Generalized Recursive Atom Ordering & Equivalence to CL-shellability

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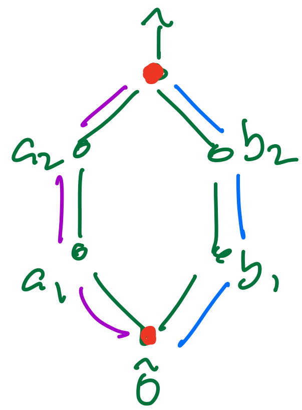
(joint work with Grace Stadnyk, Furman University)

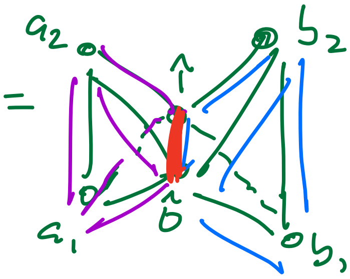
Talk Plan:

- Background
- Generalized Recursive Atom Orderings (GRAOs)
- Atom Reordering Process & Equivalence of GRAO to Having Recursive Atom Ordering (RAO)
- Properties of GRAOs (& Reordering Process)
- New Equivalences among types of Lexicographic shellability
- Application to uncrossing orders

I. Background

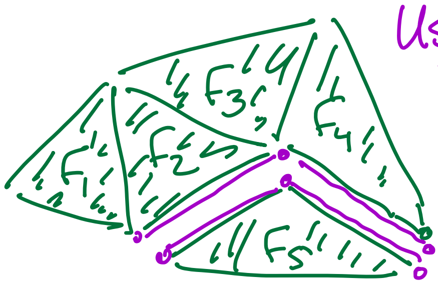
Def'n: The order complex, $\Delta(P)$, of a finite poset P is simplicial complex whose i-dimensional faces are the chains $v_0 < v_1 < \dots < v_i$ of $i+1$ comparable poset elements.

e.g. $P =$ 

$\Delta(P) =$ 

- pure means all max faces of same dimension

Def'n: A shelling of (not necessarily pure) simplicial complex Δ is total order $F_1, F_2, \dots, F_k$ on maximal faces of Δ s.t. $\bar{F}_j \cap (\bigcup_{i<j} \bar{F}_i)$ is pure codimension one subcomplex of $\bar{F}_j$ for $j = 2, 3, \dots, k$.



Usefulness:
- Implies homotopy equiv. to wedge of spheres or contractible,
- Calculates Möbius function of P via $\mu_P(u,v) = \tilde{\chi}(\Delta_P(u,v))$

Our Focus: Conditions on a poset P giving rise to a shelling for $\Delta(P)$, specifically constructive equivalences among some such conditions, new & old.

Poset Terminology:

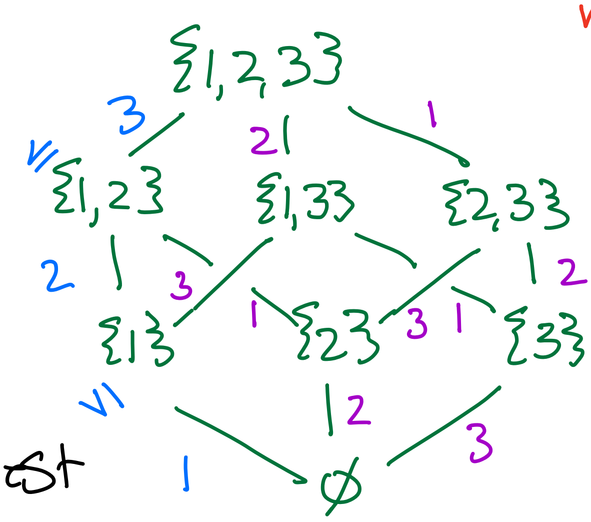
- A poset is bounded if it has unique minimal element "$\hat{0}$" & unique max elt "$\hat{1}$"
- A cover relation $u \lessdot v$ means $u < v$ s.t. there is no intermediate z with $u < z < v$
- An atom of bounded poset P is an element $a \in P$ s.t. $\hat{0} \lessdot a$
- A chain is any $u_1 < \cdots < u_k$ in a poset P.
- A chain is maximal if no additional elements may be added to it
- A saturated chain from u to v in P is any $u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_k \lessdot v$
- The length of P is max # cover relations in any maximal chain of P

Def'n (Björner, Björner–Wachs): An **EL-labeling** of a finite bounded poset P is a labeling λ of its cover relations $x \lessdot y$ with labels $\lambda(x,y)$ s.t. for each $u<v$ in P:

(i) there is a unique saturated chain $u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_k \lessdot v$ from u to v s.t. $\lambda(u,u_1) \le \lambda(u_1,u_2) \le \lambda(u_2,u_3) \le \cdots \le \lambda(u_k,v)$

(ii) the label sequence $(\lambda(u,u_1), \lambda(u_1,u_2), \ldots, \lambda(u_k,v))$ is lexicographically smaller than the label sequence of any other saturated chain u to v

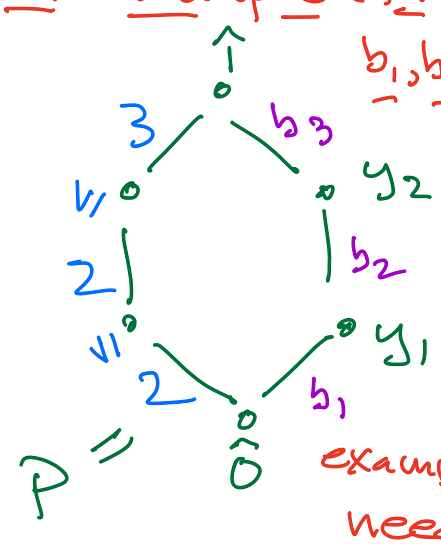
e.g.



123

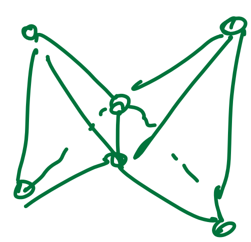
lexic. smallest

non-example (for all b_1, b_2, b_3)



example needs $b_1 \le b_2$; $b_2 \le b_3$

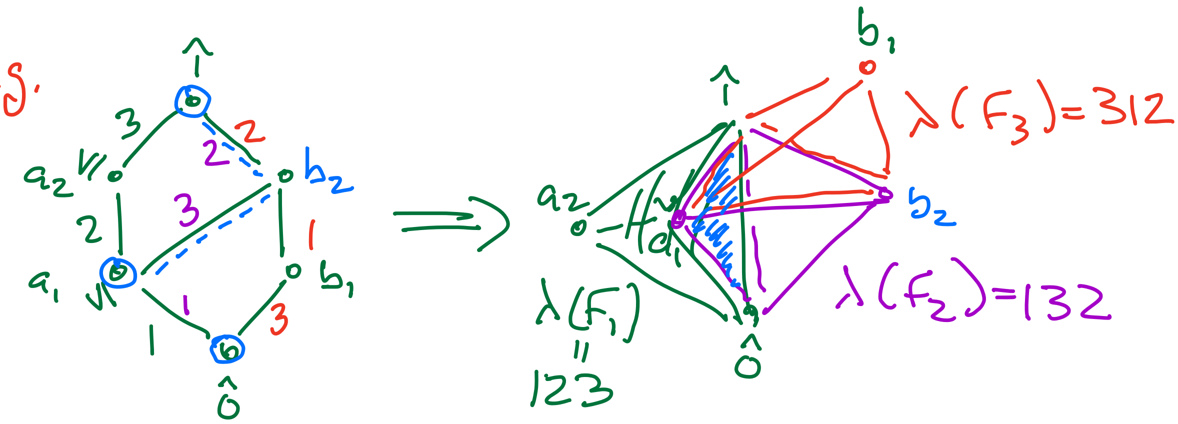
$\Delta(P) =$



not shellable

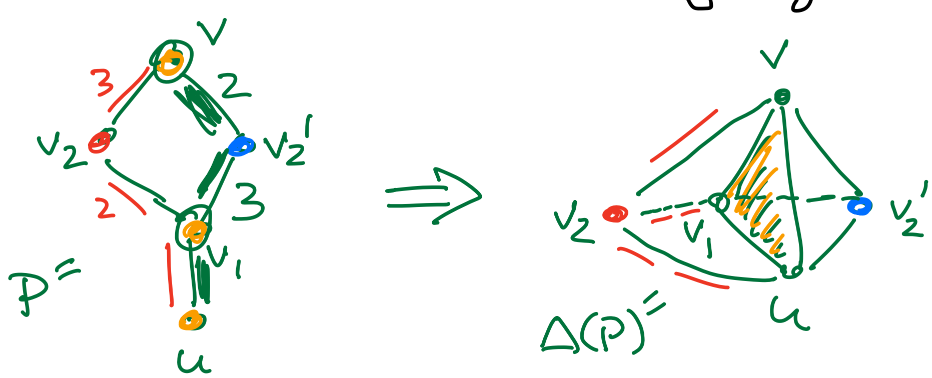
Thm (Björner, Björner-Wachs): EL-labeling of P $\Rightarrow$ Shelling for $\Delta(P)$ by ordering label sequences lexicographically, breaking ties in any manner.

e.g.



$123 <_{lex} 132 <_{lex} 312$

Idea: descent in consec. labels of chain $\leftrightarrow$ codim. one face shared with earlier facet (by replacing descent with lex. earlier ascending sequence)



Aside: Stephen Lacina's thesis studies partial order on max chains in P derived from any EL-labeling

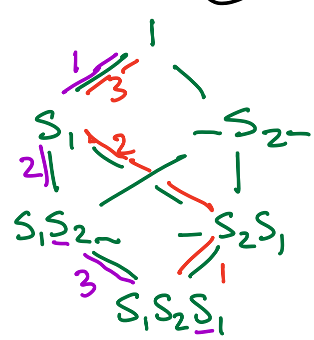
Def'n (Björner–Wachs): A CL-labeling of finite bounded poset P is a generalization of EL-labeling. The label on $u \lessdot v$ is allowed to depend not just on u and v but also depends on choice of "root" $r = \hat{0} \lessdot u_1 \lessdot \cdots \lessdot u$ from $\hat{0}$ to u. Otherwise same def'n.

That is, each "rooted interval" $[u,v]_r$ has unique saturated chain u to v with weakly ascending labels, and this is lex'ly smallest label sequence in $[u,v]_r$.

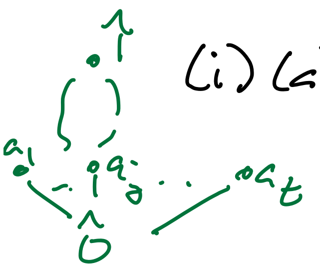
Thm (Björner & Wachs): If finite bounded poset P has CL-labeling, then this induces "lexicographic" shelling on $\Delta(P)$ by ordering label sequences lexicographically (breaking ties in any manner).

Example that Motivated B-W to Introduce CL-Labelings:

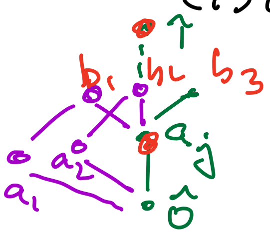
(Dual to) Bruhat order



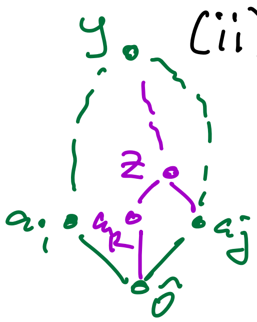
Defn (Björner–Wachs): A recursive atom ordering (RAO) on a finite bounded poset P of length > 1 is a total order $a_1, a_2, \ldots, a_t$ on the atoms of P s.t.:



(i)(a) For all $j = 1, \ldots, t$, the closed interval $[a_j, \hat{1}]$ admits an RAO $b_1, b_2, \ldots, b_s$



(i)(b) For $j \neq 1$, the first atoms $b_1, \ldots, b_i$ in RAO $b_1, \ldots, b_s$ for $[a_j, \hat{1}]$ are those b_ℓ s.t. $a_i < b_\ell$ for some atom a_i with $i < j$.



(ii) For each $i < j$ and for each $y \in P$ s.t. $a_i < y$ & $a_j < y$, there exists $k < j$ & $z \in P$ s.t. $a_j \lessdot z \leq y$ and $a_k < z \leq y$.

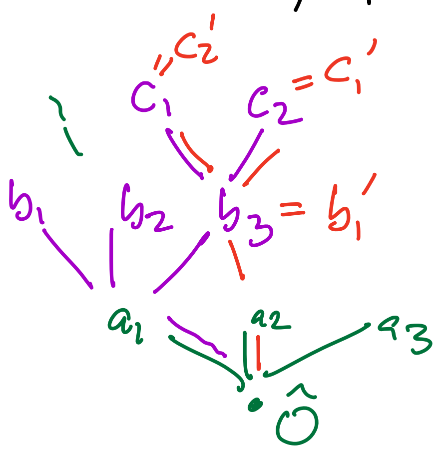
Any P of length 1 has an RAO.

Thm (Björner & Wachs): A finite bounded poset P is CL-shellable $\iff$ P admits RAO.

"Chain-atom orderings"

Def'n: A chain-atom ordering in a finite, bounded poset P is a total order $a_1, a_2, \ldots, a_t$ on the atoms of each rooted interval $[u, \hat{1}]_r$.

e.g.



- RAO is a type of total order on atoms of finite, bounded poset P that can be extended to chain atom-ordering in nice way.

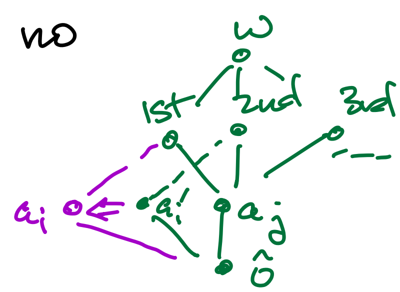
- We often think of an RAO (+ a "Generalized Recursive Atom Ordering" (defined next) as type of chain-atom ordering rather than as atom ordering.

II. Generalized Recursive Atom Orderings (GRAOs)

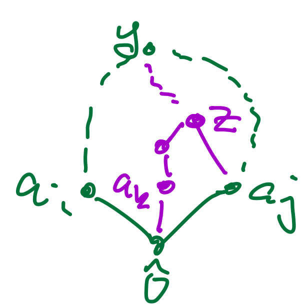
Def'n (H.-Stadnyk): A finite, bounded poset P admits a **generalized recursive atom ordering (GRAO)** if: length$(P) > 1$ with an ordering $a_1, a_2, \ldots, a_t$ on the atoms of P s.t.

(i)(a) $[a_j, \hat{1}]$ admits a GRAO for $j = 1, 2, \ldots, t$

(i)(b) $\forall\, w \in P$ such that $a_j \lessdot x \lessdot w$ for some $x \in P$, either 1st atom of $[a_j, w]$ is greater than a_i for some $i < j$, or no atom of $[a_j, w]$ is greater than any such a_i.



(ii) $a_i < y$ & $a_j < y$ for $i < j$ implies there exists $z \in P$ s.t. $a_k < z \le y$ & $a_j \lessdot z \le y$ for some $k < j$.

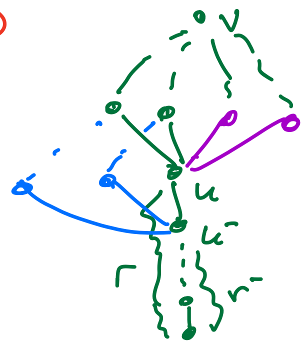


or length$(P) = 1$.

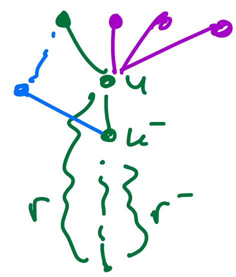
"GRAO" vs "RAO":

1. For $u < v$ and each root r:

RAO



GRAO



- all atoms of $[u,v]_r$ that are above earlier atoms than u in $[u^-,v]_{r^-}$ come before all other atoms

- 1st atom of $[u,v]_r$ above earlier atom if any is

2. GRAO only requires above for $u < v$ and r s.t. $u \lessdot x \lessdot v$ for some x

Thm (H.-Stadnyk): If finite, bounded poset admits a GRAO, then it also admits an RAO via "atom reordering process" applied to the GRAO to produce an RAO.

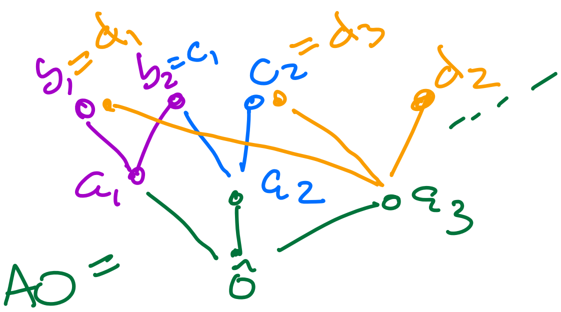
Conversely: Every RAO is a GRAO

Upshot: Admitting GRAO equiv. to CL-shellability

III. Atom Reordering Process

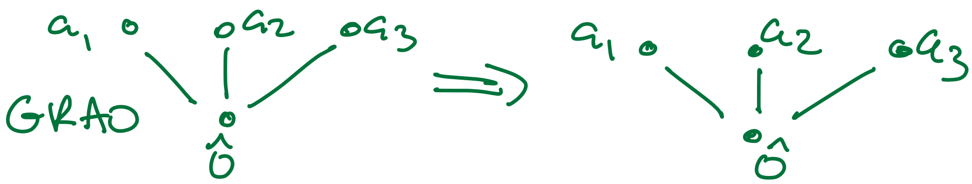
- Start with any GRAO (regarded as chain-atom ordering)

e.g. GRAO =



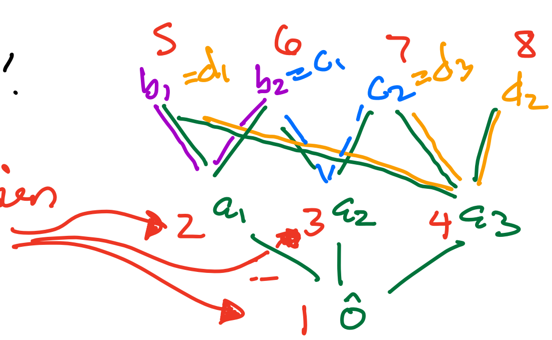
- Leave order of atoms of P alone

e.g.

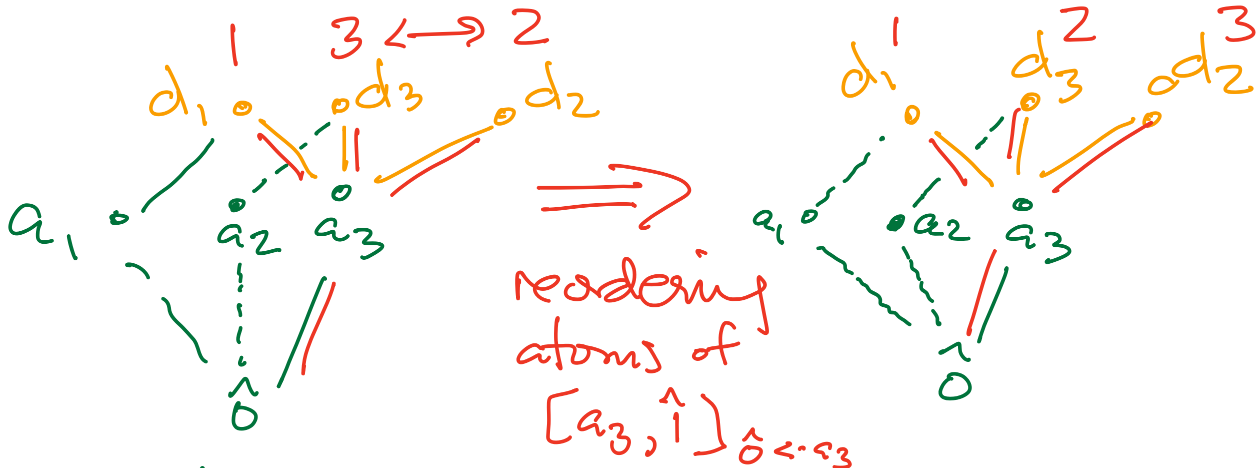


- Choose linear extension $u_1, u_2, \ldots, u_{|P|}$ of P.
- Proceed to each $u_i \in P$ in turn in this linear order, doing "atom reordering" described next for atoms of $[u, \hat{1}]_r$ for each root r after reordering atoms of each $[u', \hat{1}]_{r'}$, for $u' < u$ & each root r'.

e.g. linear extension of P



e.g. atom reordering for $[a_3,\hat{1}]_{\hat{0}\lessdot a_3}$ swaps



d_2, d_3 since earlier a_2 below d_3 while no earlier atom below d_2.

- **General Reordering Step:** For $r = \hat{0} \lessdot \cdots \lessdot u^- \lessdot u$, reorder atoms of $[u,\hat{1}]_r$ as follows:

1. Let

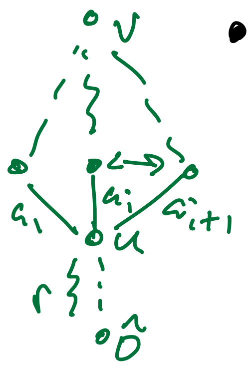
$$F_r(u) := \left\{ \begin{array}{l} \text{atoms } a \in [u,\hat{1}]_r \text{ s.t. } a > u' \\ \text{for some } u' \in [u^-,\hat{1}]_{r^-} \text{ earlier} \\ \text{atom than } u \text{ in chain-atom} \\ \text{order just prior to this step} \end{array} \right\}$$

$$G_r(u) := \left\{ \begin{array}{l} \text{atoms of } [u,\hat{1}]_r \text{ not} \\ \text{in } F_r(u) \end{array} \right\}$$

2. Move atoms in $F_r(u)$ ahead of those in $G_r(u)$, otherwise preserving order of atoms of $[u,\hat{1}]_r$

IV. Properties of GRAO's & of Atom Reordering Process

Properties of GRAO's



- Restricts to GRAO on any rooted interval
- Given GRAO for P, swapping order of consec. atoms a_i and a_{i+1} in rooted interval $[u,\hat{1}]_r$ s.t. neither a_i nor a_{i+1} is 1st atom in any $[u,v]_r$ with $a_i, a_{i+1} \in [u,v]$, then resulting chain-atom ordering is still a GRAO.

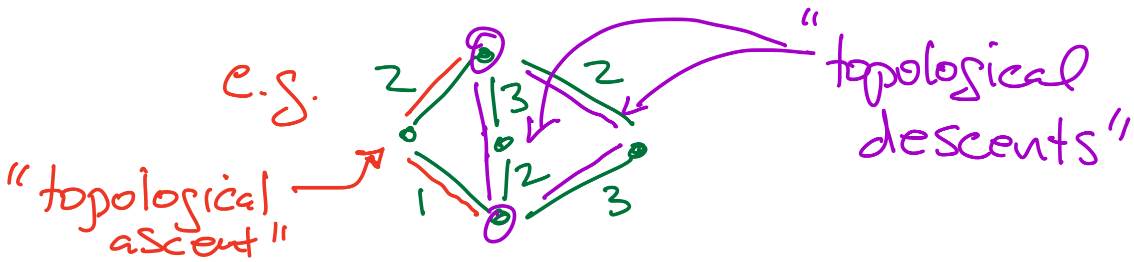
Useful Features of Atom Reordering Process

- Thm (H.-Stadnyk): it preserves which atom is 1st in $[u,v]_r$ $\forall u,v \in P$ for all roots r
- Thm (H.-Stadnyk): may be accomplished by series of rooted interval atom swaps $a_i \leftrightarrow a_{i+1}$ of type above. Still have GRAO after each such step (by result above).
- Cor: reordering GRAO yields a GRAO
- Thm (H.-Stadnyk): Reordering a GRAO yields an RAO, so poset is CL-shellable.

V. Equivalences among types of Lexicographic Shellability (but not the shellings themselves)

Defn (H.-Kleinberg): Given an edge labeling λ of a finite poset, λ has topological descent at $u \lessdot v \lessdot w$ if there is lexicographically earlier $u \lessdot v_1' \lessdot \cdots \lessdot w$, whereas λ has topological ascent at $u \lessdot v \lessdot w$ otherwise.

e.g.



Intuitive idea of EC-labelings & CC-labelings:

- Relax EL-labeling (resp. CL-labeling) requirements by replacing ascents with topol. ascents & descents with topol. descents

Def'n (Kozlov, as reformulated by H.-Kleinberg):

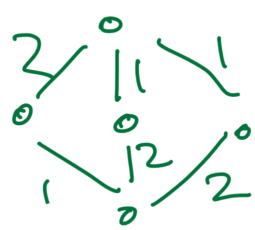
An edge labeling (resp. chain labeling) of finite bounded poset P is an EC-labeling (resp. CC-labeling) if:

(i) Satisfies requirements of EL-labeling (resp. CL-labeling) but with ascents & descents replaced by topol. ascents & descents.

(ii) for each $[u,v]_r$, distinct maximal chains of $[u,v]_r$ have distinct label sequences.

(iii) for each $[u,v]_r$, no maximal chain of $[u,v]_r$ has label sequence that is prefix of any other.

Word of Caution: Many well-known EL-labelings & CL-labelings such as

e.g. 

fail condition (ii) and/or (iii).

Thm (Kozlov): An EC-labeling (resp. CC-labeling) of a finite bounded poset P induces lexicographic shelling $\Delta(P)$

Variation on CC-Shellability:

Def'n (H.-Stadnyk): A TCL-labeling is a chain labeling s.t.

(1) each rooted interval $[u,v]_r$ has unique saturated chain with lex'ly smallest label sequence

(2) every other saturated chain in $[u,v]_r$ has a topological descent

Thm (H.-Stadnyk): TCL-labeling of finite bounded poset P induces lexicographic shelling of $\Delta(P)$.

Thm (H.-Stadnyk): Let P be a finite bounded poset. Then the following are equivalent:

(1) P admits recursive atom ordering

(2) P admits generalized recursive atom ordering

(3) P admits CL-labeling

(4) P admits "self-consistent" CC-labeling

(5) P admits CC-labeling with "UE property"

(6) P admits CL-labeling with "UE property"

(7) P admits "self-consistent" TCL-labeling

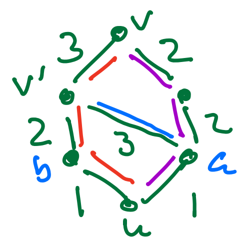
(8) P admits TCL-labeling with "UE property"

Moreover, all of these implications are obtained constructively.

"CC-Analogue" of CL-shellable $\iff$ RAO

Def'n (H.-Stadnyk): A chain-labeling for P is **self-consistent** if: for a the atom in lex 1st saturated chain of $[u,v]_r$ & b any other atom of $[u,v]_r$, then for $v' > u$ s.t. $a, b \in [u, v']$ all saturated chains of $[u,v']_r$ containing a have lex smaller label sequence than all saturated chains of $[u,v']_r$ containing b.

e.g. not

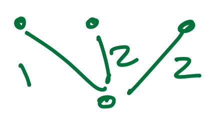


u to v': $12 <_{lex} 13$

u to v: $122 <_{lex} 123$

Def'n (H.-Stadnyk): A chain-labeling has **UE property** if each $[u,v]_r$ has smallest label that appears on any $u \lessdot a$ only appearing on one such cover relation.

e.g.



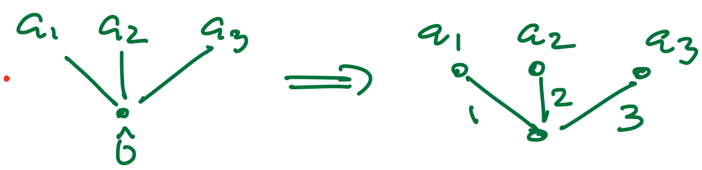
Prop'n: UE property $\Rightarrow$ self-consistent

Prop'n: All CL-labelings have UE property

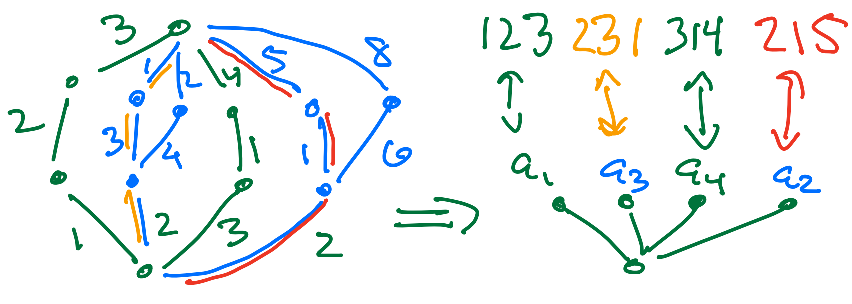
Thm (H.–Stadnyk): A finite, bounded poset P admits self-consistent CC-labeling $\iff$ P admits GRAO

Sketch of (Constructive) Proof:

($\Leftarrow$) Use GRAO to specify chain-labeling on saturated chains

e.g. 

($\Rightarrow$) order atoms of each $[u,\hat{1}]_r$ based on which is contained in lex'ly earlier saturated chain

e.g. 

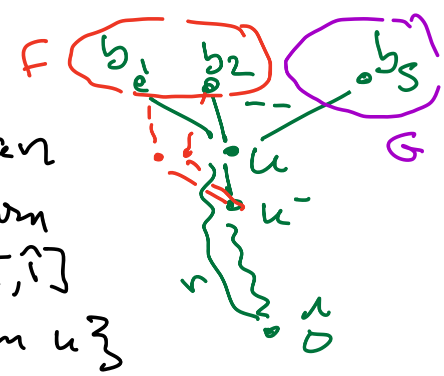
Rk: Inspired by Björner-Wachs constructions for CL-shellable $\iff$ RAO result, but much simpler due to added flexibility

How to Obtain CL-Labeling from any Recursive Atom Ordering (Björner-Wachs)

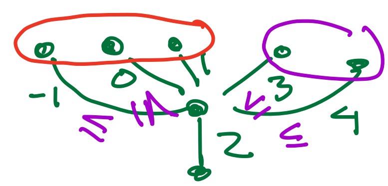
Step 1: For atoms ordered $a_1, a_2, \ldots, a_t$, let
$$\lambda(\hat{0}, a_i) = i \text{ in CL-labeling}$$

Step 2: Work upward in poset, so once we've labeled all cover rel'ns below $u \in P$ for all choices of "root" r from $\hat{0}$ to u, use atom ordering $b_1, b_2, \ldots, b_s$ for $[u, \hat{1}]_r$ to label cover rel'ns upward from u as follows:

- Temporarily set $\lambda_r(u, b_i) = i$ for $i = 1, 2, \ldots, s$



- for $b_1, \ldots, b_j \in F_r(u) = \{b_i$ above an earlier atom of $[u^-, \hat{1}]$ than $u\}$ shift label down enough to make descent with label $\lambda(u^-, u)$ below



- for $b_{j+1}, \ldots, b_s \in G_r(u)$, shift label up enough to get ascent w/ $\lambda(u^-, u)$

Recap (Pulling Above Results Together...):

Thm (H.-Stadnyk): Let P be a finite bounded poset. Then the following are equivalent:

(1) P admits recursive atom ordering

(2) P admits generalized recursive atom ordering

(3) P admits CL-labeling

(4) P admits CL-labeling with UE property

(5) P admits self-consistent CC-labeling

(6) P admits CC-labeling with UE property

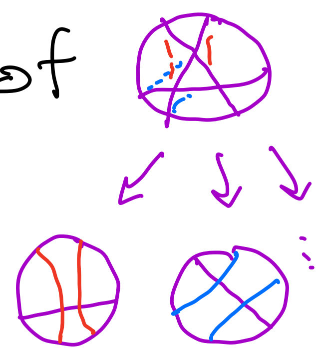
(7) P admits self-consistent TCL-labeling

(8) P admits TCL-labeling with UE property

Moreover, all of these implications are obtained constructively.

VI. Application to Uncrossing Order

- Thomas Lam proved uncrossing order is Eulerian & conjectured it to be (dual) lexicographically shellable
- motivated by interpretation as face poset for stratified space of planar electrical networks



- H.-Kenyon proved uncrossing order dual EC-shellable

Thm (H.-Stadnyk): Dual CC-shelling by H.-Kenyon satisfies UE-property, hence constructively yields dual GRAO. Thus, uncrossing order has dual RAO & hence is dual CL-shellable.

THANKS!

An Announcement:

Cascade Lectures in Combinatorics:
(CALICO)

- May 18 & 19, 2024
- Eugene, Oregon at Univ. of Oregon
- Speakers: Christian Gaetz
  Rick Kenyon
  Allen Knutson
  Bridget Tenner
  Cynthia Vinzant
- Poster session
- Some travel funding - for grad students & other early career mathematicians
  - simple application process