The principle of stationary action[1]

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1 Introduction

We have considered a system of particles J with masses m_J in the presence of a potential

$$V(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N,t) = \frac{1}{2}\sum_{JK} V_{JK}(|\boldsymbol{x}_J - \boldsymbol{x}_K|) + \sum_J V_J^{(e)}(\boldsymbol{x}_J,t). \tag{1}$$

Here I use boldface for vectors in three dimensions and I set $V_{KJ} = V_{JK}$ and $V_{JJ} = 0$. For this system, the equations of motion are

$$m_I\,\ddot{\boldsymbol{x}}_J = -\vec{\nabla}_J V(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N,t). \tag{2}$$

We now reformulate the theory.

Let us use the notation $x(t)$ to stand for the entire collection of coordinates $\{\boldsymbol{x}_1(t),\dots,\boldsymbol{x}_N(t)\}$ at time t. Consider paths that the system might take between a fixed initial position $x(t_i)$ at time t_i and a fixed final position $x(t_f)$ at time t_f. We use x to label the whole function describing the path, the function whose value at time t is $x(t)$. We define the action

$$S[x] = \int_{t_i}^{t_f} dt\left\{\sum_J \frac{1}{2}\,m_J\,\dot{\boldsymbol{x}}_J^2 - V(x(t),t)\right\}. \tag{3}$$

Thus the action is a function defined on the space of paths.

We consider how the action changes if we change from one path x to another path $x+\delta x$ that is infinitesimally different and has the same endpoints:

$$\delta x(t_i) = \delta x(t_f) = 0. \tag{4}$$

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Define the variation $\delta S[x]$ of the action to be the part of $S[x+\delta x]-S[x]$ that is linear in δx, omitting contributions proportional to the square or higher powers of δx. The principle of stationary action says that the path that the system takes to get from $x(t_i)$ to $x(t_f)$ is the path that makes

$$\delta S[x] = 0 \tag{5}$$

for all variations δx in the path obeying Eq. (4).

To see what the implications of the principle of stationary action are, we must evaluate δS:

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f} dt \sum_J \{m_J\, \dot{\boldsymbol{x}}_J(t)\cdot \delta\dot{\boldsymbol{x}}_J(t) - \delta\boldsymbol{x}_J(t)\cdot \boldsymbol{\nabla}_J V(x(t),t)\} \\
&= \int_{t_i}^{t_f} dt \sum_J \left\{m_J\, \dot{\boldsymbol{x}}_J(t)\cdot \frac{d}{dt}\delta\boldsymbol{x}_J(t) - \delta\boldsymbol{x}_J(t)\cdot \boldsymbol{\nabla}_J V(x(t),t)\right\} \\
&= \int_{t_i}^{t_f} dt \sum_J \left\{-m_J \left[\frac{d}{dt}\,\dot{\boldsymbol{x}}_J(t)\right]\cdot \delta\boldsymbol{x}_J(t) - \delta\boldsymbol{x}_J(t)\cdot \boldsymbol{\nabla}_J V(x(t),t)\right\} \\
&\quad + \sum_J m_J\, \{\dot{\boldsymbol{x}}_J(t_f)\cdot\delta\boldsymbol{x}_J(t_f) - \dot{\boldsymbol{x}}_J(t_i)\cdot\delta\boldsymbol{x}_J(t_i)\} \\
&= \int_{t_i}^{t_1} dt \sum_J \{-m_J\ddot{\boldsymbol{x}}_J(t) - \boldsymbol{\nabla}_J V(x(t),t)\}\cdot\delta\boldsymbol{x}_J(t).
\end{aligned} \tag{6}$$

In the last step, we have used the assumption that $\delta x(t)$ vanishes at $t = t_i$ and at $t = t_f$. In order for δS to vanish for for *all* variations δx in the path obeying Eq. (4), the quantity in braces in the last line must vanish for all J and all t in the interval $t_i < t < t_f$. That is

$$m_J\, \ddot{\boldsymbol{x}}_J(t) = -\boldsymbol{\nabla}_J V(x(t),t). \tag{7}$$

The reverse holds too: if Eq. (7) holds, then $\delta S = 0$ for all δx obeying Eq. (4).

We conclude that when the force is derived from a potential, the differential equation "$F = ma$" is equivalent to the variational principle $\delta S = 0$. We thus have two ways to formulate classical mechanics, or at least a subset of classical mechanics.

Although "$F = ma$" and $\delta S = 0$ are equivalent, there may be advantages to one or the other formulation. The "$F = ma$" formulation has the advantage that it allows the inclusion of forces of friction, which do not come from a potential. In the following sections, we explore some of the advantages of the $\delta S = 0$ formulation.

2 Connection to quantum mechanics.

The action principle is important in connecting classical mechnics to quantum mechanics. This connection is a big subject. For the moment, I merely point out the connection to the path integral formulation of quantum mechanics. Consider the amplitude

$$\langle x_f | U(t_f, t_i) | x_i \rangle \tag{8}$$

for the system that starts at time t_i at position x_i to be found at a later time t_f at position x_f. This amplitude can be written as

$$\langle x_f | U(t_f, t_i) | x_i \rangle = \int \mathcal{D}x \ \exp\left(iS[x]\right). \tag{9}$$

Here we integrate over all paths x such that $x(t_i) = x_i$ and $x(t_f) = x_f$ and $S[x]$ is the action for that path. Here integrating over all paths means that one divides the time interval up into small segments of size Δt and integrates the position at the end of the nth interval, $x(t_i + n\Delta t)$, from $-\infty$ to $+\infty$, while rewriting the action in a discrete approximation so that it is a function only of the variables $x(t_i + n\Delta t)$. Then one supplies a suitable normalizing factor and takes the limit $\Delta t \to 0$. Note that *all* paths contribute to the integral, not just the classical path that makes $S[x]$ stationary. For certain situations, say the path of a baseball, only paths x that are near to the classical path are important. Then we recover the predictions of classical mechanics.

I leave it to the class on quantum mechanics to explore this connection further.

3 Change of coordinates

One practical advantage of the variational principle formulation of mechanics is that it is easy to change coordinates. Suppose that we start with our cartesian coordinates $\{x_I^1, x_I^2, x_I^3\}$, with $I = 1, \dots, N$, and we want to change to some new coordinates q_J, with $J = 1, \dots, 3N$. The new coordinates are functions of the old coordinates, and vice versa. The relations could involve the time explicitly: $x_I^j(t) = X_I^j(q_1(t), \dots, q_{3N}(t); t)$. It is typically pretty easy to write the action as a function of the path expressed as a path in the space of the new coordinates, $S[q]$. Then the allowed paths are those that make $\delta S[q] = 0$. It is then straightforward to write this as a differential equation for the $q_J(t)$. We could, of course, directly change the equations $F = ma$ into the new coordinates, but that is usually harder.

For example, consider a two dimensional harmonic oscillator,

$$S = \int dt \, \left\{ \frac{m}{2} \left[(\dot{x}^1)^2 + (\dot{x}^2)^2 \right] - \frac{1}{2} k \left[(x^1)^2 + (x^2)^2 \right] \right\}. \qquad (10)$$

Perhaps we would like to know the equations of motion in polar coordinates $\{r, \theta\}$ where

$$x^1 = r \cos\theta, \qquad x^2 = r \sin\theta. \qquad (11)$$

Simple geometry gives

$$S = \int dt \, \left\{ \frac{m}{2} \left[\dot{r}^2 + r^2 \dot{\theta}^2 \right] - \frac{1}{2} k r^2 \right\}. \qquad (12)$$

Let's find the equations of motion using a general formulation that will apply to all of the cases that we might want to consider. Our action has the form

$$S[q] = \int dt \; L(\dot{q}(t), q(t)) \qquad (13)$$

where the $q_J(t)$ are our coordinates and the $\dot{q}_J(t)$ are their time derivatives. For cases in which the forces are given by the gradients of a potential energy,[3] we set $L = T - V$ where T is the kinetic energy and V is the potential energy, both written as functions of the $\dot{q}_J(t)$ and the $q_J(t)$. We know that this is correct because it gives $F = ma$ when the $q_J(t)$ are cartesian components of the particle positions in an inertial reference frame. All that we are doing is keeping S the same and changing the variables that we use to describe the path. Using the general coordinates and repeating the previous derivation, we have

$$\begin{aligned} \delta S[q] &= \int dt \, \sum_J \left\{ \frac{\partial L(\dot{q}(t), q(t))}{\partial \dot{q}_J} \, \delta \dot{q}_J(t) + \frac{\partial L(\dot{q}(t), q(t))}{\partial q_J} \, \delta q_J(t) \right\} \\ &= \int dt \, \sum_J \left\{ - \frac{d}{dt} \frac{\partial L(\dot{q}(t), q(t))}{\partial \dot{q}_J} + \frac{\partial L(\dot{q}(t), q(t))}{\partial q_J} \right\} \delta q_J(t). \end{aligned} \qquad (14)$$

[3]Later, we will generalize our treatment to include particles that interact with an external magnetic field. Then, the force is not just the gradient of a potential and we will need a new form for $L(\dot{q}(t), q(t))$.

Here we have integrated by parts and there is no surface term because $\delta q(t)$ vanishes at the integration endpoints (which, for simplicity of notation, have not been indicated). In order for δS to vanish for every allowed δq we must have

$$0 = -\frac{d}{dt}\frac{\partial L(\dot{q}(t), q(t))}{\partial \dot{q}_J} + \frac{\partial L(\dot{q}(t), q(t))}{\partial q_J} \tag{15}$$

These are called the Euler-Lagrange equations of motion.

We can apply Eq. (15) to the example described above to get

$$\begin{aligned} 0 &= -\frac{d}{dt}\, m\dot{r} + mr\dot{\theta}^2 - kr \\ 0 &= -\frac{d}{dt}\,(mr^2\dot{\theta}). \end{aligned} \tag{16}$$

That is

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 - kr \\ \ddot{\theta} &= -\frac{2}{r}\,\dot{r}\dot{\theta}. \end{aligned} \tag{17}$$

Exercise 3.1: Derive this result by changing variables directly in the differential equation $\boldsymbol{F} = m\boldsymbol{a}$.

Exercise 3.2: Derive the equations of motion in spherical polar coordinates for a particle in three dimensional space in a gravitational potential centered at the origin. Use whichever method you think simplest.

In our example, we considered a change of variables from cartesian coordinates x in an inertial reference frame to some other coordinates q that are functions of the x only. There is, however, no reason that the relation between q and x cannot involve the time t. All that we have to do is to express the action in terms of the q and then use $\delta S = 0$ find the allowed paths in q space.

An important example is provided by changing to a rotating reference frame. For simplicity of notation, we consider a single particle of mass m in a reference frame rotating with angular velocity $\boldsymbol{\omega}$ with respect to an inertial reference frame. We have

$$x^i = R(t)^{ij}\; q^j \tag{18}$$

where $R(t)$ is the rotation matrix. For example, if the rotation axis is the 3-axis, we have

$$R = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Expressing the potential in terms of q and using simple geometry for the velocity gives

$$L = \frac{1}{2}\,m\left(\frac{d\boldsymbol{q}}{dt} + \boldsymbol{\omega}\times\boldsymbol{q}\right)^2 - V(\boldsymbol{q},t). \tag{20}$$

With use of vector identities (easily derived using the ϵ^{ijk} tensor) we have

$$L = \frac{1}{2}\,m\left\{\left(\frac{d\boldsymbol{q}}{dt}\right)^2 + 2\,\frac{d\boldsymbol{q}}{dt}\cdot(\boldsymbol{\omega}\times\boldsymbol{q}) + \omega^2\boldsymbol{q}^2 - (\boldsymbol{\omega}\cdot\boldsymbol{q})^2\right\} - V(\boldsymbol{q},t). \tag{21}$$

Then $\delta S = 0$ gives

$$0 = -m\frac{d}{dt}\left(\frac{d\boldsymbol{q}}{dt} + \boldsymbol{\omega}\times\boldsymbol{q}\right) - m\,\boldsymbol{\omega}\times\frac{d\boldsymbol{q}}{dt} + m[\omega^2\boldsymbol{q} - (\boldsymbol{\omega}\cdot\boldsymbol{q})\,\boldsymbol{\omega}] - \boldsymbol{\nabla}V(\boldsymbol{q},t). \tag{22}$$

To get the second term here, it is convenient to use

$$\frac{d\boldsymbol{q}}{dt}\cdot(\boldsymbol{\omega}\times\boldsymbol{q}) = -\boldsymbol{q}\cdot(\boldsymbol{\omega}\times\frac{d\boldsymbol{q}}{dt}). \tag{23}$$

Performing the differentiation with respect to t, we have

$$0 = -m\left(\frac{d^2\boldsymbol{q}}{dt^2} + \boldsymbol{\omega}\times\frac{d\boldsymbol{q}}{dt}\right) - m\,\boldsymbol{\omega}\times\frac{d\boldsymbol{q}}{dt} + m[\omega^2\boldsymbol{q} - (\boldsymbol{\omega}\cdot\boldsymbol{q})\,\boldsymbol{\omega}] - \boldsymbol{\nabla}V(\vec{q},t). \tag{24}$$

We can simplify this to

$$m\frac{d^2\boldsymbol{q}}{dt^2} = -2m\,\boldsymbol{\omega}\times\frac{d\boldsymbol{q}}{dt} + m\omega^2\boldsymbol{q}_\perp - \boldsymbol{\nabla}V(\boldsymbol{q},t) \tag{25}$$

where

$$\boldsymbol{q}_\perp = \boldsymbol{q} - \frac{1}{\omega^2}(\boldsymbol{\omega}\cdot\boldsymbol{q})\,\boldsymbol{\omega} \tag{26}$$

is the part of $\boldsymbol{q}$ perpendicular to $\boldsymbol{\omega}$. The term

$$-2m\,\boldsymbol{\omega}\times\frac{d\boldsymbol{q}}{dt} \tag{27}$$

is the coriolis force. The term

$$m\omega^2 \boldsymbol{q}_\perp \tag{28}$$

is the centrifugal force.

Exercise 3.3: Weather patterns on the surface of the earth are influenced by the Coriolis force, which is a pseudo-force that arises when we consider the motion of the atmosphere in a reference frame that rotates with the earth.

In Salem (45^o North), what is the acceleration due to the Coriolis force on a wind with velocity 50 m/s towards the east? Do storms in the Northern Hemisphere centered on a region of low pressure circulate clockwise or counterclockwise? Why?

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4 Constraints

The lagrangian method is especially useful for problems with constraints. The simplest case is probably that of a block that moves on a smooth frictionless table. The constraint is that the block can't move *into* the table. Thus if the cartesian coordinates of the block are $\{x^1, x^2, x^3\}$ and h is the height of the table, the constraint is $x^3 \geq h$. The constraint is enforced by the "normal force" F_N that the table exerts on the block. The magnitude of F_N is whatever it needs to be to keep the block from moving into the table. If the only other force on the block with a vertical component is the gravitational force, with magnitude mg, then $F_N^3 = mg$, so that the net z-component of force on the block is $F_N^3 - mg = 0$.

There is a tricky point that arises right from the start. The normal force points up. It can't point down. Thus if there is some other force (say electrical) that pulls the block up, it could happen that the electrical force overcomes the gravitational force mg and the block leaves the table. To see if this happens, one would solve a modified problem with a constraint given as an equality, $x^3 = h$. Then we solve for F_N and see if its 3-component is ever negative. If F_N^3 does become negative at some time t_1, then for $t > t_1$ we have solved the wrong problem. We should switch problems to a problem in which the x^3-motion is unconstrained. Sometimes one sees problems like this. Note that if the block hits the table again, we have a mess. Does it bounce? Does it have an inelastic collision in which kinetic energy goes into heating the block and table? Evidently, we need more information.

For the rest of this section, we will deal with constrainst that are expressed as equalities, such as $x^3 = h$.

Let us deal with the problem of motion with constraints using a quite general notation. Let $\boldsymbol{x}_I$ be the cartesian coordinates of the particles, with $I = 1, \dots, N$. Suppose that the constraints are of the form

$$C_K(x(t), t) = 0, \qquad K = 1, \dots, N_C. \tag{29}$$

What we need to do is to think of the functions $C_K(x, t)$ as specifying coordinates c_K,

$$c_K = C_K(x(t), t) \qquad K = 1, \dots, N_C. \tag{30}$$

Suppose that we find some more coordinates

$$q_J(t) = Q_J(x(t), t), \qquad J = 1, \dots, 3N - N_C. \tag{31}$$

What we need is that the c_K and the q_J together form a complete coordinate system, so that, given the q's and the c's we can find the $\boldsymbol{x}$'s.

For the simple example, we could take $q^1 = x^1$, $q^2 = x^2$, and $c^1 = x^3 - h$. For the slightly more complicated example of a particle constrained to move on the surface of a sphere of radius R, we could take $q^1 = \theta$, $q^2 = \phi$, and $c^1 = r - R$, where $\{r, \theta, \phi\}$ are the usual polar coordinates of the particle.

We can express the idea of a constraint by supposing that the constraint is enforced by having an enormous cost in potential energy if the particles leave the required surface $c_K = 0$. That is, for each K we can imagine that there is a term in the potential energy

$$\frac{1}{2} k c_K^2 \tag{32}$$

with an enormous value for the spring constant k. Then we imagine writing the lagrangian as a function of the q's and $\dot{q}$'s and of the c's and $\dot{c}$'s. (This is just something to imagine, we won't do it.) We will get equations of motion that contain the forces of constraint. Qualitatively, we will have $F_K \sim k c_K$. Then the deviation of the special coordinates c_K from 0 will be $c_K \sim F_K/k$. The forces of constraint are not large, but k is large, so the c_K are small. The potential energy associate with the Kth constraint is of order $(1/2) F_K^2/k$ so that this potential energy is small. Also the terms in the kinetic energy that are proportional to time derivatives of the c_K will be small for large k. This rough physical argument leads to the prescription, Set $c_K = \dot{c}_K = 0$ in the lagrangian.

We can be more explicit about this. The kinetic energy will be a function $T(q, \dot{q}, c, \dot{c})$ of the coordinates and their time derivatives. The potential energy will be a function $V(q, c, t)$ of the coordinates if we don't include the constraints. To model how the constraint comes about, we suppose that there is an additional potential energy

$$\frac{1}{2}\, k \sum c_K^2. \tag{33}$$

Then the lagrangian is

$$L = T(q, \dot{q}, c, \dot{c}, t) - V(q, c, t) - \frac{1}{2}\, k \sum c_K^2. \tag{34}$$

We consider this in the limit $k \to \infty$. To do this, change variables from c_K to $z_K = k c_K$. Then

$$L = T(q, \dot{q}, z/k, \dot{z}/k, t) - V(q, z/k, t) - \frac{1}{2k} \sum z_K^2. \tag{35}$$

The equations of motion are

$$\begin{aligned}
0 &= -\frac{d}{dt}\, \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial \dot{q}_J} + \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial q_J} - \frac{\partial V(q, z/k, t)}{\partial q_J} \\
0 &= -\frac{d}{dt} \frac{1}{k} \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial \dot{c}_K} + \frac{1}{k} \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial c_K} \\
&\quad - \frac{1}{k} \frac{\partial V(q, z/k, t)}{\partial c_K} - \frac{1}{k} z_K,
\end{aligned} \tag{36}$$

or, multiplying the second equation by k,

$$\begin{aligned}
0 &= -\frac{d}{dt}\, \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial \dot{q}_J} + \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial q_J} - \frac{\partial V(q, z/k, t)}{\partial q_J} \\
z_K &= g_K(q, \dot{q}, z/k, \dot{z}/k, t).
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
g_K(q, \dot{q}, z/k, \dot{z}/k, t) &= -\frac{d}{dt} \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial \dot{c}_K} + \frac{\partial T(q, \dot{q}, z/k, \dot{z}/k, t)}{\partial c_K} \\
&\quad - \frac{\partial V(q, z/k, t)}{\partial c_K}.
\end{aligned} \tag{38}$$

For $k \to \infty$ we have

$$z_K \approx g_K(q, \dot{q}, 0, 0, t). \tag{39}$$

These are the small deviations that produce the required constraint forces. The equations for the unconstrained coordinates then become

$$0 \approx -\frac{d}{dt}\,\frac{\partial T(q, \dot{q}, 0, 0, t)}{\partial \dot{q}_J} + \frac{\partial T(q, \dot{q}, 0, 0, t)}{\partial q_J} - \frac{\partial V(q, 0, t)}{\partial q_J}. \tag{40}$$

That is, we set $c_K = \dot{c}_K = 0$ in T and V and throw away the potential that gives the constraints.

To be careful about this, we should note that if we consider the full equations of motion including the harmonic oscillator forces that we imagine express the constraints, then for any finite value of the spring constant k, there will be solutions of the equations of motion in which there is a lot of energy in the constraint oscillators so that the resulting c_K are not small. For example, we would normally consider a car travelling on a road to be constrained to stay on the road. But the body of the car is really connected to the wheels and thus the road by springs in the suspension of the car. Thus our oscillator model is not so bad if we consider the usual damping force built into an automobile suspension to be absent. Now, with a fixed value of k, if we deliberately try, we can get the car moving with large vertical oscillations in which a lot of energy is stored in the suspension and then released in each oscillation cycle. However, the derivation presented above shows that it is consistent to consider a limit in which $k \to \infty$ and all of the c_K are infinitesimal.

When we carry out this procedure sketched above and take $k \to \infty$, the lagrangian is expressed in terms of coordinates q and their time derivatives $\dot{q}$. The lagrangian is (kinetic energy) $-$ (potential energy). Here the potential energy is the potential energy of the system written as a function of the q_J for $c_K = 0$. The potential energy does not include the potentials that give the forces of constraint. The kinetic energy is

$$\sum \frac{1}{2} m_I\, \dot{\boldsymbol{x}}_I^2 \tag{41}$$

for particles that obey the constraint conditions $c_K = 0$.

The nice feature of this is that you can write the equations of motion for cases with quite complicated constraints. In addition, you never have to worry about the forces of constraint.

Exercise 4.1: A block of mass m rests on an inclined plane, with an inclination angle θ with respect to the horizontal direction. The surface of the plane is covered with ice, so that the block is free to slide without friction. Let the distance of the block from the top of the plane down and to the right be x. So far, this is a problem that you have seen before, but now comes a twist in the plot. The inclined plane itself is made of a triangular block of steel, has a mass M, and is free to slide without friction on a horizontal tabletop. Let the horizontal displacement of the inclined plane along the table to the right be y. The system starts from rest at time $t = 0$. Find the subsequent motion $\{x(t), y(t)\}$ until the block reaches the lower end of the plane.