Symmetries and conserved quantities[1]

D. E. Soper[2]
University of Oregon

Physics 611, Theoretical Mechanics
8 October 2012

1 The lagrangian

We consider a mechanical system described by coordinates

$$\{q(t)\}_N = \{q_1(t), q_2(t), \dots, q_N(t)\}$$

and the corresponding velocities

$$\{\dot{q}(t)\}_N = \{\dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_N(t)\}.$$

The system evolves according to the lagrangian $L(\{q(t), \dot{q}(t)\}_N, t)$:

$$\frac{d}{dt}\frac{\partial L(\{q,\dot{q}\}_N, t)}{\partial \dot{q}_J} = \frac{\partial L(\{q,\dot{q}\}_N, t)}{\partial q_J} \; . \tag{1}$$

We have a special interest in the case in which the coordinates are the cartesian components of the positions of particles,

$$\{\boldsymbol{x}(t)\}_n = \{\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \dots, \boldsymbol{x}_n(t)\}.$$

In this case, each particle is described by three components of its position, (x_J^1, x_J^2, x_J^3), so there are $N = 3n$ coodinate variables. In this case, we suppose that $L = T - V$ where

$$T = \sum_J \frac{1}{2}\, m_J \dot{\boldsymbol{x}}_J^2 \tag{2}$$

and

$$V = \frac{1}{2}\sum_{JK} V_{JK}(|\boldsymbol{x}_J - \boldsymbol{x}_K|) + \sum_J V_J^{(e)}(\boldsymbol{x}_J, t) \tag{3}$$

with $V_{KJ} = V_{JK}$ and $V_{JJ} = 0$.

[2]soper@uoregon.edu

2 Symmetries

We want to investigate what happens when the lagrangian has a symmetry under an infinitesimal change of coordinates $\{q(t)\}_N \to \{q(t)+\delta q(t)\}_N$. Here δq_J is specified by a definite rule giving it as a function of the $\{q, \dot{q}\}_N$ and t and is proportional to a parameter $\delta\epsilon$ that we take to be infinitesimal. This can best be explained by giving several examples, for which we use the cartesian components of vector positions as the coordinates:

- *Translation in space.* Each position is translated through a distance $\delta\epsilon$ in the direction of a unit vector $\boldsymbol{n}$:

$$\delta \boldsymbol{x}_J = \boldsymbol{n}\, \delta\epsilon\ . \tag{4}$$

- *Rotation.* Each position is rotated through an angle $\delta\epsilon$ about an axis the direction a unit vector $\boldsymbol{n}$:

$$\delta \boldsymbol{x}_J = \boldsymbol{n} \times \boldsymbol{x}_J\, \delta\epsilon\ . \tag{5}$$

- *Translation in time.* We translate the time variable through a shift $\delta\epsilon$, so that

$$\boldsymbol{x}(t) + \delta\boldsymbol{x}(t) + \mathcal{O}((\delta\epsilon)^2) = \boldsymbol{x}(t+\delta\epsilon) = \boldsymbol{x}(t) + \dot{\boldsymbol{x}}(t)\, \delta\epsilon + \mathcal{O}((\delta\epsilon)^2)\ . \tag{6}$$

  Thus we take

$$\delta\boldsymbol{x}(t) = \dot{\boldsymbol{x}}(t)\, \delta\epsilon\ . \tag{7}$$

- *Boost to a new reference frame.* We change to a new reference frame moving with velocity $\delta\epsilon$ in the direction of a unit vector $-\boldsymbol{n}$, so that $\boldsymbol{x}_J + \delta\boldsymbol{x}_J$ differs from $\boldsymbol{x}_J$ by an amount that grows with time:

$$\delta \boldsymbol{x}_J = \boldsymbol{n}\, t\, \delta\epsilon\ . \tag{8}$$

In fact, these are the symmetries of mechanics in the non-relativistic approximation that hold as long as no external forces are imposed on the system. By symmetry here, we mean that the symmetry transformation either leaves the lagrangian invariant or changes it by a total time derivative:

$$\delta L(\{q,\dot{q}\}_N, t) = \frac{d}{dt} B(\{q,\dot{q}\}_N, t)\, \delta\epsilon\ . \tag{9}$$

Here B may be zero. If Eq. (9) holds for the coordinate transformation, then corresponding change δS in the action depends only on the coordinates and velocities at the starting and ending times. This implies that if $\{q(t)\}_N$ solves the equations of motion, then $\{q(t)+\delta q(t)\}_N$ also solves the equations of motion.

Let's see if the transformations that we looked at are indeed symmetries in this sense for the lagrangian speciefied in Eqs. (2) and (3) with $V_J^{(e)} = 0$:

- *Translation in space.*
$$\delta \boldsymbol{x}_J = \boldsymbol{n}\,\delta\epsilon\ . \tag{10}$$
Then $\delta\dot{\boldsymbol{x}}_J = 0$ so $\delta T = 0$ and $\delta(\boldsymbol{x}_J - \boldsymbol{x}_K) = 0$ so $\delta V = 0$. Thus
$$\delta L = 0\ . \tag{11}$$

- *Rotation.*
$$\delta \boldsymbol{x}_J = \boldsymbol{n} \times \boldsymbol{x}_J\,\delta\epsilon\ . \tag{12}$$
Then $\delta\dot{\boldsymbol{x}}_J = \boldsymbol{n} \times \dot{\boldsymbol{x}}_J\,\delta\epsilon$ so $\dot{\boldsymbol{x}}_J \cdot \delta\dot{\boldsymbol{x}}_J$ so $\delta T = 0$ and also $\delta|\boldsymbol{x}_J - \boldsymbol{x}_K| = 0$ so $\delta V = 0$. Thus
$$\delta L = 0\ . \tag{13}$$

- *Translation in time.*
$$\delta \boldsymbol{x}(t) = \dot{\boldsymbol{x}}(t)\,\delta\epsilon\ . \tag{14}$$
Then
$$\begin{aligned} L(\{q(t),\dot q(t)\}_N) &+ \delta L(\{q(t),\dot q(t)\}_N) + \mathcal{O}((\delta\epsilon)^2) \\ &= L(\{q(t+\delta\epsilon),\dot q(t+\delta\epsilon)\}_N) \\ &= L(\{q(t),\dot q(t)\}_N) + \frac{d}{dt}L(\{q(t),\dot q(t)\}_N)\delta\epsilon \\ &\quad + \mathcal{O}((\delta\epsilon)^2)\ . \end{aligned} \tag{15}$$
Thus
$$\delta L(\{q(t),\dot q(t)\}_N) = \frac{d}{dt}L(\{q(t),\dot q(t)\}_N)\delta\epsilon\ , \tag{16}$$
or
$$B(\{q(t),\dot q(t)\}_N, t) = L(\{q(t),\dot q(t)\}_N)\ . \tag{17}$$

- *Boost to a new reference frame.*

$$\delta \boldsymbol{x}_J = \boldsymbol{n}\, t\, \delta\epsilon \; . \tag{18}$$

Then

$$\delta \dot{\boldsymbol{x}}_J = \boldsymbol{n}\, \delta\epsilon \; , \tag{19}$$

so

$$\delta T = \sum_J m_J \dot{\boldsymbol{x}}_J \cdot \boldsymbol{n}\, \delta\epsilon = \frac{d}{dt} \sum_J m_J \boldsymbol{x}_J \cdot \boldsymbol{n}\, \delta\epsilon = \frac{d}{dt} M \boldsymbol{R}(t) \cdot \boldsymbol{n}\, \delta\epsilon \; . \tag{20}$$

On the other hand, $\delta(\boldsymbol{x}_J - \boldsymbol{x}_K) = 0$ so $\delta V = 0$. Thus

$$\delta L = \frac{d}{dt} M \boldsymbol{R}(t) \cdot \boldsymbol{n}\, \delta\epsilon \; . \tag{21}$$

That is,

$$B(\{q(t), \dot{q}(t)\}_N, t) = M \boldsymbol{R}(t) \cdot \boldsymbol{n} \; . \tag{22}$$

3 Nöther's theorem

There is a theorem due to Emmy Nöther[3] that says that to every invariance of the lagrangian there corresponds a conserved quantity. The theorem also constructs the corresponding conserved quantity.

The proof is simple. Consider any symmetry operation $\{q(t)\}_N \to \{q(t) + \delta q(t)\}_N$ such that Eq. (9) holds,

$$\delta L(\{q, \dot{q}\}_N, t) = \frac{d}{dt} B(\{q, \dot{q}\}_N, t)\, \delta\epsilon \; . \tag{23}$$

We can calculate δL by relating it to $\{\delta q(t), \delta \dot{q}(t)\}_N$:

$$\begin{aligned} \delta L(\{q, \dot{q}\}_N, t) = & \sum_J \left[\frac{\partial L(\{q, \dot{q}\}_N, t)}{\partial \dot{q}_J}\, \delta \dot{q}_J(t) + \frac{\partial L(\{q, \dot{q}\}_N, t)}{\partial q_J}\, \delta q_J(t) \right] \\ = & \sum_J \left[-\frac{d}{dt} \frac{\partial L(\{q, \dot{q}\}_N, t)}{\partial \dot{q}_J} + \frac{\partial L(\{q, \dot{q}\}_N, t)}{\partial q_J} \right] \delta q_J(t) \\ & + \frac{d}{dt} \sum_J \frac{\partial L(\{q, \dot{q}\}_N, t)}{\partial \dot{q}_J}\, \delta q_J(t) \; . \end{aligned} \tag{24}$$

[3] E. Nöther, Nachr. Ges. Wiss. Göttingen, 171 (1918).

If the coordinates obey the equations of motion, the first term on the right hand side vanishes. Then, comparing Eq. (23) to Eq. (24), we have

$$\frac{d}{dt}B(\{q,\dot{q}\}_N,t)\,\delta\epsilon = \frac{d}{dt}\sum_J \frac{\partial L(\{q,\dot{q}\}_N,t)}{\partial \dot{q}_J}\,\delta q_J(t)\ . \tag{25}$$

Let us define

$$Q(t) = \sum_J \frac{\partial L(\{q,\dot{q}\}_N,t)}{\partial \dot{q}_J}\,\delta q_J(t)/\delta\epsilon - B(\{q,\dot{q}\}_N,t)\ \ . \tag{26}$$

Then $Q(t)$ is conserved:

$$\frac{dQ(t)}{dt} = 0\ . \tag{27}$$

More precisely, $Q(t)$ is conserved when the system obeys its equations of motion.

Let's look at our examples and see what the conserved quantities are:

- *Translation in space.*

  $$\delta\boldsymbol{x}_J = \boldsymbol{n}\,\delta\epsilon\ . \tag{28}$$

  $$B = 0\ . \tag{29}$$

  Then

  $$Q(t) = \sum_J m_J\,\dot{\boldsymbol{x}}_J\cdot\boldsymbol{n} = \boldsymbol{P}\cdot\boldsymbol{n}\ \ . \tag{30}$$

  The component of the momentum in the direction $\boldsymbol{n}$ is conserved.

- *Rotation.*

  $$\delta\boldsymbol{x}_J = \boldsymbol{n}\times\boldsymbol{x}_J\,\delta\epsilon\ . \tag{31}$$

  $$B = 0\ . \tag{32}$$

  $$Q(t) = \sum_J m_J\,\dot{\boldsymbol{x}}_J\cdot(\boldsymbol{n}\times\boldsymbol{x}_J) = \left[\sum_J m_J\,(\boldsymbol{x}_J\times\dot{\boldsymbol{x}}_J)\right]\cdot\boldsymbol{n}\ \ . \tag{33}$$

  The component of the angular momentum in the direction $\boldsymbol{n}$ is conserved.

- *Translation in time.*

$$\delta \boldsymbol{x}(t) = \dot{\boldsymbol{x}}(t)\,\delta\epsilon\ . \tag{34}$$

$$B = L\ . \tag{35}$$

Then

$$Q(t) = \sum_J m_J\, \dot{\boldsymbol{x}}_J \cdot \dot{\boldsymbol{x}}_J - L = T + V\ \ . \tag{36}$$

The energy is conserved.

- *Boost to a new reference frame.*

$$\delta \boldsymbol{x}_J = \boldsymbol{n}\, t\, \delta\epsilon\ . \tag{37}$$

$$B = M\boldsymbol{R}(t)\cdot\boldsymbol{n}\ . \tag{38}$$

Then

$$Q(t) = \sum_J m_J\, \dot{\boldsymbol{x}}_J \cdot \boldsymbol{n}\, t - M\boldsymbol{R}(t)\cdot\boldsymbol{n} = [\boldsymbol{P}\, t - M\boldsymbol{R}(t)]\cdot\boldsymbol{n}\ \ . \tag{39}$$

The equation $Q(t) = Q(0)$ is then

$$\boldsymbol{R}(t)\cdot\boldsymbol{n} = \boldsymbol{R}(0)\cdot\boldsymbol{n} + \frac{1}{M}\,\boldsymbol{P}\cdot\boldsymbol{n}\ t\ . \tag{40}$$

That is, the commponent along the direction $\boldsymbol{n}$ of the center of mass position moves with constant velocity. (One can derive this from momentum conservation, but here it emerges as a separate law.)

One common case of a symmetry and consequent conserved quantity is when the lagrangian is expressed as a function of coordinates $\{q(t)\}_N$ and velocities $\{\dot{q}(t)\}_N$ but does not have an explicit time dependence. For instance, this happens when any external force is independent of time and when any constraints are independent of time. Then we have a lagrangian $L(\{q(t),\dot{q}(t)\}_N)$ that changes with time only because $\{q(t)\}_N$ and $\{\dot{q}(t)\}_N$ change with time. In this case, the system is invariant under time translation, $\delta q_J(t) = \dot{q}_J(t)\delta\epsilon$, with $B = L$. Then the quantity

$$h(\{q(t),\dot{q}(t)\}_N) = \sum_J \left[\frac{\partial L(\{q(t),\dot{q}(t)\}_N)}{\partial \dot{q}_J}\,\dot{q}_J(t)\right] - L(\{q(t),\dot{q}(t)\}_N) \tag{41}$$

is conserved. This is a generalization of the case of time translation invariance for cartesian coordinates $\boldsymbol{x}_J(t)$. We normally call the conserved quantity that results from time translation invariance the energy.

Another common case of a symmetry and consequent conserved quantity is when the lagrangian does not depend on one of the coordinates, say q_K. Then it is invariant under translation of that one coordinate, $\delta q_J(t) = \delta_{JK}\delta\epsilon$. In this case the conserved quantity is

$$Q(t) = \frac{\partial L(\{q(t), \dot{q}(t)\}_N)}{\partial \dot{q}_K} \; . \tag{42}$$

We often call this the momentum conjugate to the coordinate q_K. If q_K is an angle, then this is physically an angular momentum.