the necessity of considering the groups $G^{\mathfrak{F}(p)}$ or C_p , as defined by CARTER and HAWKES. On the other hand, it seems difficult to get certain detailed information about G directly from the simpler definition. We shall have occasion to use both points of view in what follows.

2. A splitting theorem. It is not always true that $G^{\mathfrak{F}}$ is complemented in G . For example, the central product of $SL_2(3)$ with a cyclic group of order 4 with subgroups of order 2 amalgamated has the quaternion group Q_8 as uncomplemented $\mathfrak{R}$ -residual subgroup. For odd p a similar example of order $2p^4$ exists in which the $\mathfrak{R}$ -residual subgroup is extra-special of order p^3 and exponent p . The main result of this section, Theorem 2.2, can be viewed as saying that these examples are typical of the groups for which $G^{\mathfrak{F}}$ is uncomplemented.

To get a complement for $G^{\mathfrak{F}}$ we shall impose conditions on its Sylow subgroups. A p -group is called *modular* in case its lattice of subgroups is a modular lattice. (Of course every abelian group is modular.)

HUPPERT has shown the following result ([9], Satz 3.2).

Theorem A. If G is a finite group, if p is an odd prime and if P is a modular p -Sylow subgroup of G , then $P \cap Z(G) \cap G^{\mathfrak{R}} = 1$.

A 2-group is called *quaternion-free* in case it has no section isomorphic to the quaternion group Q_8 of order 8. DORNHOFF ([3], Theorem 2.8) was able to show the following result.

Theorem B. If G is a finite solvable group with quaternion-free 2-Sylow subgroup P then $P \cap Z(G) \cap G^{\mathfrak{R}} = 1$.

Neither of these theorems is easy to prove, but the following weaker result can be proved by a routine computation of the transfer of G into P .

Theorem C. If G is a finite group with abelian Sylow subgroup P , then

$$P \cap Z(G) \cap G' = 1.$$

The first result of this section shows some of the influences of Sylow structure of $G^{\mathfrak{F}}$ on chief factors of G .

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Theorem 2.1. *Let $\mathfrak{F}$ be a saturated formation with support π . Let G be a finite solvable group. Let p be a prime in π , let P be a p -Sylow subgroup of $G^{\mathfrak{F}}$ and suppose that P is quaternion-free if $p = 2$ and is modular if p is odd. Then every p -chief factor of G lying below $G^{\mathfrak{F}}$ is $\mathfrak{F}$ -eccentric.*

Proof. Suppose the theorem is false and let G be a minimal counterexample. Let A be a minimal normal subgroup of G . Then G/A satisfies the hypothesis of the theorem, so that every p -chief factor of G between A and $AG^{\mathfrak{F}}$ is $\mathfrak{F}$ -eccentric. It follows that $A \leq G^{\mathfrak{F}}$, A is a p -group and A is $\mathfrak{F}$ -central. Thus $G/C_G(A) \in \mathfrak{F}(p) \subseteq \mathfrak{F}$, so that $A \leq Z(G^{\mathfrak{F}})$. If $(G^{\mathfrak{F}})^{\mathfrak{F}} > 1$ then by taking A inside $(G^{\mathfrak{F}})^{\mathfrak{F}}$ we would have $A \leq P \cap Z(G^{\mathfrak{F}}) \cap (G^{\mathfrak{F}})^{\mathfrak{F}}$, a contradiction to Theorem A or Theorem B. Thus $G^{\mathfrak{F}}$ is nilpotent, from which it easily follows that $G^{\mathfrak{F}} = P$.

Let $K/P = \mathbf{O}_{p'}(G/P)$. Then $G/K \in \mathfrak{F}(p)$, and since $G/C_G(A) \in \mathfrak{F}(p)$ we have $K \cap C_G(A) \geq G^{\mathfrak{F}(p)}$. Since K/P has a normal p -complement, so does $(K \cap C_G(A))/P$. If $K \cap C_G(A)$ were a p -group then $G^{\mathfrak{F}(p)}$ would be one and all chief factors below $G^{\mathfrak{F}}$ would be $\mathfrak{F}$ -central, which is not the case. Thus G has a p' -chief factor L/P with $L \leq K \cap C_G(A)$. Then $A \leq Z(L) \cap P$ and an application of Theorem A or Theorem B to L shows that $L^{\mathfrak{F}} = 1$. But then L has a non-trivial normal p -complement, contradicting the fact that all minimal normal subgroups of G are p -groups. The proof is complete.

The following splitting theorem is an immediate consequence of Theorem 2.1 and the covering and avoidance properties of $\mathfrak{F}$ -normalizers.

Theorem 2.2. *Let $\mathfrak{F}$ be a saturated formation, and let G be a finite solvable group. Let τ be the set of primes dividing $|G:G^{\mathfrak{F}}|$. Suppose that for each p in τ the p -Sylow subgroups of $G^{\mathfrak{F}}$ are quaternion-free if $p = 2$ and are modular if p is odd. Let D be an $\mathfrak{F}$ -normalizer of G and D_{τ} a τ -Hall subgroup of D . Then $D_{\tau} \cdot G^{\mathfrak{F}} = G$ and $D_{\tau} \cap G^{\mathfrak{F}} = 1$. Moreover, D_{τ} is an $(\mathfrak{F} \cap \mathfrak{G}_{\tau})$ -normalizer of G .*

Proof. The fact that D_{τ} complements $G^{\mathfrak{F}}$ is an immediate consequence of Theorem 2.1 and the covering and avoidance properties of $\mathfrak{F}$ -normalizers established in section 1. If E is an $(\mathfrak{F} \cap \mathfrak{G}_{\tau})$ -normalizer of G , then by applying the theorem to $\mathfrak{F} \cap \mathfrak{G}_{\tau}$ instead of $\mathfrak{F}$ we see that E also complements $G^{\mathfrak{F}}$. The argument on p. 177 of [2] extends to show that if $\mathfrak{F}$ is locally induced by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteq \mathfrak{F}$ for each p , then $\mathfrak{F} \cap \mathfrak{G}_{\tau}$ is locally induced by $\{\mathfrak{F}(p) \cap \mathfrak{G}_{\tau} \mid p \in \tau\}$. By the definition of E , G has a Sylow system Σ with $E = \Sigma_{\tau} \cap \bigcap_{p \in \tau} N_G(G^{\mathfrak{F}(p)} \mathbf{O}_{\tau}(G) \cap \Sigma^p)$. Say $\mathfrak{F}$ has support σ . For $p \in \sigma - \tau$,

$$E \leq \Sigma_{\tau} \leq \Sigma^p \leq N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p),$$

while for $p \in \tau$,

$$E \leq N_G(G^{\mathfrak{F}(p)} \mathbf{O}_{\tau}(G) \cap \Sigma^p) \leq N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p).$$

Hence, $E \leq \Sigma_{\tau} \cap \bigcap_{p \in \sigma} N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p)$, a conjugate of the τ -Hall subgroup D_{τ} of D , by Proposition 1.1. Since D_{τ} and E have the same order, we conclude that D_{τ} and E are conjugate.

One consequence of Theorem 2.2 is a generalization for solvable groups of a result due to GASCHÜTZ ([5]) and reproved by ROSE ([11]).

Corollary 2.3. *Let G be a solvable group and π a set of primes. Suppose that for p in π the p -Sylow subgroups of $O^\pi(G)$ are modular for p odd and quaternion-free for p even. Then $O^\pi(G)$ is complemented by a $\mathfrak{G}_\pi$ -normalizer of G .*

The following corollary can be proved using Theorem C in place of Theorems A and B.

Corollary 2.4. *Let $\mathfrak{F}$ be a saturated formation and G a solvable group. Suppose that $G^\mathfrak{F}$ has abelian Sylow subgroups for the primes dividing $|G:G^\mathfrak{F}|$. Then $G^\mathfrak{F}$ is complemented by a Hall subgroup of an $\mathfrak{F}$ -normalizer of G .*

3. Some conditions for conjugacy of complements to $G^\mathfrak{F}$. A group may satisfy the hypotheses of the splitting theorem of the last section without having all complements of $G^\mathfrak{F}$ conjugate. For example, let H be the primitive solvable group of degree 8 and order 168 and let $G = H \times \langle a \rangle$, where $\langle a \rangle$ has order 2. Then G has abelian Sylow subgroups. Let D_0 be a system normalizer of H , so that D_0 is of order 3. Then D_0 centralizes an involution, z , of H . The subgroups $D_0 \times \langle a \rangle$ and $D_0 \times \langle az \rangle$ both complement $G^\mathfrak{F}$ but are not conjugate.

In this example $G^\mathfrak{F}$ is not itself nilpotent. If G is a group in which $G^\mathfrak{F} \in \mathfrak{N}$, then $G = N_G(G^\mathfrak{F} \cap \Sigma)$, so that by Theorem 1.6 the $\mathfrak{F}$ -normalizers of G are $\mathfrak{F}$ -covering subgroups of G . If also $G^\mathfrak{F}$ is complemented by an $\mathfrak{F}$ -normalizer, then G satisfies the hypotheses of the following.

Theorem 3.1. *Let $\mathfrak{F}$ be a saturated formation. Suppose that G is a solvable group in which $G^\mathfrak{F}$ is complemented by an $\mathfrak{F}$ -covering subgroup. Then all complements of G are conjugate.*

Proof. Suppose that G is a counterexample of minimal order. Let E be an $\mathfrak{F}$ -covering subgroup of G and H some other complement of $G^\mathfrak{F}$ not conjugate to E . Let A be a minimal normal subgroup of G contained in $G^\mathfrak{F}$. Then AE and AH complement $G^\mathfrak{F}$ mod A , so that by minimality of G we may assume that $AE = AH$. Now E is an $\mathfrak{F}$ -covering subgroup of AE , so that E covers $AE/(AE)^\mathfrak{F}$. Since E avoids A , $A = (AE)^\mathfrak{F}$. Minimality of G implies that $G = AE = AH$ so that H itself is an $\mathfrak{F}$ -covering subgroup of G . This contradiction establishes the theorem.

Corollary 3.2. *Let $\mathfrak{F}$ be a saturated formation. Let G be a solvable group and π the set of primes dividing $|G:G^\mathfrak{F}|$. Suppose that for each prime p in π , $G^\mathfrak{F}$ has a normal p -complement and that the p -Sylow subgroups of $G^\mathfrak{F}$ are modular for p odd and quaternion-free for $p = 2$. Then $G^\mathfrak{F}$ is complemented by an $(\mathfrak{F} \cap \mathfrak{G}_\pi)$ -normalizer of G and all complements of $G^\mathfrak{F}$ are conjugate.*

Proof. Let K be the normal π -complement of $G^\mathfrak{F}$, and let D be an $\mathfrak{F} \cap \mathfrak{G}_\pi$ -normalizer of G . By Theorem 2.2, D complements $G^\mathfrak{F}$. Since $G/K \in \mathfrak{N} \mathfrak{F}$, Theorem 3.1 shows that every complement to $G^\mathfrak{F}/K$ in G/K is conjugate to DK/K . An application of the Schur-Zassenhaus theorem shows that all complements to $G^\mathfrak{F}$ in G are conjugate to D .

Instead of placing restrictions directly on $G^{\mathfrak{F}}$, we may impose conditions on the action of an $\mathfrak{F}$ -normalizer on $G^{\mathfrak{F}}$ in order to try to avoid the centralization which allowed nonconjugate complements in the example at the beginning of this section. Such conditions on the action in turn imply more-or-less hidden restrictions on $G^{\mathfrak{F}}$ and on the structure of G . The next theorem gives an indication of what can be done in this direction.

If the group G acts on the group S we say G acts *fixed-point-freely* (fpf) on S in case no element of G other than 1 fixes an element of $S - 1$.

Theorem 3.3. *Let $\mathfrak{F}$ be a saturated formation and let D be an $\mathfrak{F}$ -normalizer of the solvable group G . Suppose that for each prime p the p -Sylow subgroups of D act fpf by conjugation on each p' -chief factor of G lying below $G^{\mathfrak{F}}$. Then $G^{\mathfrak{F}}$ is complemented by D . Moreover, if the 2-Sylow subgroups of D are quaternion-free, then all complements of $G^{\mathfrak{F}}$ are conjugate.*

Proof. The proof here is technical but is simple in outline. After we show that D complements $G^{\mathfrak{F}}$ a reduction leads to the case $\mathfrak{F} = \mathcal{G}_p$. The minimal counterexample is severely restricted and must in fact be a special kind of group of permutations of a finite field. A result from field theory then completes the argument.

Notice first that the hypotheses of the theorem are satisfied by homomorphic images of G . Moreover, if $K \trianglelefteq G$ and $K \leq G^{\mathfrak{F}}$, then p -Sylow subgroups of D act fpf on the p' -chief factors of DK lying below K .

We show first that $D \cap G^{\mathfrak{F}} = 1$, so that D complements $G^{\mathfrak{F}}$. If not, let G be a minimal counterexample. If A is a minimal normal subgroup of G , then by minimality of G , $D \cap G^{\mathfrak{F}} \leq A$, and since D does not avoid A , $D \cap G^{\mathfrak{F}} = A$. Hence $D \cap G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G and is $\mathfrak{F}$ -central. Say $D \cap G^{\mathfrak{F}}$ is a p -group. Then $D \cap G^{\mathfrak{F}}$ acts fpf on all p' -chief factors of G below $G^{\mathfrak{F}}$ and at the same time centralizes $G^{\mathfrak{F}}$. Consequently, $G^{\mathfrak{F}} \leq \mathbf{O}_p(G)$. Since $G^{\mathfrak{F}(p)}$ centralizes $D \cap G^{\mathfrak{F}}$, $D \cap G^{\mathfrak{F}(p)}$ must be a p -group. Since $D \cdot \mathbf{O}_p(G) \geq D \cdot G^{\mathfrak{F}} = G$, D contains a p -complement of G and $G^{\mathfrak{F}(p)} \leq \mathbf{O}_p(G)$. But then $G \in \mathfrak{F}$ and $G^{\mathfrak{F}} = 1$, a contradiction.

Now suppose that G is minimal such that for some $\mathfrak{F}$ and $\mathfrak{F}$ -normalizer D of G the hypotheses of the theorem are satisfied but $G^{\mathfrak{F}}$ has a complement, E , not conjugate to D . Let A be a minimal normal subgroup of G . Suppose that $A \cap G^{\mathfrak{F}} = 1$. Then A is $\mathfrak{F}$ -central, so that $A \leq D$. Say A is a p -group. Since A centralizes $G^{\mathfrak{F}}$ and acts fpf on p' -chief factors of $G^{\mathfrak{F}}$, $G^{\mathfrak{F}} \leq \mathbf{O}_p(G)$. Thus $G \in \mathfrak{N}\mathfrak{F}$ and D , an $\mathfrak{F}$ -normalizer of G , is $\mathfrak{F}$ -covering in G , by Theorem 1.6. By Theorem 3.1, all complements to $G^{\mathfrak{F}}$ are conjugate, a contradiction. It follows that each minimal normal subgroup of G is contained in $G^{\mathfrak{F}}$. In particular, if A is minimal normal in G , then EA/A complements G/A . Thus $\text{core}(D) = \text{core}(E) = 1$, $D \cap A = 1$ (since D does not cover A), A is $\mathfrak{F}$ -eccentric, and we may assume $EA = DA$. It follows that $A \leq G^{\mathfrak{F}}$ and that $E \cap A = 1$. Say A is a p -group. Then D is not a p' -group, since if it were D and E would be (conjugate to) p -complements of DA .

Let M be an $\mathfrak{F}$ -critical maximal subgroup of G such that D is an $\mathfrak{F}$ -normalizer of M . (The theory of $\mathfrak{F}$ -critical maximal subgroups in [2] does not depend on having $\mathfrak{F} \supseteq \mathfrak{N}$.) Since $M^{\mathfrak{F}} \subseteq M \cap G^{\mathfrak{F}}$, M also satisfies the hypotheses of the theorem. Since

$M = D \cdot M^{\mathfrak{S}} = D \cdot (G^{\mathfrak{S}} \cap M)$ it follows that $M^{\mathfrak{S}} = G^{\mathfrak{S}} \cap M$. If A were contained in M , then E would be conjugate to D , by minimality of G . Thus $A \cap M = 1$, $\text{core}(M) = 1$, A is the unique minimal normal subgroup of G and $A = C_G(A)$.

Now $A < G^{\mathfrak{S}}$, for if $A = G^{\mathfrak{S}}$ then all complements of $G^{\mathfrak{S}}$ are conjugate by Theorem 3.1 and the remarks preceding it. Let L/A be a chief factor of G with $L \leq G^{\mathfrak{S}}$. Let $Q = M \cap L$, so that Q is a q -Sylow subgroup of L for some prime $q \neq p$ and $D \leq M = N_G(Q)$.

If $A \leq K < G^{\mathfrak{S}}$ with $K \triangleleft G$, and if $\mathfrak{S}$ is a saturated formation, then we claim D cannot be an $\mathfrak{S}$ -normalizer of DK . For if it were, then DK would satisfy the hypotheses of the theorem for the formation $\mathfrak{S}$ and would be smaller than G . Now $E \leq DA \leq DK$. Since E complements $K = (DK)^{\mathfrak{S}}$ in DK , E would be conjugate to D . The claim follows.

Since the p -Sylow groups of D act f.p.f. on L/A and since for $r \neq p$ the r -Sylow groups of D act f.p.f. on A , each Sylow subgroup of D is cyclic or generalized quaternion, ([12], Theorem 12.6.15). We now assume that D has quaternion-free 2-Sylow subgroups. Then D is metacyclic with derived group a Hall subgroup ([12], Theorem 12.6.7).

Suppose that D is p -nilpotent. The formation $\mathfrak{R}$ of p -nilpotent groups is locally induced by $\{\mathfrak{R}(r)\}$, where $\mathfrak{R}(p) = \{1\}$ and $\mathfrak{R}(r) = \mathfrak{R}$ for $r \neq p$. We have just seen that D is not a $\mathfrak{R}$ -normalizer of DA , yet D is in $\mathfrak{R}$ and $D \cdot F(DA) = DA$. By Lemma 1.3 and Theorem 1.6 (or by Theorem 5.8 of [2]) D is contained in a $\mathfrak{R}$ -normalizer, K , of DA . Since $D < K$, DA has a $\mathfrak{R}$ -central chief factor below A . Such a factor must be central in DA , so that D must be a p -subgroup in this case.

If D is not p -nilpotent, then p divides $|D'|$. In this case, too, D has a normal p -Sylow subgroup, and D belongs to the formation $\mathfrak{S}$ of groups G for which $G/F(G)$ is a p' -group.

Now $\mathfrak{S}$ is locally induced by $\{\mathfrak{S}(r)\}$ where $\mathfrak{S}(r) = \mathfrak{S}_{p'}$ for each r . By Theorem 5.8 of [2] applied to the group DQ , D is contained in an $\mathfrak{S}$ -normalizer H of DQ . The p -Sylow subgroups of D act f.p.f. on Q , so that no chief factors of DQ lying below Q are $\mathfrak{S}$ -central. Thus $D = H$ and $(DQ)^{\mathfrak{S}} = Q$. Then $(DL)^{\mathfrak{S}} \cdot A = (DQ)^{\mathfrak{S}} \cdot A = QA = L$, and $Q \leq (DL)^{\mathfrak{S}}$. Let Σ be a Sylow system of DL such that $Q = (DL)^{\mathfrak{S}} \cap \Sigma_q$. Then $N_{DL}((DL)^{\mathfrak{S}} \cap \Sigma) = N_{DL}(Q) = DL \cap M = DQ$. Since D is an $\mathfrak{S}$ -covering subgroup of DQ , by Theorem 1.6 D is an $\mathfrak{S}$ -normalizer of DL . We have seen that this cannot happen if $L < G^{\mathfrak{S}}$. Hence, $G^{\mathfrak{S}} = L$ and $M = DQ$.

Let D_p and E_p be p -Sylow subgroups of D and E , respectively. Since D_p acts f.p.f. on Q and $(D_p L) \cap N_G(Q) = D_p Q$, D_p is a system normalizer of $D_p L$.

Suppose that D_p and E_p are conjugate. We may assume that $D_p = E_p$. Then $E \leq N_G(E_p) = N_G(D_p)$ and $N_G(D_p) \cap L = C_L(D_p) \leq A$, so that if $D_{p'}$ and $E_{p'}$ are p -complements of D and E , respectively, then $D_{p'}$ and $E_{p'}$ are p -complements of $N_G(D_p)$. But then $D_{p'}$ and $E_{p'}$ are conjugate by an element normalizing D_p and hence D and E are conjugate. This contradiction means that D_p and E_p are not conjugate. However, D_p is a system normalizer of $D_p L$, and $D_p L$ satisfies the hypothesis of the theorem. Since we may assume that $D_p L = E_p L$, we conclude that $D_p L = G$, $D_p = D$, D is cyclic and a system normalizer of G .

Apply Clifford conjugation and

Suppose first that t is a restricted homogenous equation where D_0 is a

If $D_0 > 1$, then E is conjugate to D that $E \cap G_0 = 1$ and $x^r = (xa)^r$ equation may

Let the indices of $(A_r)x = A_1$. The exponents of the

Thus

for an element x so that $xa = b$. It follows that x is a tation.

A lemma of [10] we can apply a to a group of transformations to the group of multiplications in its prime field. by the automorphism then $0 = 0(\sigma \cdot a)$ ([10], p. 215) $\sigma \cdot a = \sigma \cdot b \sigma^{-1}$ final contradiction

4. A special case Sylow subgroup of G that $G^{\mathfrak{S}}$ is a p -factor of G . Then make D act on

Our first result conjugation, as

Theorem 4.1. free 2-Sylow sub

Apply Clifford's theorem to the irreducible representation of DQ on A by conjugation and to the normal subgroup Q of QD .

Suppose first that there is more than one homogeneous component of the representation restricted to Q . Write $A = A_1 \oplus \cdots \oplus A_r$, where the groups A_i are the homogeneous components. Let G_0 be the stability group of A_1 in G . Then $G_0 = L \cdot D_0$ where D_0 is a subgroup of index r in D .

If $D_0 > 1$, then D_0 is a system normalizer of G_0 and by minimality of G , $E \cap G_0$ is conjugate to D_0 . If $D_0 = 1$, we also have $E \cap G_0$ and D_0 conjugate. We may assume that $E \cap G_0 = D_0$. Let $D = \langle x \rangle$. Then E has a generator of form xa with $a \in A$ and $x^r = (xa)^r$. If we view A as a right D -module and use additive notation this equation may be rewritten

$$0 = (a)x^{r-1} + \cdots + (a)x + a.$$

Let the indices on $A_1, \dots, A_r$ be chosen such that $(A_i)x = A_{i+1}$ for $1 \leq i < r$ and $(A_r)x = A_1$. Write $a = a_1 + \cdots + a_r$ with $a_i \in A_i$. By considering the A_1 -components of the two sides of the equation above we find that

$$0 = (a_1) + (a_r)x + (a_{r-1})x^2 + \cdots + (a_2)x^{r-1}.$$

Thus

$$a = (a_2)(1 - x^{r-1}) + \cdots + (a_r)(1 - x) = b(1 - x)$$

for an element b of A . In multiplicative notation again, we have $a = b^{-x} \cdot b = x^{-1}b^{-1}xb$, so that $xa = b^{-1}xb$ and E and D have conjugate generators and are hence conjugate. It follows that there is only one homogeneous component of the restricted representation.

A lemma of GREEN (Lemma 2.2.3 of [6]) shows that Q acts irreducibly on A . Now we can apply a lemma of HUPPERT (Hilfssatz 2, [8]) to conclude that G is isomorphic to a group of transformations $x \rightarrow a \cdot (x\sigma) + b$ on a finite field F . Here A corresponds to the group of all translations $\mathbf{b}: x \rightarrow x + b$, Q corresponds to a group of field multiplications and D corresponds to a subgroup of the Galois group of the field over its prime field. We identify G with this group of transformations. If D is generated by the automorphism σ , then E has a generator of form $\sigma \cdot \mathbf{a}$ for $a \in F$. If $|D| = s$, then $0 = 0(\sigma \cdot \mathbf{a})^s = a\sigma^{s-1} + \cdots + a\sigma + a$. The additive form of Hilbert's Theorem 90 ([10], p. 215) shows that there is an element b of F with $a = b - b\sigma$. But then $\sigma \cdot \mathbf{a} = \sigma \cdot \mathbf{b}\sigma^{-1} \cdot \mathbf{b} = \mathbf{b}^{-1} \cdot \sigma \cdot \mathbf{b}$, and again D and E have conjugate generators. This final contradiction completes the proof of the theorem.

4. A special case. The hypotheses of Theorem 3.3 say very little about how the Sylow subgroups of an $\mathfrak{F}$ -normalizer D of G act on each other. In the extreme case that $G^{\mathfrak{F}}$ is a p -group, D may very well have a p -Sylow subgroup which is a direct factor of G . The results of this section show that strengthening the hypotheses to make D act well on all chief factors of G severely limits the structure of G .

Our first result concerns those solvable groups having a subgroup that acts, under conjugation, as much like a group of Frobenius automorphisms as is possible.

Theorem 4.1. *Let G be a solvable group with subgroup H such that H has quaternion-free 2-Sylow subgroups. Suppose that for each prime p a p -Sylow subgroup of H acts*

fpf by conjugation on every p' -chief factor of G . Then H is contained in a system normalizer of G .

Proof. Let p be a prime for which G has a central p -chief factor. Then H is a p -group. Moreover, a system normalizer D of G must also be a p -group. Then $G^{\mathfrak{N}} = \mathbf{O}_p(G)$, and $H \cap G^{\mathfrak{N}} = 1$, since $H \cap G^{\mathfrak{N}}$ centralizes each chief factor $G^{\mathfrak{N}}/K$ of $G^{\mathfrak{N}}$. If $\mathbf{O}_{p'}(G^{\mathfrak{N}}) > 1$, then by taking a suitable system normalizer D we may suppose inductively that $H \cdot \mathbf{O}_{p'}(G^{\mathfrak{N}}) \leq D \cdot \mathbf{O}_{p'}(G^{\mathfrak{N}})$ and Sylow's theorem will give the desired result.

Suppose that $\mathbf{O}_{p'}(G^{\mathfrak{N}}) = 1$. Let N be a minimal normal subgroup of G contained in $G^{\mathfrak{N}}$. (If $G^{\mathfrak{N}} = 1$, there is nothing to prove.)

By induction we may suppose $HN \leq DN$. Hence if $N \leq D$ we are done. Suppose that N is eccentric. Let Q be a p -complement in a Sylow system of G normalized by D . Then D normalizes a system of Q , and D is a p -Sylow subgroup of $N_G(Q)$. Since $H \leq DN \leq N_G(QN)$, $T = NQH$ is a group. Since $D \cap N = 1$, $N \cap C_G(Q) = 1$ and hence $T \cap D$ is a system normalizer of T . Now $T \cap D$ and H are complements in T of $T^{\mathfrak{N}} = NQ$ and act fpf on p' -chief factors of T . Theorem 3.3 now implies that H and $D \cap T$ are conjugate, as desired.

Theorem 4.2. *Let $\mathfrak{F}$ be a saturated formation and G a solvable group. Let D be an $\mathfrak{F}$ -normalizer of G , and suppose that for each prime p the p -Sylow subgroups of D act fpf by conjugation on every p' -chief factor of G . Then D is a p -group for some prime p , D is a system normalizer of G complementing $G^{\mathfrak{N}}$ and $G^{\mathfrak{N}}$ has a normal p -Sylow subgroup.*

Proof. As in the beginning of the proof of Theorem 4.1 we see that D is a p -group for some prime p and that a system normalizer, E , of G must also be a p -group. The first part of the proof of Theorem 3.3 shows that $D \cap G^{\mathfrak{F}} = 1$, so that no p -chief factor below $G^{\mathfrak{F}}$ is central. It follows that $|D| = |E|$. Now for a suitable Sylow system Σ , $D = \Sigma_p \cap N_G(G^{\mathfrak{F}(p)} \cap \Sigma^p) \geq \Sigma_p \cap N_G(\Sigma^p)$, a conjugate of E . Hence D is a system normalizer of G . Clearly $G^{\mathfrak{F}} = G^{\mathfrak{N}} = \mathbf{O}_p(G)$.

To show that $G^{\mathfrak{N}}$ has a normal p -Sylow subgroup we suppose not and let G be a minimal counterexample. Then G has a unique minimal normal subgroup A , A is a q -group for some prime $q \neq p$, and if P is a p -Sylow subgroup of $G^{\mathfrak{N}}$, then $PA \leq G$. By the Frattini argument $G = N_G(P) \cdot A$, and hence $N_G(P)$ is a coreless complement to A , and $A = C_G(A)$. Let Q be a p -complement of $G^{\mathfrak{N}}$ normalized by D . Then DQ is a Frobenius group with kernel Q , so that Q is nilpotent ([14]) and D is cyclic or generalized quaternion. Since $A = C_G(A)$, Q is a q -group.

Let L/A be a chief factor of G with $L \leq PA$. Since all chief factors of G below $G^{\mathfrak{F}}$ are eccentric and L/A is centralized by P , L/A is an eccentric chief factor of DLQ . Thus D is a system normalizer of DLQ . By minimality of G , if $DLQ < G$ then $1 < \mathbf{O}_p(DLQ) \leq C_G(A)$, a contradiction. Thus $G = DLQ$ and $L = AP$.

Let $G_0/G^{\mathfrak{N}}$ be a chief factor of G . Then $|G_0 : G^{\mathfrak{N}}| = p$. Moreover, Clifford's theorem and the fpf action of D ensure that G_0 has no central chief factors below $G^{\mathfrak{N}}$. Consequently $G_0^{\mathfrak{N}} = G^{\mathfrak{N}}$, and by minimality of G we must have $G = G_0$. Thus $|D| = p$.

We may suppose that D normalizes P , so that DP is a p -Sylow subgroup of G . If DP is abelian then $D \leq C_G(PA/A) \leq G$, whence $PA/A \leq Z(G/A)$, contradicting

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the fact that $G^{\mathfrak{F}}$ is hyper-eccentric. Thus $(DP)' \neq 1$ and A has an irreducible DP -submodule N which is not centralized by $(DP)'$. Restrict the conjugation representation of DP on N to the normal subgroup P . By Clifford's Theorem, since $|D| = p$, $N = N_1 \oplus \cdots \oplus N_p$ with $N_1, \dots, N_p$ non-isomorphic irreducible P -modules permuted cyclically by D .

Say $D = \langle d \rangle$, and let n be a non-zero element of N_1 . Then $n + nd + \cdots + nd^{p-1}$ is a non-identity element of A fixed by D , contradicting the fpf action of D on A . The proof is complete.

Groups satisfying the hypotheses of Theorem 4.2 have several obvious resemblances to Frobenius groups. The corollary below lists properties of such groups which follow from the theorems of this section.

Corollary 4.3. *Let the formation $\mathfrak{F}$, the group G and the $\mathfrak{F}$ -normalizer D of G satisfy the hypotheses of Theorem 4.2. If $D \neq G$, then D and G have the following properties.*

- (1) D is a p -group for some prime p .
- (2) $G^{\mathfrak{N}}$ has a normal p -Sylow subgroup P , and G/P is a Frobenius group with Frobenius kernel $G^{\mathfrak{N}}/P$.
- (3) D is a system normalizer of G and complements $G^{\mathfrak{N}}$.
- (4) D is cyclic or generalized quaternion. If D is cyclic, then all complements of $G^{\mathfrak{N}}$ are conjugate to D .
- (5) The p -Sylow subgroups of G are Carter subgroups of G .
- (6) If G has regular p -Sylow subgroups, then G is a Frobenius group with kernel $G^{\mathfrak{N}}$.

Proof. Statements (1), (2), (3), and (4) follow immediately from Theorems 3.3, 4.1, and 4.2. A Carter subgroup contains a system normalizer and hence (by fpf action of D) must be a p -group. Since Carter subgroups are self-normalizing, (5) holds. By a theorem of CARTER [1], G has a normal p -complement if its p -Sylow subgroups are regular. Thus (6) follows from (2).

As a final comment, we remark that we know of no examples of groups to show that the quaternion-free hypotheses is necessary in Theorem 3.3 or Theorem 4.1. However, our proof of Theorem 3.3 uses the assumption at two key points.

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