Free rotation of a rigid body[1]

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1 Introduction

In this section, we describe the motion of a rigid body that is free to rotate about its center of mass. If the body is just floating in space with no forces actding on it, then the center of mass moves with constant velocity. If the body is in free fall in the gravitational field of the earth, then the center of mass moves along a parabolic trajectory. In both cases, the motion of the center of mass is not very interesting, so we just eliminate it from consideration by supposing that the center of mass is fixed.

What the body can do is rotate, so we have to figure out how to describe an arbitrary rotation.

2 Rotations

Let $\bar{x}_I$ be the coordinates of a point in the inertial frame in which we suppose that the center of mass of the body is fixed. We will call this the lab frame. Let x_i be the coordinates of the same point in a reference frame fixed on the body. (I use capital letters for components in the lab frame, lower case letters for components in the body frame.) Then the two coordinates are related by a rotation matrix,

$$\bar{x}_I = R_{Ij}\, x_j. \tag{1}$$

The matrix R tells the rotation that the body has undergone. Thus the dynamical question is to determine how R changes with time. But we have a problem: a 3×3 matrix has nine components, but there are only three degrees of freedom. Thus we need to write R in terms of three variables.

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One straightforward way to do this (not mentioned in your book) is to say that R is a rotation through angle ψ about an axis $\boldsymbol{n}$ whose components are

$$(n_1, n_2, n_3) = (\bar{n}_1, \bar{n}_2, \bar{n}_3) = (\sin\theta\ \cos\phi, \sin\theta\ \sin\phi, \cos\theta). \tag{2}$$

(To think about: why does $(n_1, n_2, n_3) = (\bar{n}_1, \bar{n}_2, \bar{n}_3)$?). The rotation matrix is

$$R_{Ij} = \cos\psi\ (\delta_{Ij} - n_I n_j) - \sin\psi\ \epsilon^{Ijk} n_k + n_I n_j \tag{3}$$

Where did this come from? It is the standard result in the case that n_i is along one of the coordinate axes and it is covariant under rotations of the coordinate axes in the lab and body system made before we rotate one of these systems with respect to the other.

This establishes that R can be written in terms of three coordinates ψ, θ, and ϕ. However, people usually use three different coordinates, called the Euler angles.

The Euler angles are defined by writing R as a product of three simple matrices:

$$R = R_{(1)} R_{(2)} R_{(3)}. \tag{4}$$

The matrix $R_{(1)}$ is

$$R_{(1)} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

The matrix $R_{(2)}$ is

$$R_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{6}$$

The matrix $R_{(3)}$ is

$$R_{(3)} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

The product is

$$R = \tag{8}$$

$$\begin{pmatrix} \cos\phi\,\cos\psi - \cos\theta\,\sin\psi\,\sin\phi & -\sin\psi\,\cos\phi - \cos\theta\,\sin\phi\,\cos\psi & \sin\theta\,\sin\phi \\ \cos\psi\,\sin\phi + \cos\theta\,\cos\phi\,\sin\psi & -\sin\psi\,\sin\phi + \cos\theta\,\cos\phi\,\cos\psi & -\sin\theta\,\cos\phi \\ \sin\theta\,\sin\psi & \sin\theta\,\cos\psi & \cos\theta \end{pmatrix}.$$

To describe this in physical terms, think of the relation giving the coordinates x_i in the body system of a point with coordinates $\bar{x}_i$ in the lab system:

$$x_i = [R^T_{(3)} R^T_{(2)} R^T_{(1)}]_{iI} \bar{x}_I. \tag{9}$$

We begin with a rotation of coordinates about the body 3-axis (which at the start is the same as the lab 3-axis) through angle $-\phi$:

$$R^T_{(1)} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

(That is, the coordinate axes attached to the body rotate though angle $+\phi$ about the body 3-axis.) Then we perform a rotation of coordinates about the body 1-axis through angle $-\theta$:

$$R^T_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{11}$$

(That is, the coordinate axes attached to the body rotate though angle $+\theta$ about the body 1-axis.) Finally, we rotate the coordinates about the body 3-axis through angle $-\psi$:

$$R^T_{(3)} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

(That is, the coordinate axes attached to the body rotate though angle $+\psi$ about the body 3-axis.)

Exercise 2.1: Multiply the three matrices together to show that this result holds. You might want to use Mathematica to do this.

Exercise 2.2: What if we defined the Euler′ angles as follows. We subject the body to three successive rotations. We begin with a rotation about the lab 3-axis through angle ϕ. Then we perform a rotation about the lab 1-axis through angle θ. Finally, we rotate about the lab 3-axis through angle

ψ. The body coordinates would then be related to the lab coordinates by $\bar{x}_I = R'_{Ij}\,x_j$, but with a different matrix R'. Please write R' as a matrix product of three matrices, but do not perform the matrix multiplication.

Exercise 2.3: What if we defined the Euler″ angles as follows. We subject the body to three successive rotations. We begin with a rotation about the lab 3-axis through angle ϕ. Then we perform a rotation about the body 1-axis through angle θ. Finally, we rotate about the lab 1-axis through angle ψ. The body coordinates would then be related to the lab coordinates by $\bar{x}_I = R''_{Ij}\,x_j$, but with a different matrix R''. Please write R'' as a matrix product of three matrices, but do not perform the matrix multiplication.

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3 The lagrangian

Now that we know how to describe a rotation, we can write down the lagrangian. It is

$$L = \sum_p \tfrac{1}{2} m_p \bar{v}_I^{(p)} \bar{v}_I^{(p)} \tag{13}$$

where we have a sum over an index p that labels the particles in the rigid body and where $\bar{v}_I^{(p)}$ is the Ith component of the velocity of particle p in the lab system. There is an implicit sum over the indices I. We have

$$\bar{x}_I^{(p)} = R_{Ij}\,x_j^{(p)} \tag{14}$$

where R_{Ij} depends on the time t but $x_j^{(p)}$ does not. Thus

$$\bar{v}_I^{(p)} = \dot{R}_{Ij}\,x_j^{(p)} \tag{15}$$

Thus

$$L = \sum_p \tfrac{1}{2} m_p \dot{R}_{Ij}\,x_j^{(p)}\,\dot{R}_{Ik}\,x_k^{(p)} \tag{16}$$

We define the components of the inertia tensor in the body frame by

$$I_{jk} = \sum_p m_p \left[(x_l^{(p)} x_l^{(p)})\delta_{jk} - x_j^{(p)} x_k^{(p)} \right]. \tag{17}$$

Note that the components of this tensor do not change as the body moves because we are taking its components along the body fixed axes. We actually don't have quite this tensor in the lagrangian. What we have is

$$\tilde{I}_{jk} = \sum_p m_p \, x_j^{(p)} x_k^{(p)}. \tag{18}$$

We can relate these. We have

$$I_{jk} = \tilde{I}_{ll} \, \delta_{jk} - \tilde{I}_{jk} \tag{19}$$

so

$$I_{jj} = \tilde{I}_{ll} \times 3 - \tilde{I}_{jj} = 2 \, \tilde{I}_{jj} \tag{20}$$

Thus

$$\tilde{I}_{jk} = \tfrac{1}{2} I_{ll} \, \delta_{jk} - I_{jk}. \tag{21}$$

This gives us the lagrangian

$$L = \tfrac{1}{2} \, \dot{R}_{Ij} \, \dot{R}_{Ik} \, (\tfrac{1}{2} I_{ll} \, \delta_{jk} - I_{jk}). \tag{22}$$

Here R is a function of the three Euler angles, so L is a function of the three Euler angles and their time derivatives. We simply have to multiply everything together to express L as a simple function of the three Euler angles and their time derivatives, then differentiate to obtain the equations of motion. However, a little bit of thought may be better than blind algebra. We undertake this in the next section.

4 Analysis of the lagrangian

In order to understand the lagrangian better, let us write $\dot{R}_{Ij}$ in the form

$$\dot{R}_{Ij} = R_{Ik} \, \Omega_{kj} \tag{23}$$

where this equation defines the matrix Ω: $\Omega = R^{-1} \, \dot{R}$. The matrix Ω has a special structure. To see what it is, recall that because R is a rotation matrix we have

$$R^T \, R = 1 \tag{24}$$

Thus $R^{-1} = R^T$, so

$$\Omega = R^T \, \dot{R}. \tag{25}$$

Differentiating (24) with respect to t we have

$$\dot{R}^T R + R^T \dot{R} = 0 \tag{26}$$

or

$$\Omega^T + \Omega = 0. \tag{27}$$

This says that Ω is antisymmetric, so we can write it as

$$\Omega_{ij} = -\epsilon_{ijk}\,\omega_k. \tag{28}$$

Let's see how to interpret ω_l. To do this, consider a fixed point with coordinates $\bar{x}_I$ in the lab. The body coordinates of this point are

$$x_i = R^T_{iJ}\,\bar{x}_J \tag{29}$$

Of course, the body coordinates of the point are changing because R is always changing. Let's see how fast the coordinates x_i are changing. We use a matrix notation in order to keep the number of indices down:

$$\dot{x} = \dot{R}^T\,\bar{x} = \dot{R}^T R\, R^T \bar{x} = \Omega^T\, x = -\Omega\, x. \tag{30}$$

That is

$$\dot{x}_i = \epsilon_{ijk}\,\omega_k\; x_j = -\epsilon_{ikj}\,\omega_k\; x_j \tag{31}$$

or, in a vector notation,

$$\frac{d}{dt}\boldsymbol{x} = -\boldsymbol{\omega}\times\boldsymbol{x}. \tag{32}$$

That is, an observer on the body sees a point in the lab rotating with angular velocity $-\boldsymbol{\omega}$. Thus the components ω_i are the components in the body fixed coordinates of the instantaneous angular velocity of the body.

Since Ω is so nice, let's write the Lagrangian in terms of it. We have

$$\begin{aligned}
L &= \tfrac{1}{2}\,R_{Im}\,\Omega_{mj}\,R_{In}\,\Omega_{nk}\;(\tfrac{1}{2}I_{ll}\,\delta_{jk} - I_{jk}) \\
&= \tfrac{1}{2}\,\delta_{mn}\,\Omega_{mj}\,\Omega_{nk}\;(\tfrac{1}{2}I_{ll}\,\delta_{jk} - I_{jk}) \\
&= \tfrac{1}{2}\,\Omega_{nj}\,\Omega_{nk}\;(\tfrac{1}{2}I_{ll}\,\delta_{jk} - I_{jk}) \\
&= \tfrac{1}{2}\,\epsilon_{nji}\,\omega_i\,\epsilon_{nkm}\,\omega_m\;(\tfrac{1}{2}I_{ll}\,\delta_{jk} - I_{jk}) \\
&= \tfrac{1}{2}\,(\delta_{jk}\,\delta_{im} - \delta_{jm}\,\delta_{ik})\,\omega_i\,\omega_m\;(\tfrac{1}{2}I_{ll}\,\delta_{jk} - I_{jk}) \\
&= \tfrac{1}{2}\,\omega_i\,\omega_i\;(\tfrac{1}{2}I_{ll}\,\delta_{jj} - I_{jj}) - \omega_k\,\omega_j\;(\tfrac{1}{2}I_{ll}\,\delta_{jk} - I_{jk}) \\
&= \tfrac{1}{2}\,\omega_i\,\omega_i\;(\tfrac{3}{2}I_{ll} - I_{ll}) - \omega_i\,\omega_i\;\tfrac{1}{2}I_{ll} + \omega_k\,\omega_j\;I_{jk} \\
&= \tfrac{1}{2}\,\omega_j\,\omega_k\;I_{jk}.
\end{aligned} \tag{33}$$

We may as well simplify things a little more. We did not specify how to choose the body axes. Since I is a real symmetric matrix, we can choose axes such that it is diagonal:

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{34}$$

The values I_1, I_2, I_3 are known as the principle moments of inertia. Using this choices of axes in the body, we have

$$L = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \tag{35}$$

Of course, we knew how L had to turn out. The lagrangian is the kinetic energy and the kinetic energy is $\frac{1}{2}I\omega^2$ with suitable tensor indices. So what we have that is really new is the relation of $\boldsymbol{\omega}$ to the angles that define the orientation of the body in the lab. Using the definitions in Eqs. (25) and (28) one finds

$$\begin{aligned} \omega_1 &= \dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi \\ \omega_2 &= \dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi \\ \omega_3 &= \dot{\phi}\,\cos\theta + \dot{\psi}. \end{aligned} \tag{36}$$

Exercise 4.1: Use Eqs. (35) and (36) to find the equations of motion for the three Euler angles.

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5 The angular velocity and the Euler angles

Actually, Eq. (36) is amazingly simple. In principle, we differentiate R with respect to time, then form $\Omega = R^T\dot{R}$. Finally ω_1 is $-\Omega_{23}$, *etc.*. But that's pretty messy. Couldn't we be more clever? Landau gives a nice geometrical argument. But we can do it with algebra, too, and that would teach us something.

We have

$$\Omega = R_{(3)}^T R_{(2)}^T R_{(1)}^T [R_{(1)}R_{(2)}\dot{R}_{(3)} + R_{(1)}\dot{R}_{(2)}R_{(3)} + \dot{R}_{(1)}R_{(2)}R_{(3)}]. \tag{37}$$

Let's consider the three terms separately. Begin with the first. Using $R^T R = 1$ we have

$$R^T_{(3)} R^T_{(2)} R^T_{(1)} R_{(1)} R_{(2)} \dot{R}_{(3)} = R^T_{(3)} \dot{R}_{(3)}. \tag{38}$$

That's pretty easy to compute:

$$R^T_{(3)} \dot{R}_{(3)} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\psi} \tag{39}$$

Multiplying these and using $\sin^2\psi + \cos^2\psi = 1$, we get

$$R^T_{(3)} \dot{R}_{(3)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\psi} \tag{40}$$

That is

$$\left[R^T_{(3)} \dot{R}_{(3)} \right]_{ij} = -\dot{\psi}\ \epsilon_{ijk}\ \hat{z}_k \tag{41}$$

where $\hat{z}$ is a unit vector in the 3-direction: $\hat{z}_k = \delta_{k3}$. We now have one third of our answer.

Now, try the second term:

$$R^T_{(3)} R^T_{(2)} R^T_{(1)} R_{(1)} \dot{R}_{(2)} R_{(3)} = R^T_{(3)} R^T_{(2)} \dot{R}_{(2)} R_{(3)}. \tag{42}$$

As in the previous analysis, we get

$$\left[R^T_{(2)} \dot{R}_{(2)} \right]_{ij} = -\dot{\theta}\ \epsilon_{ijk}\ \hat{x}_k \tag{43}$$

where $\hat{x}_k = \delta_{k1}$ represents a unit vector in the 1-direction. But now we have this matrix wedged in between $R^T_{(3)}$ and $R_{(3)}$. We need a deep truth.

Here is a deep truth. The epsilon tensor is invariant under rotations:

$$R_{ia} R_{jb} R_{kc} \epsilon_{abc} = \epsilon_{ijk} \tag{44}$$

for any rotation matrix R. To see this, note that the left hand side of the equation is ϵ_{ijk} times the determinant of R. But if $R^T R = 1$ we have $(\det R^T)(\det R) = 1$ or $(\det R)^2 = 1$ so $\det R = \pm 1$. In fact "mirror reflections" have determinant -1, but any rotation that you can reach by a continuous path from the unit matrix must have determinant $+1$, since the unit matrix has determinant $+1$ and if we change R continuously from the unit matrix, its determinant can't jump to -1.

We can apply this result by inserting one $R^T R$, which is the unit matrix, into our previous result, then using the invariance of the epsilon tensor under rotations:

$$\begin{aligned} \left[R^T_{(3)} R^T_{(2)} \dot{R}_{(2)} R_{(3)}\right]_{ij} &= -\dot{\theta}\ R^{(3)}_{ai} R^{(3)}_{bj} \epsilon_{abk}\, \hat{x}_k \\ &= -\dot{\theta}\ R^{(3)}_{ai} R^{(3)}_{bj} \epsilon_{abc} R^{(3)}_{cl} R^{(3)}_{kl}\, \hat{x}_k \\ &= -\dot{\theta}\ \epsilon_{ijl} R^{(3)}_{kl}\, \hat{x}_k \end{aligned} \tag{45}$$

The vector $u_l = R^{(3)}_{kl}\, \hat{x}_k$ is what you get if you rotate $\hat{x}$ about the 3-axis through angle $-\psi$:

$$R^{(3)}_{kl}\, \hat{x}_k = \cos\psi\ \hat{x}_k - \sin\psi\ \hat{y}_k. \tag{46}$$

(That's most easily seen with a picture, but you could just multiply the vector by the matrix to verify it.) Thus

$$\left[R^T_{(3)} R^T_{(2)} \dot{R}_{(2)} R_{(3)}\right]_{ij} = -\dot{\theta}\ \epsilon_{ijk} \left[\cos\psi\ \hat{x}_k - \sin\psi\ \hat{y}_k\right]. \tag{47}$$

Now we are ready for the last term.

$$R^T_{(3)} R^T_{(2)} R^T_{(1)} \dot{R}_{(1)} R_{(2)} R_{(3)}. \tag{48}$$

We find

$$[R^T_{(1)} \dot{R}_{(1)}]_{ij} = -\dot{\phi}\ \epsilon_{ijk} \hat{z}_k \tag{49}$$

Then

$$R^T_{(3)} R^T_{(2)} R^T_{(1)} \dot{R}_{(1)} R_{(2)} R_{(3)} = -\dot{\phi}\ \epsilon_{ijk} u_k \tag{50}$$

where

$$u_k = R^{(2)}_{jl} R^{(3)}_{lk} \hat{z}_j = R^{(3)-1}_{kl} R^{(2)-1}_{lj} \hat{z}_j. \tag{51}$$

Thus we take a unit vector in the 3-direction and rotate it about the the 1-axis through angle $-\theta$ then about the 3-axis through angle $-\psi$. The first step gives the vector

$$\cos\theta\ \hat{z} + \sin\theta\ \hat{y}. \tag{52}$$

Then the second step gives

$$\cos\theta\ \hat{z} + \sin\theta\ \sin\psi\ \hat{x} + \sin\theta\ \cos\psi\ \hat{y}. \tag{53}$$

Thus

$$R_{(3)}^T R_{(2)}^T R_{(1)}^T \dot{R}_{(1)} R_{(2)} R_{(3)} = -\dot{\phi}\ \epsilon_{ijk}[\cos\theta\ \hat{z}_k + \sin\theta\ \ \sin\psi\ \hat{x}_k + \sin\theta\ \ \cos\psi\ \hat{y}_k]. \tag{54}$$

We can now put this together

$$\Omega_{ij} = -\epsilon_{ijk}\,\omega_k, \tag{55}$$

where

$$\begin{aligned} \boldsymbol{\omega} &= \dot{\psi}\ \hat{z}_k \\ &\quad +\dot{\theta}\ [\cos\psi\ \hat{x}_k - \sin\psi\ \hat{y}_k] \\ &\quad +\dot{\phi}\ [\cos\theta\ \hat{z}_k + \sin\theta\ \ \sin\psi\ \hat{x}_k + \sin\theta\ \ \cos\psi\ \hat{y}_k]. \end{aligned} \tag{56}$$

This is the result shown in Eq. (36).

6 The moment of inertia tensor and angular momentum

Let's look at the relation between the angular momentum of the body, its angular velocity, and the moment of inertia tensor. Working, to start with, in the lab system, we have

$$\begin{aligned} \bar{L}_I &= \sum_p m_p \epsilon_{IJK} \bar{x}_p^J \dot{\bar{x}}_p^K \\ &= \sum_p m_p \epsilon_{IJK} \bar{x}_p^J \epsilon_{KLM} \bar{\omega}^L \bar{x}_p^M \\ &= (\delta_{IL}\delta_{JM} - \delta_{IM}\delta_{JL}) \sum_p m_p \bar{x}_p^J \bar{\omega}^L \bar{x}_p^M \\ &= \sum_p m_p (\bar{x}_p^J \bar{x}_p^J \delta_{IL} - \bar{x}_p^I \bar{x}_p^L)\ \bar{\omega}^L \end{aligned} \tag{57}$$

This is

$$\bar{L}_I = \bar{I}_{IL}\,\bar{\omega}^L\ , \tag{58}$$

where

$$\bar{I}_{IL} = \sum_p m_p (\bar{x}_p^J \bar{x}_p^J \delta_{IL} - \bar{x}_p^I \bar{x}_p^L)\ . \tag{59}$$

This equation is covariant under rotations, so we can write it also in the body frame:

$$L_i = I_{il}\,\omega^l \ , \tag{60}$$

where

$$I_{il} = \sum_p m_p(x_p^j x_p^j \delta_{il} - x_p^i x_p^l) \ . \tag{61}$$

This equation is the same as our definition (17).

7 Euler's equations

We can write simple equations, known as Euler's equations, for the rate of change of $\boldsymbol{\omega}$ in the body coordinate system. Knowing $\boldsymbol{\omega}$ as a function of time does not directly tell you the Euler angles as a function of time because the components of $\boldsymbol{\omega}$ are related to not just the Euler angles but also their time derivatives. Nevertheless, Euler's equations are quite useful.

We begin by recalling Eq. (32) for the rate of change of the body frame components of a vector $\boldsymbol{A}$ that is fixed in the lab system.

$$\frac{d}{dt}A_i = -\epsilon_{ijk}\,\omega_j\,A_k. \tag{62}$$

The lab frame components of the angular angular momentum vector $\boldsymbol{L}$ are fixed, so we can apply this to the angular momentum. Using Eq. (60), we have

$$\frac{d}{dt}I_{ij}\,\omega_j = -\epsilon_{ijk}\,\omega_j\,I_{kl}\,\omega_l. \tag{63}$$

Since the components of the inertia tensor are fixed, this is

$$I_{ij}\,\dot\omega_j = -\epsilon_{ijk}\,\omega_j\,I_{kl}\,\omega_l. \tag{64}$$

We have chosen our body coordinates such that I is diagonal. Thus we have three equations, the first of which is

$$I_1\,\dot\omega_1 = -\omega_2\,I_3\,\omega_3 + \omega_3\,I_2\,\omega_2, \tag{65}$$

or

$$\dot\omega_1 = \omega_2\omega_3[I_2 - I_3]/I_1. \tag{66}$$

Thus the three equations are

$$
\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_3[I_2 - I_3]/I_1 \\
\dot{\omega}_2 &= \omega_3\omega_1[I_3 - I_1]/I_2 \\
\dot{\omega}_3 &= \omega_1\omega_2[I_1 - I_2]/I_3.
\end{aligned}
\tag{67}
$$

8 The tennis racket theorem

One simple consequence of Euler's equations is that if the body is rotating about one of the "principle axes" of the inertia tensor, then it continues to do so and the angular velocity stays constant. That is, $\omega_2 = \omega_3 = 0$ at one time, then ω_2 and ω_3 stay zero and ω_1 is constant.

Is this stable against small perturbations? Suppose that at time zero, $\omega_1 > 0$ while ω_2 and ω_3 are small. Then we can write the equations to first order in the small quantities:

$$
\begin{aligned}
\dot{\omega}_1 &= 0 \\
\dot{\omega}_2 &= \omega_3\omega_1[I_3 - I_1]/I_2 \\
\dot{\omega}_3 &= \omega_1\omega_2[I_1 - I_2]/I_3.
\end{aligned}
\tag{68}
$$

The first equation says that ω_1 is constant (in this approximation, and as long as ω_2 and ω_3 stay small). We can differentiate the other equations again and substitute for $\dot{\omega}_2$ and $\dot{\omega}_3$ on the right hand sides to get

$$
\begin{aligned}
\ddot{\omega}_2 &= -\omega_1^2 \frac{[I_1 - I_2][I_1 - I_3]}{I_2 I_3}\omega_2 \\
\ddot{\omega}_3 &= -\omega_1^2 \frac{[I_1 - I_2][I_1 - I_3]}{I_2 I_3}\,\omega_3.
\end{aligned}
\tag{69}
$$

Thus we have two identical equations that describe small oscillations if $[I_1 - I_2][I_1 - I_3] > 0$ or a combination of exponential growth and decay if $[I_1 - I_2][I_1 - I_3] < 0$. If I_1 is the largest eigenvalue we have a positive sign and thus small oscillations. If I_1 is the smallest eigenvalue we have a positive sign and thus small oscillations. But if I_1 is in-between I_2 and I_3, then we have exponential growth and decay. Of course, unless the initial conditions are just right, there will be a small amplitude for exponential decay and after a little while it is the growing part that you will see. The conclusion is that rotation about the intermediate axis is unstable.

9 Conserved quantities

We can gain some insight into how the body moves by making use of conserved quantities.

Begin by considering the angular momentum, $L_i = I_{ij}\omega_j$. The components in the body frame, with our choice of axes such that the inertia tensor is diagonal, are

$$\boldsymbol{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{70}$$

We know that the laboratory components of the angular momentum remain constant, but these body-frame components are continually changing since the body is rotating. We can get $\boldsymbol{\omega}$ from $\boldsymbol{L}$ using this equation, but it is perhaps more useful to consider how the body components of $\boldsymbol{L}$ change because the lab components of this vector do not change. (In contrast, the components of $\boldsymbol{\omega}$ change in both frames.)

Although the body components of $\boldsymbol{L}$ change, the sum of their squares equals the sum of the squares of the components in the lab frame. Thus

$$L^2 = L_1^2 + L_2^2 + L_3^2 \tag{71}$$

is a constant. Thus $\boldsymbol{L}$ remains on a sphere.

We also know that the kinetic energy E is conserved, with

$$2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2. \tag{72}$$

The surface of constant E is an ellipsoid. In terms of $\boldsymbol{L}$, this is

$$2E = \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}. \tag{73}$$

Thus $\boldsymbol{L}$ must stay on this ellipsoidal surface.

Combining these results, we see that $\boldsymbol{L}$ remains on the curve that is the intersection of these two surfaces. Evidently, the motion of $\boldsymbol{L}$ is periodic.

Let us choose the labelling so that $I_1 < I_2 < I_3$. Then suppose that E is about as big as it can be for a given L^2. That is, $E \approx L^2/I_1$. Then we must have $L_1^2 \approx L^2$ and $\text{Ł}_2^2 \approx 0$ and $L_3^2 \approx 0$. That is, $\boldsymbol{L}$ moves along a little path around the 1-axis. On the other hand, suppose that E is about as small as it could be, $E \approx L^2/I_3$. Then we must have $L_3^2 \approx L^2$ and $\text{Ł}_2^2 \approx 0$ and $L_1^2 \approx 0$. That is, $\boldsymbol{L}$ moves along a little path around the 3-axis. We see that there are motions in which the body is approximately rotating about either the axis with the smallest moment of inertia or about the axis with the largest moment of inertia. On the other hand, with $E \approx L^2/I_2$, the curve that $\boldsymbol{L}$ follows does *not* stay near the 2-axis. Motion about the 2-axis is not stable. That is what we saw in the previous section with a different style of analysis.

10 Symmetrical rigid body

An interesting special case occurs when two of the principal moments of inertia are equal, say $I_1 = I_2 \equiv I_T$. (We could have either $I_3 > I_T$ or $I_3 < I_T$.) In this case the intersection of the sphere $L^2 = const.$ with the ellipsoid $2E = const.$ is a circle about the symmetry axis, which is the 3-axis. We instantly conclude that $\boldsymbol{L}$ executes a uniform motion along this circle. The same statement holds for $\boldsymbol{\omega}$, with

$$(\omega_1, \omega_2, \omega_3) = (L_1/I_T, L_2/I_T, L_3/I_3). \tag{74}$$

We can see this by solving the Euler equations, which become

$$\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_3[I_T - I_3]/I_T \\
\dot{\omega}_2 &= -\omega_3\omega_1[I_T - I_3]/I_T \\
\dot{\omega}_3 &= 0.
\end{aligned} \tag{75}$$

The third equation says that ω_3 remains constant. The first two equations give

$$\ddot{\omega}_1 = -[\omega_3(I_T - I_3)/I_T]^2\omega_1. \tag{76}$$

The solution of this is

$$\omega_1 = A\sin(\tilde{\omega}\,t + \delta). \tag{77}$$

with

$$\tilde{\omega} = \omega_3(I_T - I_3)/I_T. \tag{78}$$

Then

$$\omega_2 = A\cos(\tilde{\omega}\,t + \delta). \tag{79}$$

That is, $\boldsymbol{\omega}$ and $\boldsymbol{L}$ move in circles with a precession velocity $\tilde{\omega}$. In the lab frame, the symmetry axis moves in circles about the (fixed) angular momentum vector with this precession velocity.

Let's see what this looks like in the lab frame. As a preliminary step, we write

$$(\omega_1, \omega_2, \omega_3) = \left(\frac{L_1}{I_T}, \frac{L_2}{I_T}, \frac{L_3}{I_T} + \left(\frac{1}{I_3} - \frac{1}{I_T} \right) L_3 \right). \tag{80}$$

That is

$$\boldsymbol{\omega} = \omega_L \, \boldsymbol{u} + \omega_n \, \boldsymbol{n} \tag{81}$$

where $\boldsymbol{n}$ is a unit vector along the *body* 3-axis, $\boldsymbol{u}$ is a unit vector along the direction of $\boldsymbol{L}$, and

$$\begin{aligned} \omega_L &= \frac{L}{I_T} \\ \omega_n &= \frac{I_T - I_3}{I_3 I_T} \, L_3. \end{aligned} \tag{82}$$

This representation will be useful to us because $\boldsymbol{u}$ is fixed in the lab and $\boldsymbol{n}$ is fixed in the body.

How do the lab components of $\boldsymbol{n}$ change with time? Since $\boldsymbol{n}$ is fixed in the body system, we have

$$\frac{d}{dt} \bar{n}_I = \epsilon_{IJK} \, \bar{\omega}_J \, \bar{n}_K. \tag{83}$$

(Recall our notation that vector names with bars and uppercase subscripts refer to lab coordinates.) We can insert our decomposition of $\boldsymbol{\omega}$ into this. The second term, proportional to $\boldsymbol{n}$ does not contribute since $\epsilon_{IJK} \, \bar{n}_J \, \bar{n}_K = 0$. Thus

$$\frac{d}{dt} \bar{n}_I = \omega_L \, \epsilon_{IJK} \, \bar{u}_J \, \bar{n}_K. \tag{84}$$

This says that $\boldsymbol{n}$ rotates about $\boldsymbol{u}$ with a uniform angular velocity ω_L.

Now while the body 3-axis is rotating about $\boldsymbol{u}$, the body as a whole is rotating somehow. To describe this motion, let's look at it in a reference frame that rotates about $\boldsymbol{u}$ with a uniform angular velocity ω_L. In this frame $\boldsymbol{u}$ is still fixed and $\boldsymbol{n}$ is also fixed. Consider a fixed point on the body with lab coordinates $\bar{x}_I$. Let its coordinates in the new frame be $\tilde{x}_a$. We have

$$\tilde{x}_a = R(t)_{aI} \, \bar{x}_I \tag{85}$$

where R is the rotation matrix relating the two frames. The time derivative of this is

$$\frac{d}{dt} \, \tilde{x}_a = R_{aI} \, \frac{d}{dt} \, \bar{x}_I - \omega_L \, \epsilon_{abc} \, \tilde{u}_b \, \tilde{x}_c. \tag{86}$$

We know what the rate of change of $\bar{x}_I$ is:

$$\frac{d}{dt}\,\bar{x}_I = \epsilon_{IJK}\,\bar{\omega}_J\,\bar{x}_K. \tag{87}$$

We can use

$$R_{aI}\,\epsilon_{IJK}\,\bar{\omega}_J\,\bar{x}_K = \epsilon_{abc}\,R_{bJ}\,R_{cK}\,\bar{\omega}_J\,\bar{x}_K = \epsilon_{abc}\,\tilde{\omega}_b\,\tilde{x}_c. \tag{88}$$

Thus

$$\frac{d}{dt}\,\tilde{x}_a = \epsilon_{abc}\,\tilde{\omega}_b\,\tilde{x}_c - \omega_L\,\epsilon_{abc}\,\tilde{u}_b\,\tilde{x}_c. \tag{89}$$

That is

$$\frac{d}{dt}\,\tilde{x}_a = \epsilon_{abc}\,(\tilde{\omega}_b - \omega_L\,\tilde{u}_b)\,\tilde{x}_c. \tag{90}$$

Recall that

$$\boldsymbol{\omega} - \omega_L\,\boldsymbol{u} = \omega_n\,\boldsymbol{n}. \tag{91}$$

Thus

$$\frac{d}{dt}\,\tilde{x}_a = \omega_n\,\epsilon_{abc}\,\tilde{n}_b\;\tilde{x}_c. \tag{92}$$

This says that (in this frame) the body rotates about the body 3-axis with a uniform angular velocity ω_n.

Thus the motion of the body is really pretty simple. It is a combination of two uniform rotations. The ratio of the two angular velocities is

$$\frac{\omega_n}{\omega_u} = \frac{I_T - I_3}{I_3}\;\frac{L_3}{L_T}. \tag{93}$$

Thus it depends on the shape of the body and on the initial conditions.

Exercise 10.1: Analyze the motion of a body with principle moments of inertia

$$\begin{aligned} I_1 &= 1\ \mathrm{kg\,m^2} \\ I_2 &= 2\ \mathrm{kg\,m^2} \\ I_3 &= 3\ \mathrm{kg\,m^2} \end{aligned} \tag{94}$$

At time zero let

$$\begin{aligned} \phi &= \pi/4 \\ \theta &= \pi/4 \\ \psi &= \pi/4 \end{aligned} \tag{95}$$

and

$$
\begin{aligned}
\dot{\phi} &= 1 \text{ s}^{-1} \\
\dot{\theta} &= 0 \\
\dot{\psi} &= 0.
\end{aligned} \tag{96}
$$

Find the location of the points with body coordinates $(1 \text{ m}, 0, 0)$, $(0, 1 \text{ m}, 0)$ and $(0, 0, 1 \text{ m})$ at times $t = 0$, $t = 1$ sec, and $t = 10$ sec.

Optionally, you can add graphs of where these three points go. If you use Mathematica, I recommend trying `ListPointPlot3D[points]`.
