From Classical to Quantum Field Theory[1]

D. E. Soper[2]
University of Oregon
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1 Classical Mechanics

Let $\phi_J(t)$, $J = 1, 2, 3$, be the position of a particle at time t. To describe the mechanics of such a particle in a potential V, we use the action (with repeated indices J implying summation)

$$S[\phi] = \int dt \left\{ \tfrac{1}{2} m\, \dot\phi_J(t) \dot\phi_J(t) - V(\phi(t)) \right\} . \tag{1}$$

The integral runs from an initial time t_I to a final time t_F. The notation $S[\phi]$ indicates that S depends on the whole function ϕ, as distinct from $\phi(t)$ at any one time t. We consider a small change in ϕ,

$$\phi_J(t) \to \phi_J(t) + \delta\phi_J(t), \tag{2}$$

where $\delta\phi(t)$ is suitably smooth and vanishes at the initial time t_I and the final time t_F. We define the *variation*, δS, of the action by

$$S[\phi + \delta\phi] = S[\phi] + \delta S[\phi, \delta\phi] + \mathcal{O}(\delta\phi^2), \tag{3}$$

where δS is linear in $\delta\phi$. This is the definition. Calculation gives

$$\begin{aligned} \delta S &= \int dt \left\{ m\, \dot\phi_J(t)\, \delta\dot\phi_J(t) - \frac{\partial V(\phi(t))}{\partial \phi_J}\, \delta\phi_J(t) \right\} \\ &= \int dt \left\{ -m\, \ddot\phi_J(t) - \frac{\partial V(\phi(t))}{\partial \phi_J} \right\} \delta\phi_J(t). \end{aligned} \tag{4}$$

There is no surface term here because $\delta\phi$ vanishes at the integration endpoints.

[2]soper@physics.uoregon.edu

This illustrates an important mathematical concept, the *functional derivative*. The functional derivative

$$\frac{\delta S}{\delta \phi_J(t)} \tag{5}$$

is a function of t such that

$$\delta S[\phi, \delta\phi] = \int dt\, \frac{\delta S}{\delta \phi_J(t)}\, \delta\phi_J(t). \tag{6}$$

Thus $\delta S/\delta\phi_J(t)$ is something like the partial derivative of S with respect to the value of ϕ_J at time t. However, this doesn't make sense if taken too literally since we can't change ϕ_J at just one time. In some applications, the functional derivative could be a distribution or generalized function that is not an ordinary function but, for instance, contains delta functions. In the present case, our calculation shows that

$$\frac{\delta S}{\delta \phi_J(t)} = -m\, \ddot{\phi}_J(t) - \frac{\partial V(\phi(t))}{\partial \phi_J}. \tag{7}$$

The *principle of stationary action* says that the function $\phi_J(t)$ actually chosen by nature is one for which $\delta S = 0$ for *every* allowed $\delta\phi$. That is

$$\frac{\delta S}{\delta \phi_J(t)} = 0. \tag{8}$$

We then obtain a second order differential equation for $\phi_J(t)$:

$$m\, \ddot{\phi}_J(t) = -\frac{\partial V(\phi(t))}{\partial \phi_J}. \tag{9}$$

That is, $F = ma$, where F is minus the gradient of the potential.

Exercise. Calculate $\delta S/\delta\phi(t)$ for

1. $S[\phi] = \int dt\, (\ddot{\phi}(t))^3$
2. $S[\phi] = \int dt\, \cos(\phi(t))\, (\dot{\phi}(t))^2$
3. $S[\phi] = \int dt\, \exp[-(\dot{\phi}(t))^2]$
4. $S[\phi] = (\phi(t_A))^2$, where t_A is a fixed time.
5. $S[\phi] = (\dot{\phi}(t_A))^2$, where t_A is a fixed time.

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2 Classical field theory

We consider a field $\phi(x) = \phi(t, \vec{x})$. This is the simplest kind of field and is analogous to the vector potential $A^\mu(x)$ in electrodynamics.

Just think of ϕ as specifying an independent dynamical variable for each value of $\vec{x}$. Thus $\vec{x}$ replaces J in the previous section. We replace $\sum_J$ by $\int d\vec{x}$. The equations of motion for ϕ can be specified by giving an action and saying that we apply the principle of stationary action. A common form of the action is

$$S[\phi] = \int dx\ \mathcal{L}\left(\partial_\mu\phi(x), \phi(x)\right). \tag{10}$$

Here $\mathcal{L}$ is called the Lagrangian density, or often just the Lagrangian even though strictly speaking the Lagrangian is the integral of $\mathcal{L}$ over $\vec{x}$. A typical example is

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu\phi\right)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4. \tag{11}$$

We consider variations $\delta\phi$ of ϕ that are nice smooth functions of x and vanish at t_F and t_I and at $|\vec{x}| \to \infty$. The functional derivative of the action with respect to $\phi(x)$ is defined by

$$S[\phi + \delta\phi] = S[\phi] + \int dx\ \frac{\delta S[\phi]}{\delta\phi(x)}\ \delta\phi(x) + \mathcal{O}(\delta\phi^2). \tag{12}$$

The principle of stationary action says that

$$\delta S = \int dx\ \frac{\delta S[\phi]}{\delta\phi(x)}\ \delta\phi(x) \tag{13}$$

vanishes for *every* allowed $\delta\phi$. That is

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0. \tag{14}$$

For an action of the form of Eq. (10) we calculate

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -\partial_\mu\ \frac{\partial\mathcal{L}(\partial_\alpha\phi(x), \phi(x))}{\partial(\partial_\mu\phi)}\ + \frac{\partial\mathcal{L}(\partial_\alpha\phi(x), \phi(x))}{\partial\phi}. \tag{15}$$

For the specific action Eq. (11) we have

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -\partial_\mu\partial^\mu\phi(x) - m^2\phi(x) - \frac{\lambda}{3!}\,\phi(x)^3. \tag{16}$$

Thus the equation of motion is the partial differential equation

$$\partial_\mu \partial^\mu \phi(x) = -m^2 \phi(x) - \frac{\lambda}{3!}\, \phi(x)^3. \tag{17}$$

We can see the analogy with electrodynamics. The parameter m would be the photon mass, but the photon mass is zero, so set $m = 0$. Then we have the wave equation with a source $-(\lambda/3!)\, \phi(x)^3$ for the waves instead of the electromagnetic current $J^\mu(x)$.

3 Symmetries and Noether's Theorem

The theorem is

An invariance of $S[\phi]$ gives a conservation law.

Instead of stating the theorem in its most general form we give three examples.
1) $\phi_J(x)$, $J = 1, 2, \ldots, N$ with

$$\mathcal{L} = \tfrac{1}{2}\, (\partial_\mu \phi_J)(\partial^\mu \phi_J) - \tfrac{1}{2} m^2 \phi_J \phi_J - \tfrac{1}{4!} \lambda\, (\phi_J \phi_J)^2. \tag{18}$$

Consider an infinitesimal $O(N)$ rotation,

$$\delta \phi_J(x) = A_{JK}\, \phi_K(x)\, \delta\theta \tag{19}$$

with $A_{JK} = -A_{KJ}$. Then one easily sees that

$$\delta \mathcal{L} = 0. \tag{20}$$

Then

$$\begin{aligned}
0 &= \delta \mathcal{L} \\
&= \frac{\partial \mathcal{L}(\partial_\alpha \phi, \phi)}{\partial(\partial_\mu \phi_J)}\, \partial_\mu \delta\phi_J(x) + \frac{\partial \mathcal{L}(\partial_\alpha \phi, \phi)}{\partial \phi_J}\, \delta\phi_J \\
&= -\partial_\mu \frac{\partial \mathcal{L}(\partial_\alpha \phi, \phi)}{\partial(\partial_\mu \phi_J)}\, \delta\phi_J + \frac{\partial \mathcal{L}(\partial_\alpha \phi, \phi)}{\partial \phi_J}\, \delta\phi_J + \partial_\mu \left\{ \frac{\partial \mathcal{L}(\partial_\alpha \phi, \phi)}{\partial(\partial_\mu \phi_J)}\, \delta\phi_J \right\} \\
&= \frac{\delta S[\phi]}{\delta \phi_J(x)}\, \delta\phi_J(x) - \partial_\mu J^\mu(x) \delta\theta
\end{aligned} \tag{21}$$

where

$$J^\mu(x)\delta\theta = -\frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\mu\phi_J)}\ \delta\phi_J. \tag{22}$$

or

$$J^\mu(x) = -\frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\mu\phi_J)}\ A_{JK}\,\phi_J(x) \tag{23}$$

We calculate

$$J^\mu(x) = -(\partial^\mu\phi_J(x))\ A_{JK}\,\phi_J(x) \tag{24}$$

for our particular lagrangian.

When the fields obey their equations of motion, $\delta S[\phi]/\delta\phi_J(x) = 0$, then $\partial_\mu J^\mu(x) = 0$. We are familiar with the fact that $\partial_\mu J^\mu(x) = 0$ is the statement that the quantity $\int d\vec{x}\, J^0(x)$ is conserved. We interpret J^0 as the density of the conserved stuff and $\vec{J}$ as the current of this stuff.

Exercise. Consider $\phi_J(t)$ with $J = 1,2,3$ and

$$S[\phi] = \int dt\left\{\tfrac{1}{2}m\,\dot\phi_J(t)\dot\phi_J(t) - V(\phi_J(t)\phi_J(t))\right\}. \tag{25}$$

The Lagrangian is invariant under a change of ϕ with

$$\delta\phi_J = A_{JK}\phi_K\,\delta\theta \tag{26}$$

with $A_{JK} = -A_{KJ}$. How does Noether's Theorem work in this case of ordinary mechanics instead of field theory? What are the conserved quantities? What is their physical interpretation?

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2) Consider an infinitesimal translation, $\phi'(x) = \phi(x-a)$, or

$$\delta\phi(x) = -a^\nu\partial_\nu\phi(x) \tag{27}$$

Then, assuming that $\mathcal{L}$ depends on ϕ and its derivatives but not directly on x, we have

$$\delta\mathcal{L}(x) = -a^\nu\partial_\nu\mathcal{L}(x). \tag{28}$$

Then

$$\begin{aligned}
0 &= \delta\mathcal{L} + a^\nu\partial_\nu\mathcal{L} \\
&= \frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\nu\phi)}\ \partial_\nu\delta\phi(x) + \frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial\phi}\ \delta\phi + a^\nu\partial_\nu\mathcal{L} \\
&= -\partial_\nu\frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\nu\phi)}\ \delta\phi + \frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial\phi}\ \delta\phi + \partial_\nu\left\{\frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\nu\phi)}\ \delta\phi + a^\nu\mathcal{L}\right\} \\
&= \frac{\delta S[\phi]}{\delta\phi(x)}\,\delta\phi(x) - a_\mu\partial_\nu T(x)^{\mu\nu}
\end{aligned} \tag{29}$$

where

$$a_\mu T(x)^{\mu\nu} = -\frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\nu\phi_J)}\;\delta\phi - a_\mu g^{\mu\nu}\mathcal{L}. \tag{30}$$

or

$$T(x)^{\mu\nu} = \frac{\partial\mathcal{L}(\partial_\alpha\phi,\phi)}{\partial(\partial_\nu\phi)}\;(\partial^\mu\phi(x)) - g^{\mu\nu}\mathcal{L}. \tag{31}$$

There are four conserved quantities,

$$P^\mu = \int d\vec{x}\;T^{\mu 0} \tag{32}$$

These are the components of the energy and momentum.

Exercise. For

$$\mathcal{L} = \tfrac{1}{2}\,(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\,\phi^4. \tag{33}$$

what is $T^{\mu\nu}$? In particular, what is the energy density T^{00}?

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3) Consider an infinitesimal Lorentz transformation, $\phi'(x) = \phi(\Lambda x) \approx \phi(x^\nu - \delta\theta A^\nu{}_\mu x^\mu)$, or

$$\delta\phi(x) = -\delta\theta A^\nu{}_\mu x^\mu \partial_\nu\phi(x) \tag{34}$$

where $A_{\nu\mu} = -A_{\mu\nu}$. Then since $\mathcal{L}$ depends on ϕ and its derivatives but not directly on x, we have

$$\delta\mathcal{L}(x) = -\delta\theta A^\nu{}_\mu x^\mu \partial_\nu\mathcal{L}(x). \tag{35}$$

Then (using the antisymmetry of A in the third line below)

$$\begin{aligned}
0 &= \delta\mathcal{L} + \delta\theta A^\nu{}_\mu x^\mu \partial_\nu\mathcal{L} \\
&= \frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\;\partial_\nu\delta\phi(x) + \frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial\phi}\;\delta\phi + \delta\theta A^\nu{}_\mu x^\mu \partial_\nu\mathcal{L} \\
&= -\partial_\nu\frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\;\delta\phi + \frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial\phi}\;\delta\phi + \partial_\nu\left\{\frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\;\delta\phi + \delta\theta A^\nu{}_\mu x^\mu \mathcal{L}\right\} \\
&= \frac{\delta S[\phi]}{\delta\phi(x)}\,\delta\phi(x) + \frac{1}{2}\,\delta\theta\,A_{\alpha\beta}\partial_\nu M(x)^{\alpha\beta\nu}
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\frac{1}{2}\,\delta\theta\, A_{\alpha\beta} M(x)^{\alpha\beta\nu} &= \frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\,\delta\phi + \delta\theta A^{\nu}{}_{\mu} x^{\mu}\mathcal{L} \\
&= -\frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\,\delta\theta A_{\alpha\beta} x^{\beta}\partial^{\alpha}\varphi + \delta\theta g^{\nu\alpha} A_{\alpha\beta} x^{\beta}\mathcal{L} \\
&= \frac{1}{2}\,\delta\theta\, A_{\alpha\beta}\Big\{\frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\,[x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha}]\,\varphi \\
&\quad -[g^{\nu\beta}x^{\alpha} - g^{\nu\alpha}x^{\beta}]\mathcal{L}\Big\} \qquad (37)
\end{aligned}$$

or

$$M(x)^{\alpha\beta\nu} = \frac{\partial\mathcal{L}(\partial\phi,\phi)}{\partial(\partial_\nu\phi)}\,[x^{\alpha}(\partial^{\beta}\varphi) - x^{\beta}(\partial^{\alpha}\varphi)] - [g^{\nu\beta}x^{\alpha} - g^{\nu\alpha}x^{\beta}]\mathcal{L}. \qquad (38)$$

There are six conserved quantities (considering the antisymmetry in $\{\alpha,\beta\}$),

$$J^{\alpha\beta} = \int d\vec{x}\; M^{\alpha\beta 0} \qquad (39)$$

These are the components of the angular momentum.

Exercise. For our sample field theory, evaluate M^{120} and M^{030}.

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4 Quantization

We have seen how to get a classical field theory from a lagrangian density $\mathcal{L}$. But how do we get a quantum field theory? The fields should have the same equations of motion as in the classical field theory. But they should be operators. The question then is, what sort of operation to they perform. It turns out that we will learn just about everything about what the operators do if we know their commutation relations with one another. Thus we investigate commutation relations.

We base the theory on classical mechanics. Consider the lagrangian

$$L = \int d\vec{x}\; \mathcal{L}(\vec{x},t). \qquad (40)$$

This is really the lagrangian, whereas $\mathcal{L}$ is the lagrangian density (even though we will sometimes be sloppy and call $\mathcal{L}$ the lagrangian). In our treatment, we consider space and time in very different roles. At any time t there is a function φ of $\vec{x}$ such that $\phi(\vec{x},t) = \varphi(\vec{x})$. The function φ, treated as a whole, is an element of a function space that represents the dynamical coordinates of our system, just like the position of a particle in three dimensional space or a particle on the surface of a sphere. For a later time t, $\phi(\vec{x},t)$ defines a *new* function φ of $\vec{x}$. Thus we have a different value of φ for each t, which we can call $\varphi(t)$. The function $\varphi(t)$, when evaluated at $\vec{x}$ is $\varphi(t;\vec{x}) = \phi(\vec{x},t)$. Mostly, however, we will omit writing the parameter t. Similarly, at any time t the function $\partial\phi(\vec{x},t)/dt$ defines a function of $\vec{x}$, which we can call $\dot{\varphi}$. This function represents the rate of change of the dynamical coordinates, the velocities. In this sense, we can consider L to be a function $L[\varphi,\dot{\varphi}]$ of these two functions. (*Cf.* $L(Q,\dot{Q})$ in classical mechanics, where Q depends on the time and may have many components, $Q_\alpha(t)$.)

Following classical mechanics, we define the canonical momenta

$$\pi(\vec{x}) = \frac{\delta L[\varphi,\dot{\varphi}]}{\delta\dot{\varphi}(\vec{x})}. \tag{41}$$

This is a functional derivative, defined by

$$\delta L[\varphi,\dot{\varphi}] \equiv \int d\vec{x} \left\{ \frac{\delta L[\varphi,\dot{\varphi}]}{\delta\varphi(\vec{x})}\ \delta\varphi(\vec{x}) + \frac{\delta L[\varphi,\dot{\varphi}]}{\delta\dot{\varphi}(\vec{x})}\ \delta\dot{\varphi}(\vec{x}) \right\}. \tag{42}$$

This is the general definition, but for our sample lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) - \tfrac{1}{2}m^2\varphi^2 - \tfrac{1}{4!}\lambda\varphi^4 \tag{43}$$

we have the simple result

$$\pi(\vec{x}) = \dot{\varphi}(\vec{x}). \tag{44}$$

In classical mechanics in the hamiltonian formulation, we solve for $\dot{\varphi}$ in terms of φ and π and then consider everything to be a function of φ and π insteat of φ and $\dot{\varphi}$. Then if A is a function of φ and π, we define variational derivatives by

$$\delta A[\varphi,\pi] \equiv \int d\vec{x} \left\{ \frac{\delta A[\varphi,\pi]}{\delta\varphi(\vec{x})}\ \delta\varphi(\vec{x}) + \frac{\delta A[\varphi,\pi]}{\delta\pi(\vec{x})}\ \delta\pi(\vec{x}) \right\}. \tag{45}$$

Note that $\varphi(\vec{y})$ defines a perfectly nice function of φ and π, so we should be able to write its functional derivatives with respect to $\varphi(\vec{x})$ and $\pi(\vec{x})$. These are, according to the definition

$$\begin{aligned}\frac{\delta\varphi(\vec{y})}{\delta\varphi(\vec{x})} &= \delta(\vec{y}-\vec{x}) \\ \frac{\delta\varphi(\vec{y})}{\delta\pi(\vec{x})} &= 0. \end{aligned} \tag{46}$$

Similarly

$$\begin{aligned}\frac{\delta\pi(\vec{y})}{\delta\varphi(\vec{x})} &= 0 \\ \frac{\delta\pi(\vec{y})}{\delta\pi(\vec{x})} &= \delta(\vec{y}-\vec{x}). \end{aligned} \tag{47}$$

Again following classical mechanics we define the Poisson bracket of any two functions A and B of φ and π,

$$\{A[\varphi,\pi], B[\varphi,\pi]\}_{\mathrm{PB}} \equiv \int d\vec{x} \left\{ \frac{\delta A[\varphi,\pi]}{\delta\varphi(\vec{x})} \frac{\delta B[\varphi,\pi]}{\delta\pi(\vec{x})} - \frac{\delta B[\varphi,\pi]}{\delta\varphi(\vec{x})} \frac{\delta A[\varphi,\pi]}{\delta\pi(\vec{x})} \right\}. \tag{48}$$

Especially important are the elementary Poisson brackets

$$\begin{aligned}\{\varphi(\vec{y}), \pi(\vec{z})\}_{\mathrm{PB}} &= \delta(\vec{y}-\vec{z}) \\ \{\varphi(\vec{y}), \varphi(\vec{z})\}_{\mathrm{PB}} &= 0 \\ \{\pi(\vec{y}), \pi(\vec{z})\}_{\mathrm{PB}} &= 0. \end{aligned} \tag{49}$$

Now comes the essential observation. The algebra of Poisson brackets is essentially the same as the algebra of commutators. For example

$$\{A, B\}_{\mathrm{PB}} = -\{B, A\}_{\mathrm{PB}} \tag{50}$$

and

$$\{A, \beta B + \gamma C\}_{\mathrm{PB}} = \beta\{A, B\}_{\mathrm{PB}} + \gamma\{A, C\}_{\mathrm{PB}} \tag{51}$$

and also

$$\{A, BC\}_{\mathrm{PB}} = B\{A, C\}_{\mathrm{PB}} + \{A, B\}_{\mathrm{PB}}\, C. \tag{52}$$

These hold also for commutators of operators. The only difference is that in the case of commutators, in the last relation the operator ordering for products of operators has to be as I wrote it, while for Poisson brackets we are talking about multiplication of numbers and the ordering of factors does not matter. Based on this observation, one replaces $\{A, B\}_{\rm PB}$ by $-i\,[A, B]$. In particular, for quantum field theory, the operators $\varphi(\vec{x})$ and $\pi(\vec{x})$ obey the commutation relations

$$\begin{aligned} [\varphi(\vec{y}), \pi(\vec{z})] &= i\delta(\vec{y} - \vec{z}) \\ [\varphi(\vec{y}), \varphi(\vec{z})] &= 0 \\ [\pi(\vec{y}), \pi(\vec{z})] &= 0. \end{aligned} \tag{53}$$

Exercise. Suppose that we start with classical mechanics, with variables ϕ_J that depend on time. How do we turn the classical mechanics into quantum mechanics?

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5 The hamiltonian equations of motion

We can construct the hamiltonian according to the usual prescription in classical mechanics:

$$H = \int d\vec{x}\, \frac{\delta L[\varphi, \dot{\varphi}]}{\delta \dot{\varphi}(\vec{x})}\, \dot{\varphi}(\vec{x}) - L. \tag{54}$$

This is

$$H = \int d\vec{x}\ \mathcal{H}(\vec{x}), \tag{55}$$

where the hamiltonian density is

$$\mathcal{H} = \frac{\delta L[\varphi, \dot{\varphi}]}{\delta \dot{\varphi}(\vec{x})}\, \dot{\varphi}(\vec{x}) - \mathcal{L}. \tag{56}$$

Exercise. Assuming that $L = \int d\vec{x}\ \mathcal{L}$ where $\mathcal{L}$ is a function of $\phi(x)$, $\partial\phi(x)/\partial t$ and $\vec{\nabla}\phi(x)$, what is the relation between $\mathcal{H}$ and the energy density T^{00} that is conserved because of Noether's Theorem?

Now to do classical mechanics in the hamiltonian formulation, we eliminate $\dot{\varphi}$ in favor of π and φ and write H in terms of these variables. Thus, for instance, with

$$\mathcal{L} = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) - \tfrac{1}{2}m^2\varphi^2 - \tfrac{1}{4!}\lambda\varphi^4 \tag{57}$$

we have

$$\mathcal{H} = \dot{\varphi}^2 - \mathcal{L} = \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) + \tfrac{1}{2}m^2\varphi^2 + \tfrac{1}{4!}\lambda\varphi^4 \tag{58}$$

and

$$\dot{\varphi} = \pi \tag{59}$$

so

$$H[\varphi,\pi] = \int d\vec{x}\left\{\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) + \tfrac{1}{2}m^2\varphi^2 + \tfrac{1}{4!}\lambda\varphi^4\right\}. \tag{60}$$

The classical equations of motion in the hamiltonian formulation are

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t;\vec{x}) &= -\left\{H[\varphi,\pi],\varphi(t;\vec{x})\right\}_{\mathrm{PB}} \\ \frac{\partial}{\partial t}\pi(t;\vec{x}) &= -\left\{H[\varphi,\pi],\pi(t;\vec{x})\right\}_{\mathrm{PB}}. \end{aligned} \tag{61}$$

With our rule about turning Poisson brackets into commutators, these become the quantum equations of motion,

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t;\vec{x}) &= i\left[H[\varphi,\pi],\varphi(t;\vec{x})\right] \\ \frac{\partial}{\partial t}\pi(t;\vec{x}) &= i\left[H[\varphi,\pi],\pi(t;\vec{x})\right]. \end{aligned} \tag{62}$$

Let's try this, using our sample theory and the elementary commutation relations. We have

$$[H[\varphi,\pi],\varphi(t;\vec{x})] = \int d\vec{y}\ \tfrac{1}{2}\left[\pi(\vec{y})^2,\varphi(t;\vec{x})\right] \tag{63}$$

$$\begin{aligned} [H[\varphi,\pi],\varphi(t;\vec{x})] &= \int d\vec{y}\ \tfrac{1}{2}\left[\pi(\vec{y})^2,\varphi(t;\vec{x})\right] \\ &= \int d\vec{y}\ \tfrac{1}{2}\left\{\pi(\vec{y})\left[\pi(\vec{y}),\varphi(t;\vec{x})\right] + \left[\pi(\vec{y}),\varphi(t;\vec{x})\right]\pi(\vec{y})\right\} \\ &= \int d\vec{y}\ \tfrac{1}{2}\left\{\pi(\vec{y})(-i)\delta(\vec{y}-\vec{x}) + (-i)\delta(\vec{y}-\vec{x})\pi(\vec{y})\right\} \\ &= -i\pi(\vec{x}). \end{aligned} \tag{64}$$

so we get

$$\frac{\partial}{\partial t}\varphi(t;\vec{x}) = \pi(t;\vec{x}). \tag{65}$$

which, we recall, was the definition of π in terms of $\dot{\varphi}$.

Exercise. Use the commutation relations to find the equation of motion for $\pi(t; \vec{x})$ in our example theory. Compare to the equation of motion from the principle of stationary action.

Exercise. In the previous exercise we found that

$$\begin{aligned} [P^0, \varphi(t; \vec{x})] &= -i\, \partial\varphi(t; \vec{x})/\partial t \\ [P^0, \pi(t; \vec{x})] &= -i\, \partial\pi(t; \vec{x})/\partial t \end{aligned} \tag{66}$$

where $P^0 = \int d\vec{x}\, T^{00}$. Use the commutation relations for our sample theory to find something similar for $[P^j, \varphi(t; \vec{x})]$ and $[P^j, \pi(t; \vec{x})]$ where $P^j = \int d\vec{x}\; T^{j0}$.