

Posets Arising as 1-skeleta of Simple Polytopes, the Nonrevisiting Path Conjecture and Poset Topology

Patricia Hersh

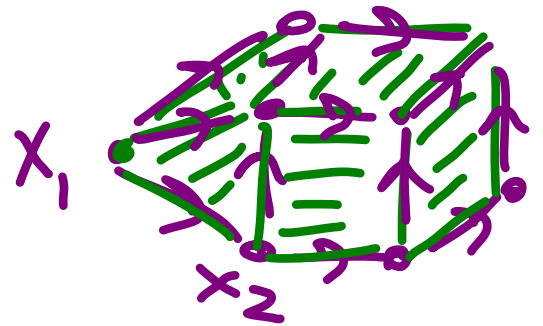
North Carolina

State University

- with thanks to Karola
Mészáros for fruitful
discussions early in project

Linear Programming

- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max \{ \vec{c} \cdot \vec{x} \mid A\vec{x} \leq \vec{b} \}$
- $\{ \vec{x} \mid A\vec{x} \leq \vec{b} \}$ is polytope P
if set is bounded //

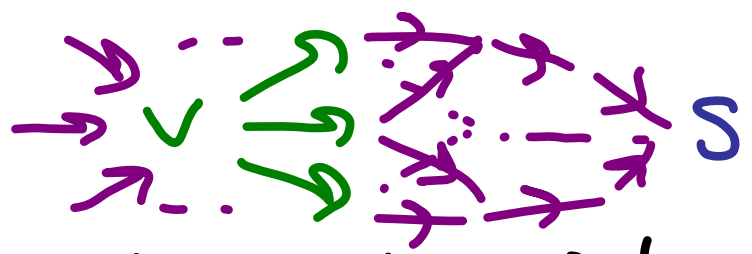


- Define $G(P, \vec{c}) :=$ directed graph on 1-skeleton of P with
 $x_1 \rightarrow x_2 \iff \vec{c} \cdot \vec{x}_1 < \vec{c} \cdot \vec{x}_2$

Solving Linear Programs via Simplex Method

- $\max \{ \vec{c} \cdot \vec{x} \mid A\vec{x} \leq \vec{b} \} = \text{Sink of } G(P, \vec{c})$

Simplex Method: walk from some vertex $v \in G(P, \vec{c})$ following arrows $v \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow s$ to sink s

Pivot Rule: method to choose which out arrow  s to follow from v towards sink s .

Key Questions: what is typical complexity of simplex method (path length)? What is worst case?

Hirsch Conjecture: For d -dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d, n)$, satisfies $\Delta(d, n) \leq n - d$.

Francisco Santos: Counterexamples!

Nonrevisiting path conjecture:

For each u, v in polytope P , there is path u to v not revisiting any facet it has left.

Non-Revis. Path Conj $\Rightarrow$ Hirsch Conj.
(by giving short path) so BOTH FALSE!

Our Plan

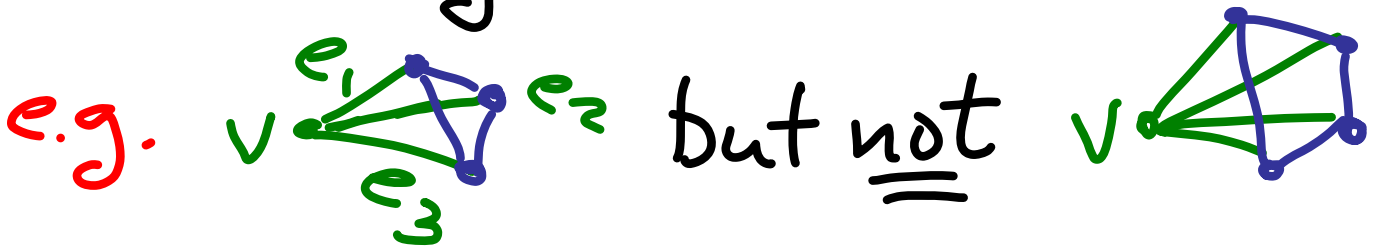
Impose further conditions on $\mathcal{P}$ and $\vec{c}$ that will imply a corollary of the following which we hope might also hold:

For each $u, v \in \mathcal{P}$, each directed path from u to v never revisits any facet it has left.

This property would make all pivot rules efficient for $\mathcal{P}$ and $\vec{c}$.

Quick Background on Polytopes

- A **polytope** in $\mathbb{R}^d$ is convex hull of finite # vertices, or equivalently a bounded set that is an intersection of half spaces.
- A polytope is **simple** if for each vertex v and each collection $e_1, e_2, \dots, e_r$ of edges emanating out from v there is an r -dim'l face containing all these edges



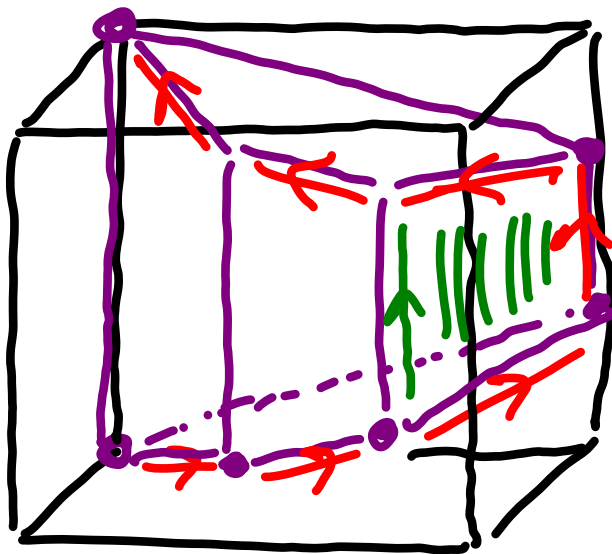
New Def'n: $G(P, \vec{c})$ has the Hasse diagram property if it is Hasse diagram of finite poset, i.e. $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_r$ for $r \geq 3$ directed path precludes $v_1 \rightarrow v_r$

Important Non-Examples:
"Klee - Minty Cubes"

e.g. $n=3$

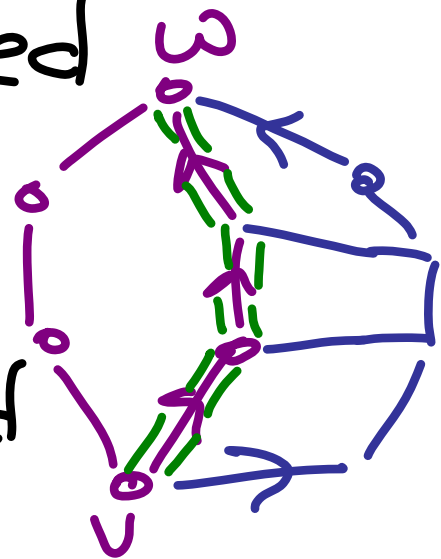
$\vec{c} = (0, 0, 1)$

- path visits all vertices!

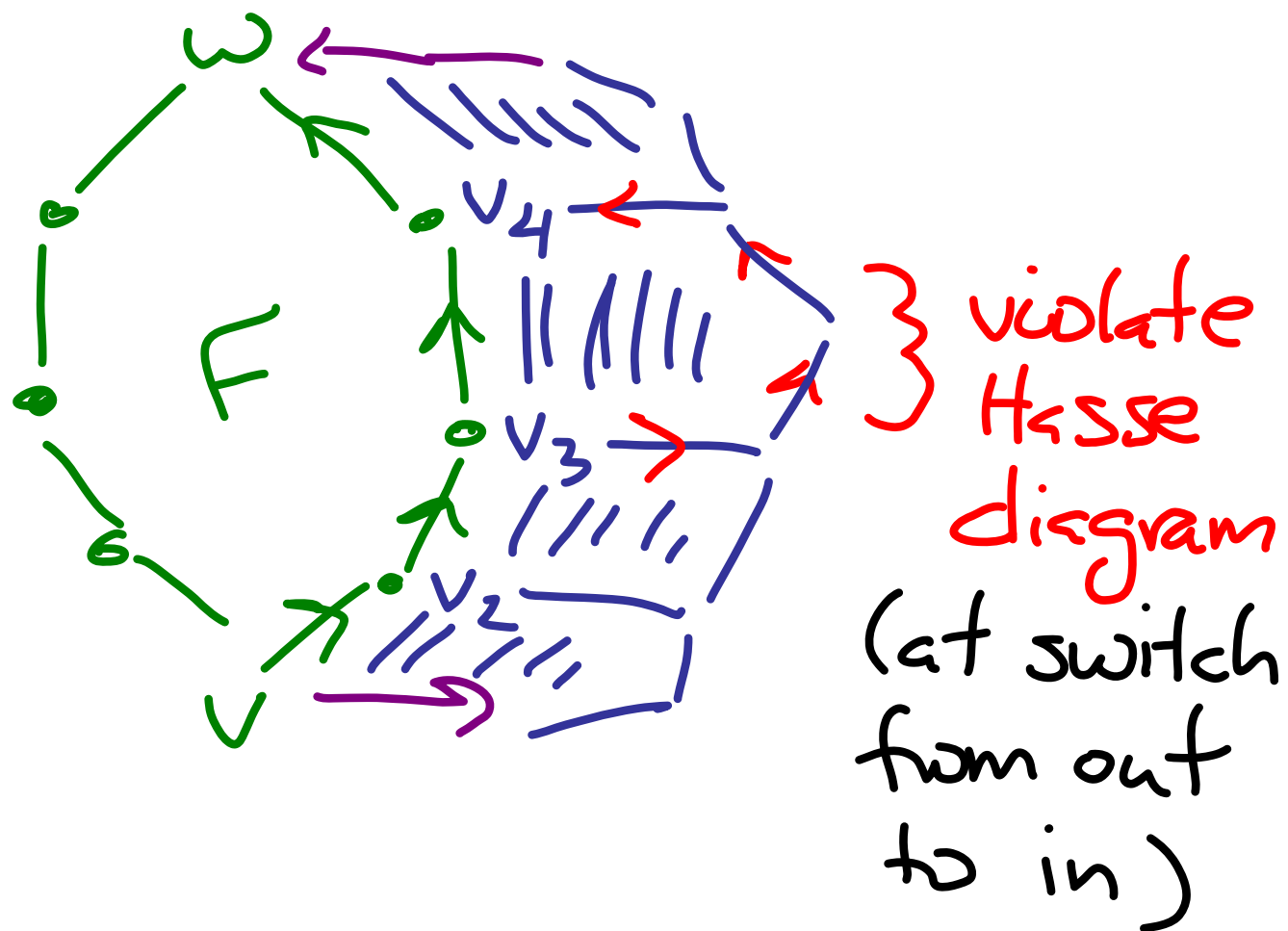


n -dim' $\mathcal{Q}$
 KM-cube $= \left\{ (x_1, \dots, x_n) \mid 0 \leq x_i \leq 1, \sum_{i=1}^{n-1} \varepsilon x_i < x_n < 1 - \sum_{i=1}^{n-1} \varepsilon x_i \right\}$
 for $i > 1, 0 < \varepsilon < \frac{1}{2}$

Lemma: Given $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope P w/ generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path P_F from v to w in F , outward oriented edge v to $G \setminus F$ and inward oriented edge $G \setminus F$ to w .



Corollary: Monotonicity of out-degrees
 $\neq$ partic. outward directions.

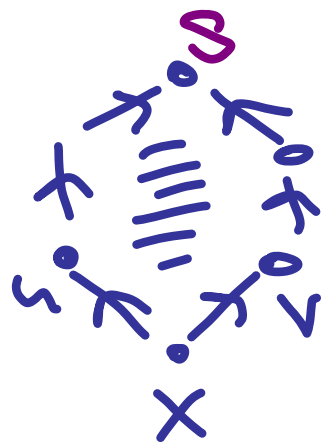


Corollary: For each face $F \subseteq P$
 with $\hat{0} \in F$ or $\hat{1} \in F$, directed
 paths cannot revisit F
 after departing from it.

Recall: A poset L is a **lattice** if for each $u, v \in L$ there exists unique least upper bound ("**join**") for u and v , denoted $u \vee v$, and unique greatest lower bound for u and v ("**meet**"), $u \wedge v$.

Note: for P simple & $G(P, \vec{e})$

Hasse diagram, an upper bound for u, v both covering x is sink s of unique 2-face containing x, u, v .



"Pseudo-joins" in a Polytope

Let P be simple polytope w/
generic cost vector $\vec{c}$ such that

$G(P, \vec{c})$ is Hasse diagram of
poset L with $x_1, x_2, \dots, x_r \in L$

s.t. there exists $u \in L$ with
 $u < x_i$ for $i=1, 2, \dots, r$. Define

pseudo-join of $x_1, x_2, \dots, x_r$ as

sink of unique r -face of

P containing $x_1, x_2, x_3, \dots, x_r$

Lemma: $S \neq T \Rightarrow \text{pseudo}(S) \neq \text{pseudo}(T)$.

Note: Since pseudo-join of $x_1, \dots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \dots \vee x_r$ to $\text{pseudo-join}(x_1, \dots, x_r)$

Thm: Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then $\text{pseudo-join}(x_1, x_2, \dots, x_r) = x_1 \vee \dots \vee x_r$

Pf: induction on r with $r=2$ base case especially tricky part.

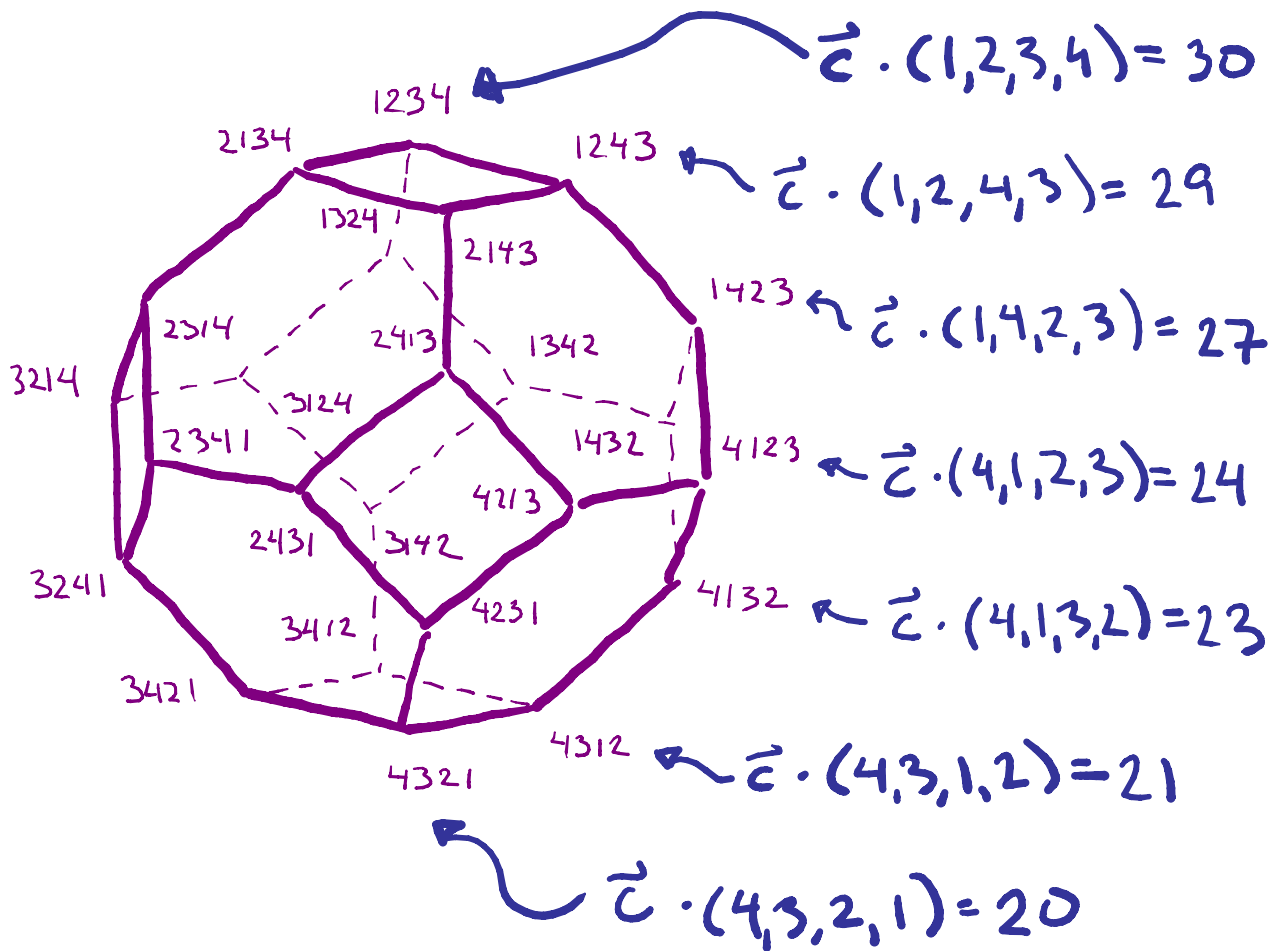
Thm: Let P be a simple polytope with generic cost vector $\vec{c}$ such that $G(P, \vec{c})$ is the Hasse diagram of a finite lattice L . Then each open interval $(u, v) = \{z \in L \mid u < z < v\}$ has order complex homotopy equivalent to a ball or a sphere

Applications:

- permutahedra $\rightsquigarrow$ weak order
- associahedra $\rightsquigarrow$ Tamari lattice
- generalized associahedra $\rightsquigarrow$ Cambrian lattices

Permutahedron as Weak Order

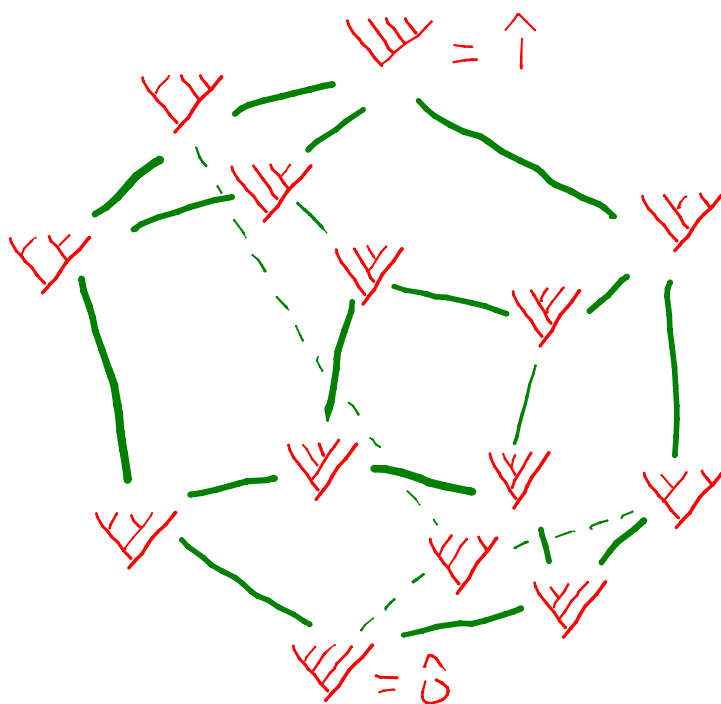
- cost vector $\vec{c}$ any strictly ascending vector such as $\vec{c} = (1, 2, 3, 4)$.



- Homotopy type 1st due to Edelman (type A) $\neq$ Björner

Associahedron ss Tamari Lattice

- Use Loday's realization
- Poset of binary trees with cover relations : $\vee \prec \vee$
 $((a,b),c)$ $(a,(b,c))$



- Homotopy type $K1$ due to Björner & Wachs via nonpure lexicographic shellability

Some Further Questions

Qn 1: Other examples?

Qn 2: Does P simple + $G(P, \vec{e})$

Hasse diagram of lattice $\Rightarrow$ no directed path can revisit face it has departed? (If not, variations?)

Thanks!!