A problem in fluid mechanics[1]

D. E. Soper[2]
University of Oregon
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For a classical field theory with fields $\Phi_K(x)$ defined from a lagrangian $\mathcal{L}(\partial_\alpha \Phi_K, \Phi_K, x)$, we define the momentum tensor

$$T_\mu{}^\nu = \partial_\mu \Phi_K \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_K)} - g_\mu^\nu \mathcal{L} \; . \tag{1}$$

When the fields obey their Euler-Lagrange equations of motion, we find

$$\partial_\nu T_\mu{}^\nu = -\frac{\partial \mathcal{L}}{\partial x^\mu} \; . \tag{2}$$

Here on the right hand side, we differentiate with respect to the x argument of $\mathcal{L}(\partial_\alpha \Phi_K, \Phi_K, x)$. Then when $\mathcal{L}$ has no such direct dependence on x, we have $\partial_\nu T_\mu{}^\nu = 0$.

For the lagrangian for a fluid, $\mathcal{L}_{\rm F} = -\rho U(\mathcal{V}, R_a)$, we defined

$$\begin{aligned}
\mathcal{V} &= 1/\rho \; , \\
P &= -\,\frac{\partial U(\mathcal{V}, R_a)}{\partial \mathcal{V}} \; , \\
\rho^2 &= J^\alpha J_\alpha \; , \\
u^\mu &= J^\mu/\rho \; , \\
J^\mu &= n(R)\epsilon^{\mu\alpha\beta\gamma}\frac{1}{3!}\epsilon_{abc}(\partial_\alpha R_a)(\partial_\beta R_b)(\partial_\gamma R_c) \; .
\end{aligned} \tag{3}$$

We found

$$T_{\rm F}^{\mu\nu} = \rho U u^\mu u^\nu + P(u^\mu u^\nu - g^{\mu\nu}) \; . \tag{4}$$

Now suppose that the fluid carries electric charge $q(R)$ per atom, so that there is an electric current

$$J_{\rm e}^\mu(x) = q(R) J^\mu(x) \; . \tag{5}$$

[2]soper@uoregon.edu

Suppose that there is an externally imposed electromagnetic field given by the vector potential $A_\mu(x)$. Then the fluid can be influenced by the electromagnetic field. This influence is described by adding a term to the lagrangian,

$$\mathcal{L}_{\rm I} = -A(x)_\alpha J_{\rm e}(x)^\alpha \quad . \tag{6}$$

Note that $A(x)_\alpha$ is fixed, not a dynmaical field. Thus $\mathcal{L}_{\rm I}(\partial_\alpha R_a, R_a, x)$ has an explicit x dependence coming from the x dependence of $A(x)_\alpha$.

1. Find the added term $T_{\rm I}^{\mu\nu}$ in $T^{\mu\nu}$ coming from $\mathcal{L}_{\rm I}$.

2. What is the resulting force density

$$f^\mu = \partial_\nu T_{\rm F}^{\mu\nu} \tag{7}$$

on the fluid?