Fibers of Maps to Totally Nonnegative Spaces

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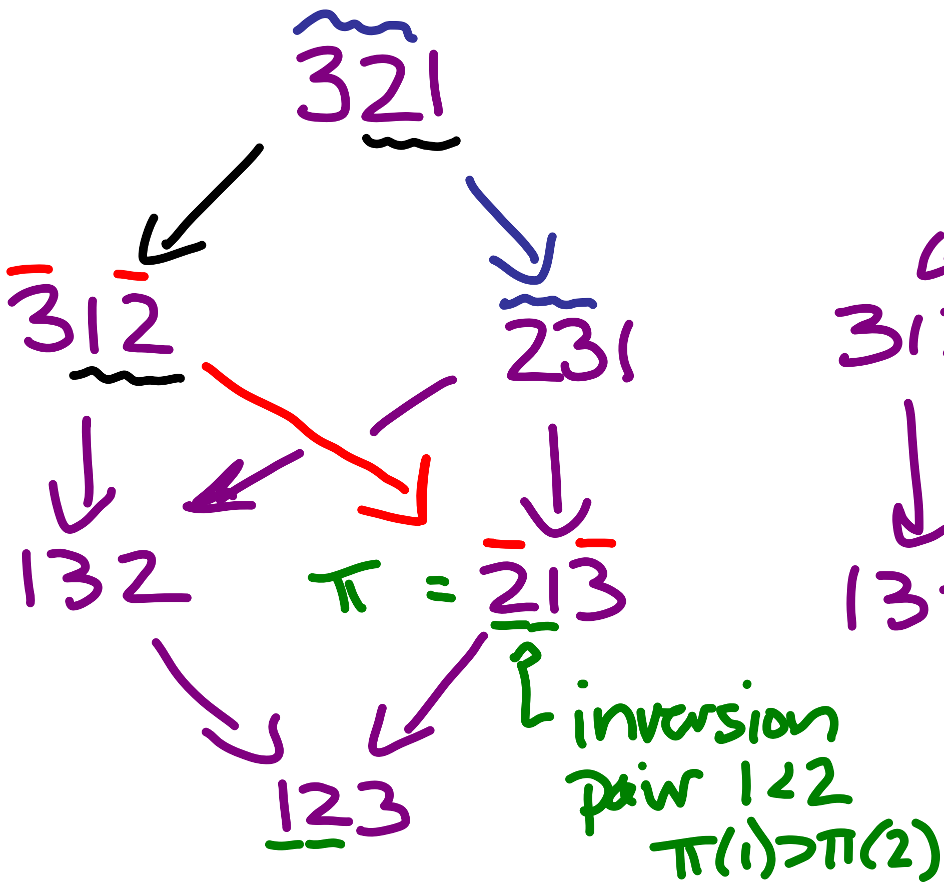
Outline for Talk

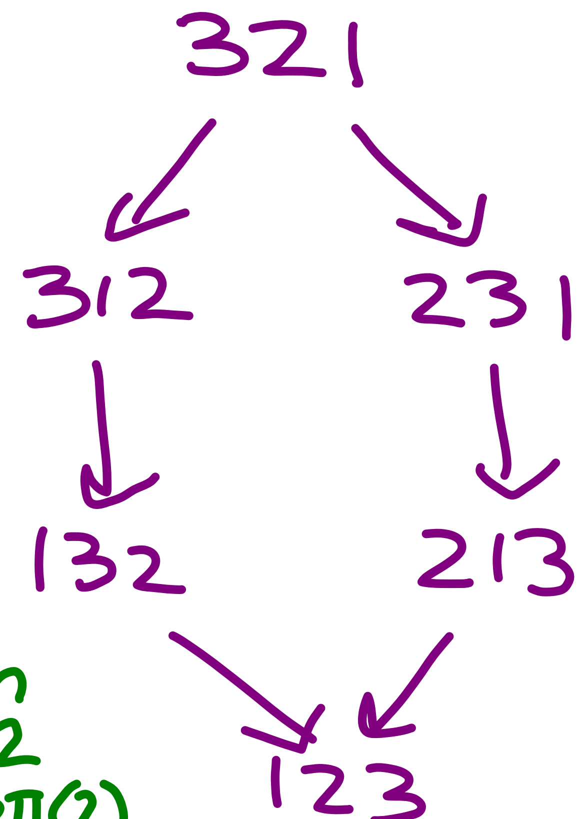
1. Background & Motivations
2. Fibers $f^{-1}_{(i_1,\dots,i_d)}(M)$ of $f_{(i_1,\dots,i_d)}$ for fixed matrix M (joint work with Jim Davis & Ezra Miller)
3. Maps with Potentially Analogous Structure & Further Questions

e.g. $$f_{(1,2,1)}(\underbrace{t_1,t_2,t_3}_{\in\, \mathbb{R}^3_{\geq 0}}) = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Partially Ordered Sets (Posets)
Describing Structure in Sorting





"Bruhat order"

"weak order"

(bubble sorting)

(sorting by swapping pairs to eliminate "inversion pairs", i.e. $i<j$ s.t. $\pi(i)>\pi(j)$)

Coxeter Groups (Generalizing S_n)

- $s_i := (i, i+1)$ = adjacent transposition
  $\Uparrow$
  S_n a.k.a. **simple** "type A" **reflection** (in W)
- $s_{i_1} \cdots s_{i_d}$ is **reduced expression** for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for d as small as possible, and $(i_1, \ldots, i_d)$ is its **reduced word**.
- **length** of w, "$\ell(w)$" := #inversion pairs in w
  $\|$
  this smallest d

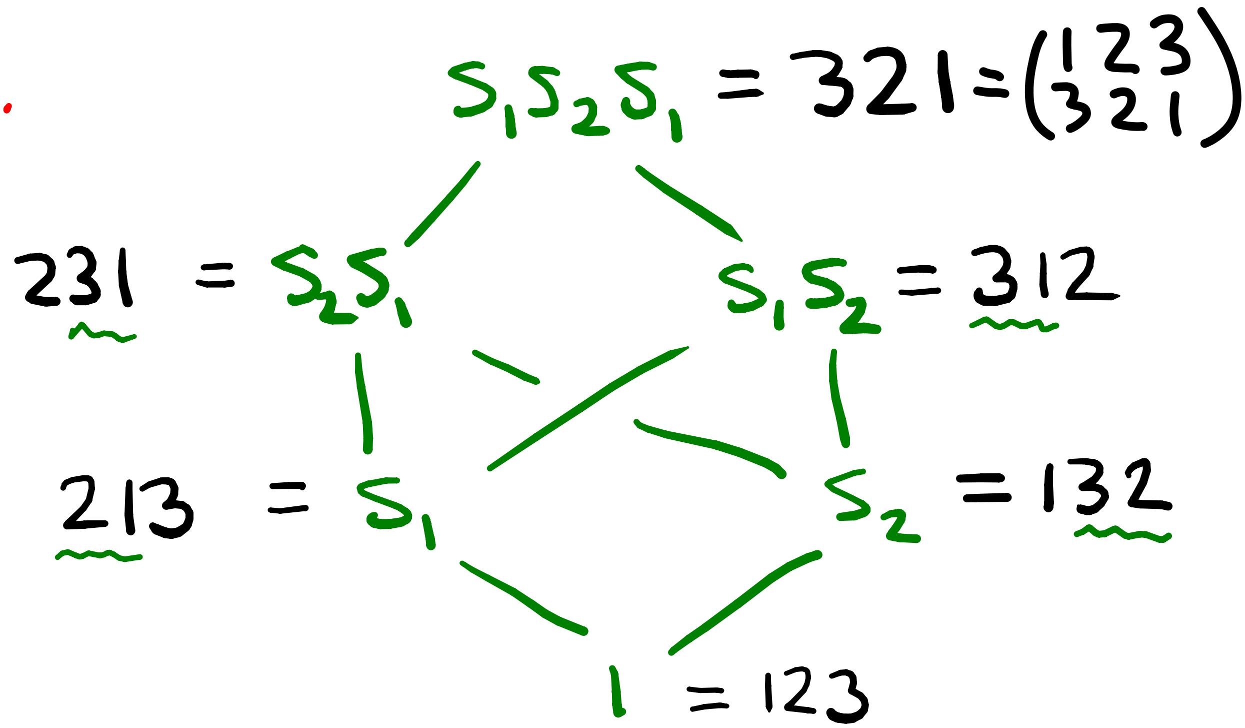
e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

$\|$

$321 \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123$

Bruhat Order for Coxeter Groups

e.g.

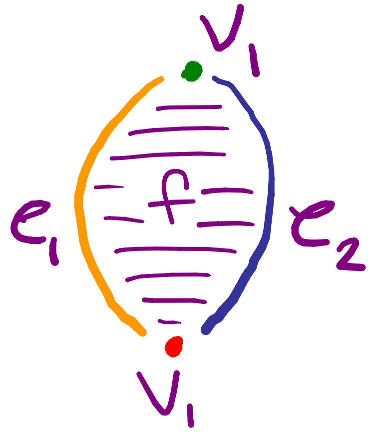


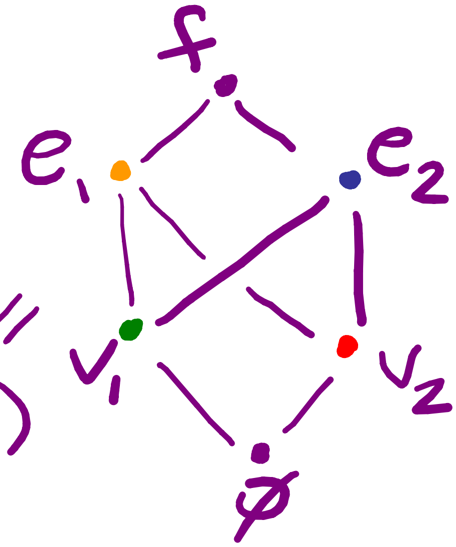
Order Relations:

$u \leq v$ for $u, v \in W \iff$ any reduced expression for v has subexpression that is reduced expression for u.

CW Complexes & their Face Posets

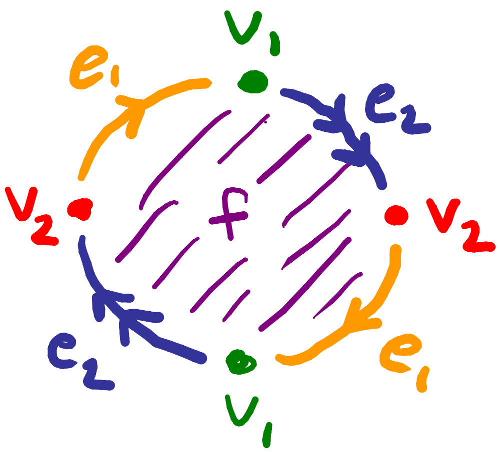
e.g.

$K =$  $\cong$ ball

$F(K) =$  $= F(K')$

"face poset"

$(u \le v \iff u \le \bar{v})$

$K' =$  $= \mathbb{RP}^2$

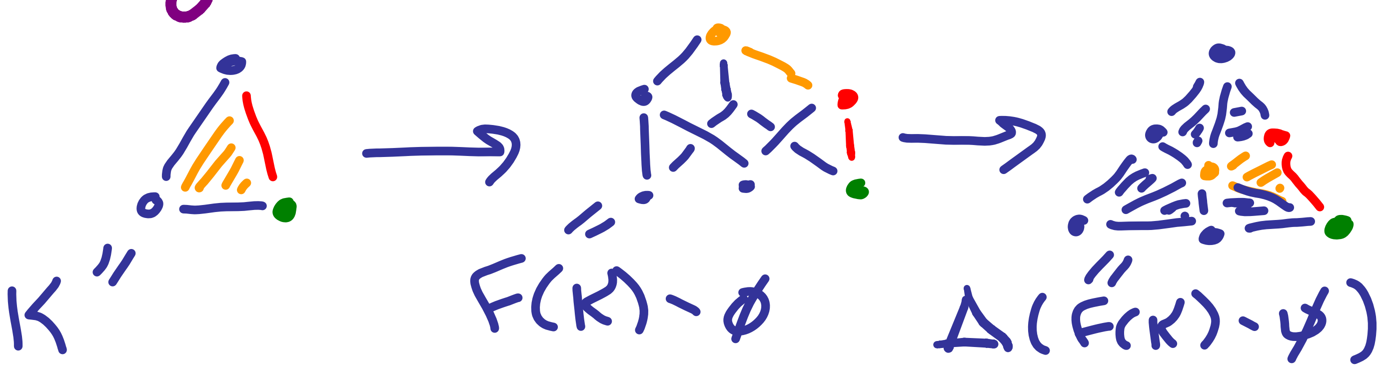
Recall: A CW complex: cells $e_\alpha \cong \mathbb{R}^{\dim(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \le \bar{\alpha}} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

Recall: 

closed ball ↗ ↖ homeomorphic

Recall: CW complex is regular if each f_α is homeomorphism.

- $\Delta(P)$ = "nerve" or "order complex" of P
  = abstract simplicial complex s.t. i-dim'l faces are the poset chains $v_0 < v_1 < \ldots < v_i$
- K regular $\Rightarrow K \cong \Delta(F(K) \setminus \emptyset) = \mathrm{sd}(K)$



Totally Nonnegative Parts of Stratified Spaces from Representation Theory

- Lusztig (94), Fomin–M. Shapiro (00)... initiated study of "totally nonnegative", real part of spaces of matrices, spaces of flags (i.e. GL_n/B) and beyond...
- totally nonnegative := all $\underbrace{\text{minors}}_{\substack{i \times i \\ \text{submatrices}}}$ nonnegative
- "flag": $\langle \vec{v}_1 \rangle \subseteq \langle \vec{v}_1, \vec{v}_2 \rangle \subseteq \dots \subseteq \mathbb{R}^n$

- These stratified spaces are conjecturally (in some cases) & provably (in other cases) regular CW complexes homeomorphic to closed balls
- Proving by studying map fibers
  - imposes restrictions on rel'n's amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases

Aside: Sharply contrasts entire flag variety, Grassmannian, etc. having rich topology (topic of Schubert calculus)

Totally Nonnegative, Real Part of "Unipotent Radical" (a Space of Matrices)

- $$x_i(t) = I_n + tE_{i,i+1} = \begin{array}{cc} \begin{array}{cccccc} & & & \text{column } i+1 & & \end{array} & \\ \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \text{row } i \\ \\ \\ \\ \end{array} \end{array}$$

  $x_i(t) = \exp(te_i)$ (general finite type, expon'd Chevalley generator)

  $I_n + tE_{i,i+1}$ (type A)

- $$f_{\underbrace{(i_1,\dots,i_d)}_{\text{reduced word}}} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

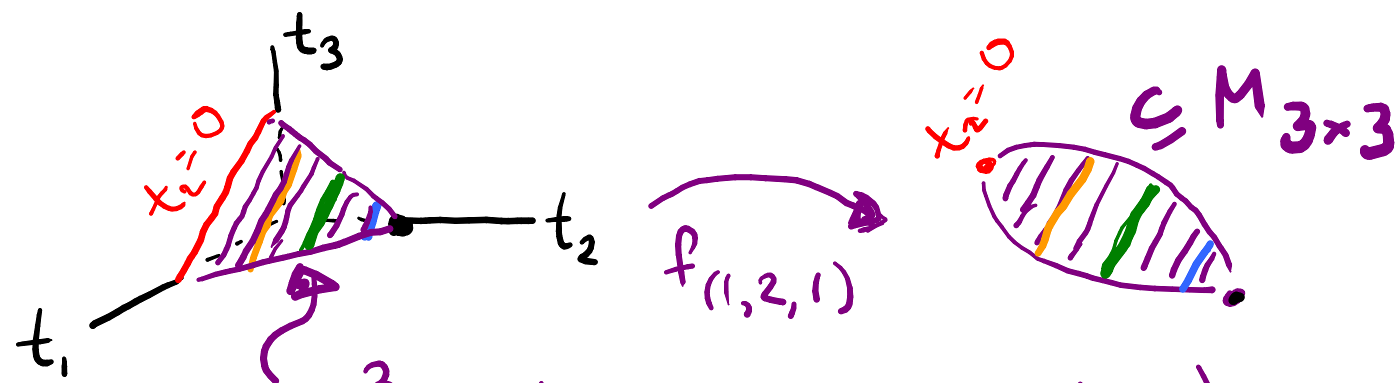
  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $$f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$$
$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

w_0 case:

$$\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{array}{l} \text{tot.} \\ \text{nonneg} \end{array} \right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\ldots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = \underbrace{K}_{\text{positive constant}} \text{ hyperplane})$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow \; t_2 = 0$$

$$f_{(1,2,1)}(t_1,0,t_3) = \overbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}^{x_1(t_1)} \cdot \overbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}^{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t > 0\} = \{x_1(t_1)x_1(t_2) \mid t_1, t_2 > 0\}$

Question (Björner, Bernstein)
Find naturally arising, rep'n theoretic regular CW complexes with Bruhat order as face poset.

Fomin-Shapiro Conjecture: The Bruhat stratification of $lk(id)$ in totally nonneg. real part of unipotent radical in Borel in algebraic group is regular CW complex homeomorphic to closed ball (w/ Bruhat order as face poset).

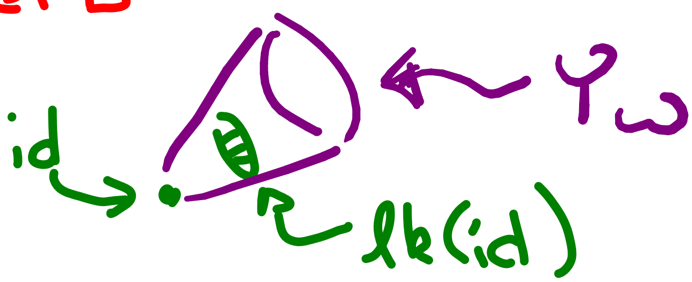
$$Y_w = \left[B^- w B^- \cap \left(\substack{\text{unipotent} \\ \text{subgp of } B} \right) \right]_{\geq 0} = \mathrm{im}\,(f_{(i_1,\dots,i_d)})$$

lower triangular opposite Borel B^-

permutation w

upper triang. w/ 1's on diagonal

totally nonneg. part

and 

Concrete Realization: Products $x_{i_1}(t_1)\cdots x_{i_d}(t_d)$ of elementary matrices, by Whitney, Loewner & Lusztig.

Theorem (H., 2014, Inventiones): Fomin-Shapiro Conjecture holds.

A Key Idea: 0-Hecke Algebra of W to Describe Strata

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

$\rightsquigarrow$ suppress parameters

$x_i x_i = x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1 t_2}{t_1+t_3}\right)$ for $t_1, t_2, t_3 > 0$

$\rightsquigarrow$ (type A)

$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$

(& analogous relations outside type A)

<u>Upshot</u>: $\mathrm{im}(f_1) = \mathrm{im}(f_2) \iff \underbrace{x(f_1) = x(f_2)}$

equal as 0-Hecke algebra elements

<u>Thm</u> (Lusztig): If $(i_1, \ldots, i_d)$ is reduced, then $f_{(i_1,\ldots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

A Motivation for Nonnegative Real Part of Unipotent Radical

- Given quantized env. alg. $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$ of Kac-Moody alg. (e.g. affine Lie alg.), then canonical basis is a basis B for U^- such that highest weight module with highest weight vector v_λ has basis $\{v_\lambda b \mid b \in B, v_\lambda b \neq 0\}$ for each λ.

- $f^{-1}_{(i_1, \ldots, i_d)}(p) \to f^{-1}_{(j_1, \ldots, j_d)}(p)$ coordinate change

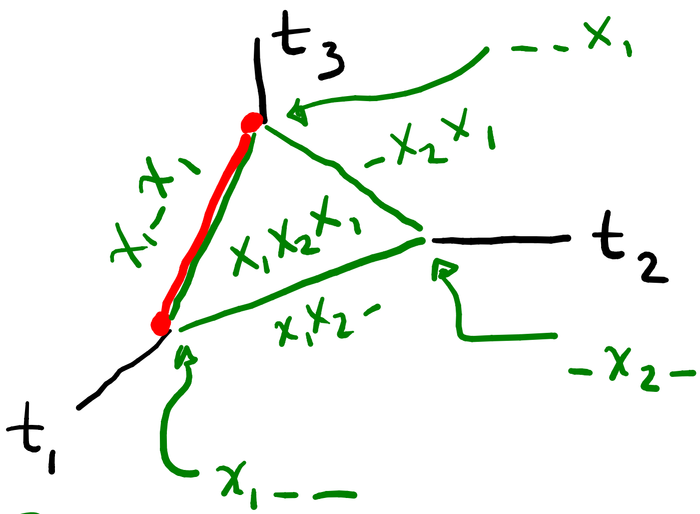
$$(t_1, t_2, t_3) \longmapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3}\right)$$

tropicalizes to coordinate change:

$$(a, b, c) \mapsto (b + c - \min(a,c), \min(a,c), a + b - \min(a,c))$$

for canonical bases w/ same braid move

Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra



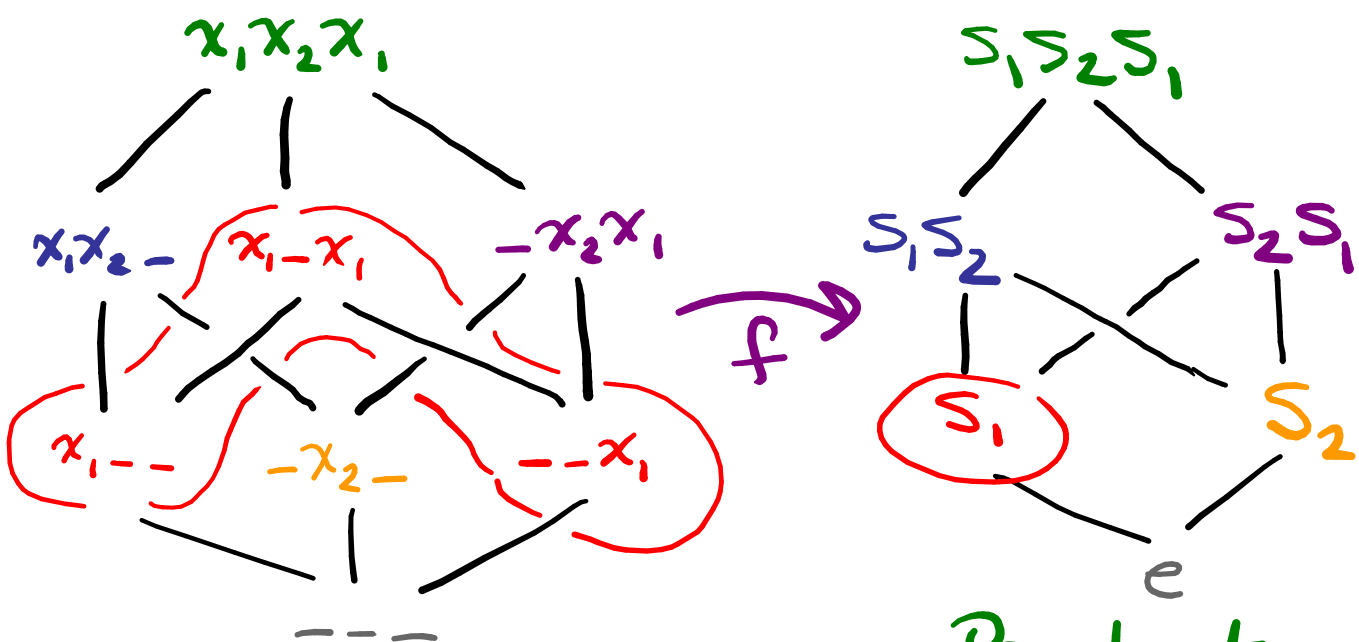
- let Y^o_w = open cell in $\mathrm{im}(f_{(i_1,\dots,i_d)})$ indexed by $w \in W$

- let $\delta(x_{i_1}\dots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1x_2x_1}_{x_2x_1x_2}x_2x_1) = \delta(x_2x_1x_2x_1) = s_1s_2s_1$

since $x_2x_2 = x_2$

Map of Face Posets Induced by Map $f_{(i_1,\ldots,i_d)}$ of Spaces

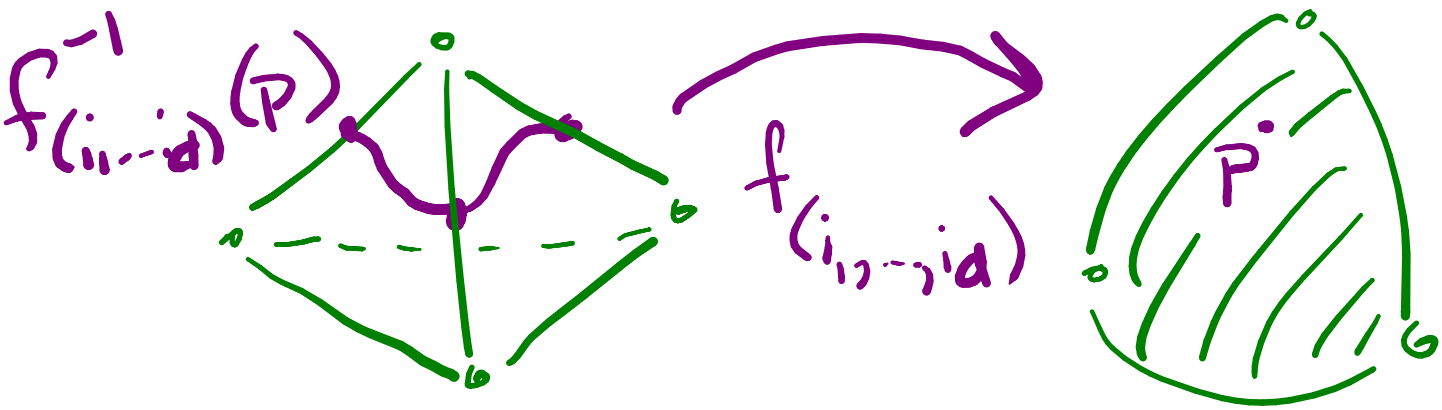


Boolean lattice B_n = Face poset of simplex

Bruhat order = $F(\mathrm{im}(f_{(i_1,\ldots,i_d)}))$

$$f: x_{i_{j_1}} \cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}} \cdots x_{i_{j_r}})$$

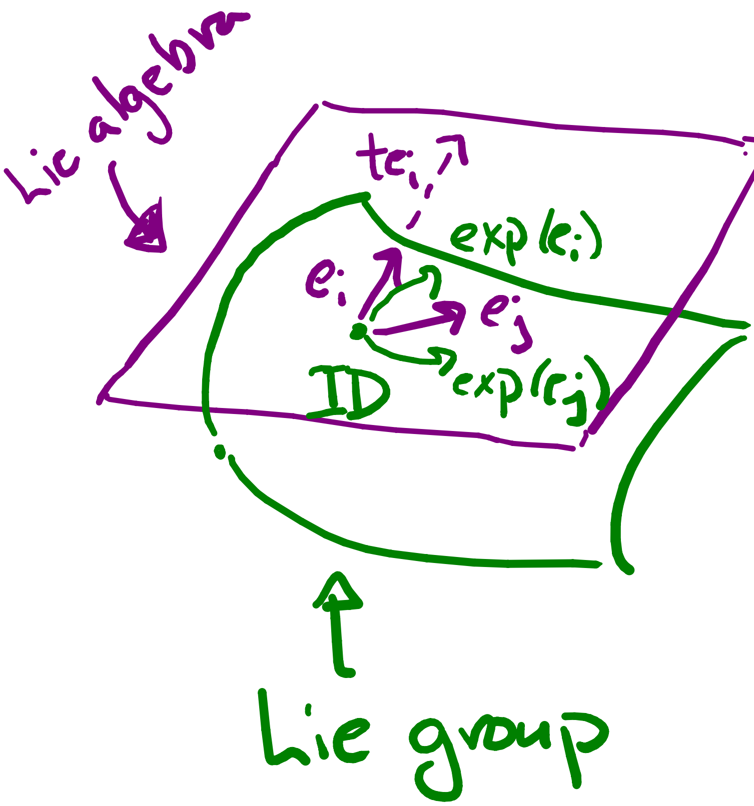
Conjecture (Davis-H-Miller): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\ldots,i_d),w)$ for $p \in Y_w^o$.



Thm (DHM): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ has cell decomposition & "correct" face poset.

Thm (DHM): Interior dual block complex of $\Delta((i_1,\ldots,i_d),w)$ is contractible.

A Motivation to Study Fibers: Relations Among (Exponentiated) Chevalley Generators



$$te_1 = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \exp(\_)$$

$$\begin{pmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\exp(t_1 e_i)\exp(t_2 e_j)$$

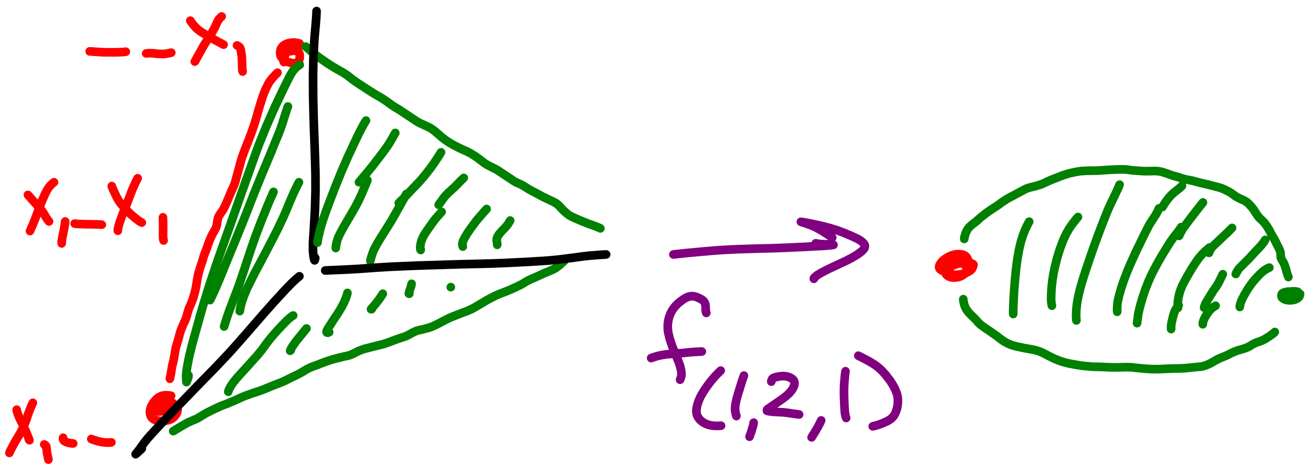
$$\|$$

$$f_{(i,j)}(t_1,t_2) = x_i(t_1)\, x_j(t_2)$$

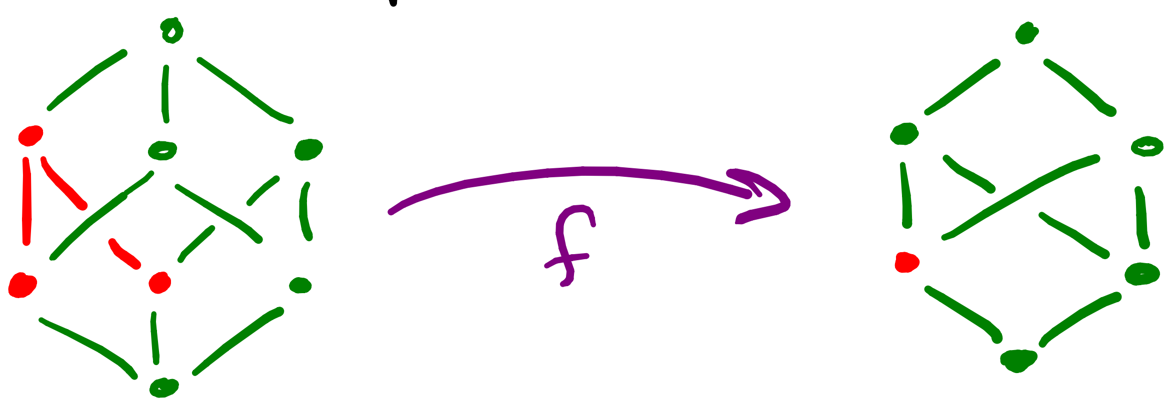
$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{t^2 \frac{e_i^2}{2}}_{0=} + \underbrace{t^3 \frac{e_i^3}{6}}_{0=} + \cdots$$

Rel'ns $\Leftrightarrow$ Elts in same fiber of $f_{(i_1,\dots,i_d)}$

Combinatorics of Fibers



Induced map of face posets:



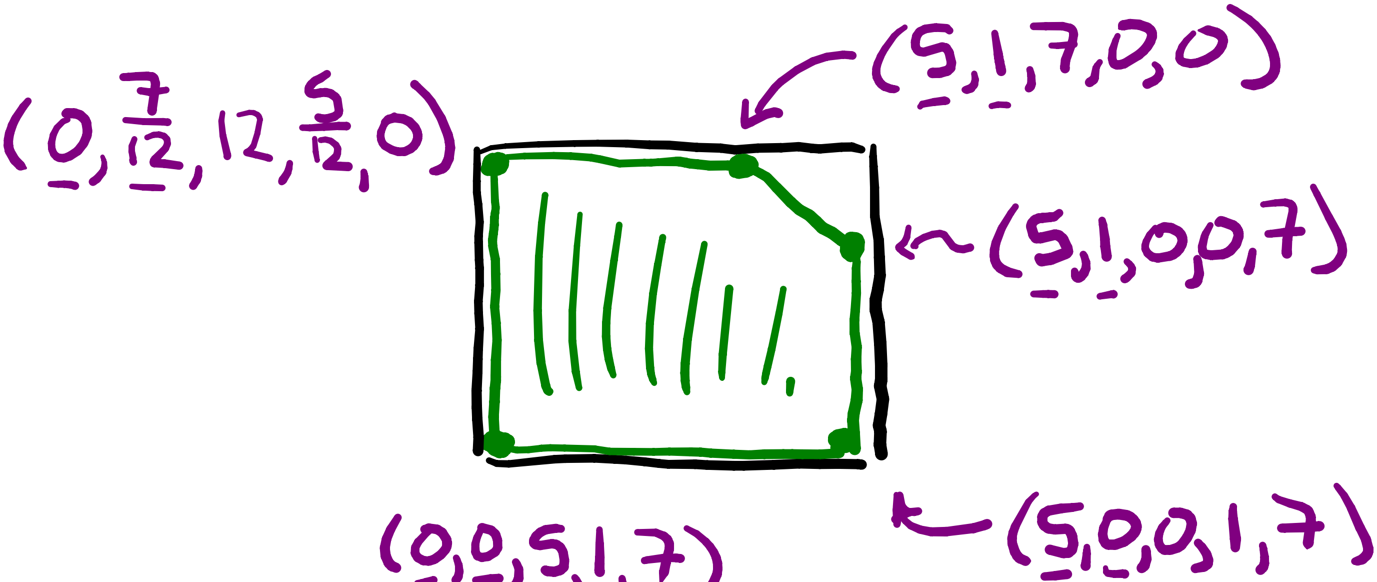
Thm (Armstrong-H., 2011): For each $u \in W$, $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1, \dots, i_d), u)$.

<u>Thm</u> (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\dots,i_d),u)$, for f induced by $f_{(i_1,\dots,i_d)}$.

e.g. $f^{-1}_{(1,2,1,2,1)}(M)$ for $M \in Y^{o}_{s_1 s_2 s_1}$

$$M = x_1(5)\,x_2(1)\,x_1(7)$$

$$f^{-1}_{(1,2,1,2,1)}(M) =$$



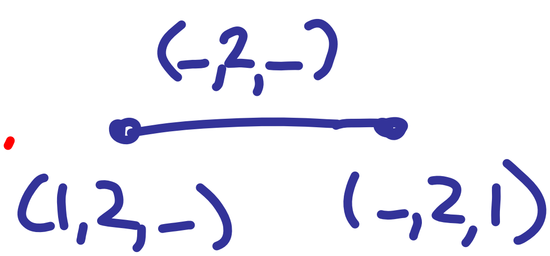
Subword Complexes

- introduced by Allen Knutson & Ezra Miller to serve as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.

$\Delta(\underset{\sim}{Q}, \underset{\sim}{w})$ = simplicial complex of subwords Q' of Q s.t. $Q \setminus Q'$ contains a reduced word for w.

Q: reduced or nonreduced word

w: Coxeter group element

e.g. 

$= \Delta((1,2,1), s_1)$

__Thm__ (Knutson-Miller): $\Delta(Q,\omega)$ is shellable.

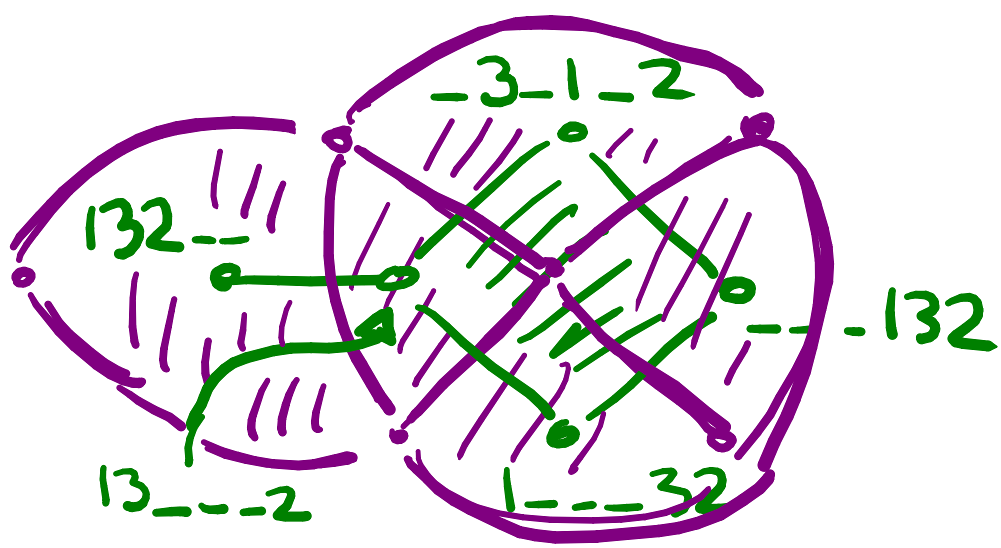
__Thm__ (Knutson-Miller): $\Delta(Q,\omega)$ is homeomorphic to ball or sphere.

__Example__ __of__ __Subword__ __Complex__ & __its__ __Interior__ __Dual__ __Block__ __Complex__

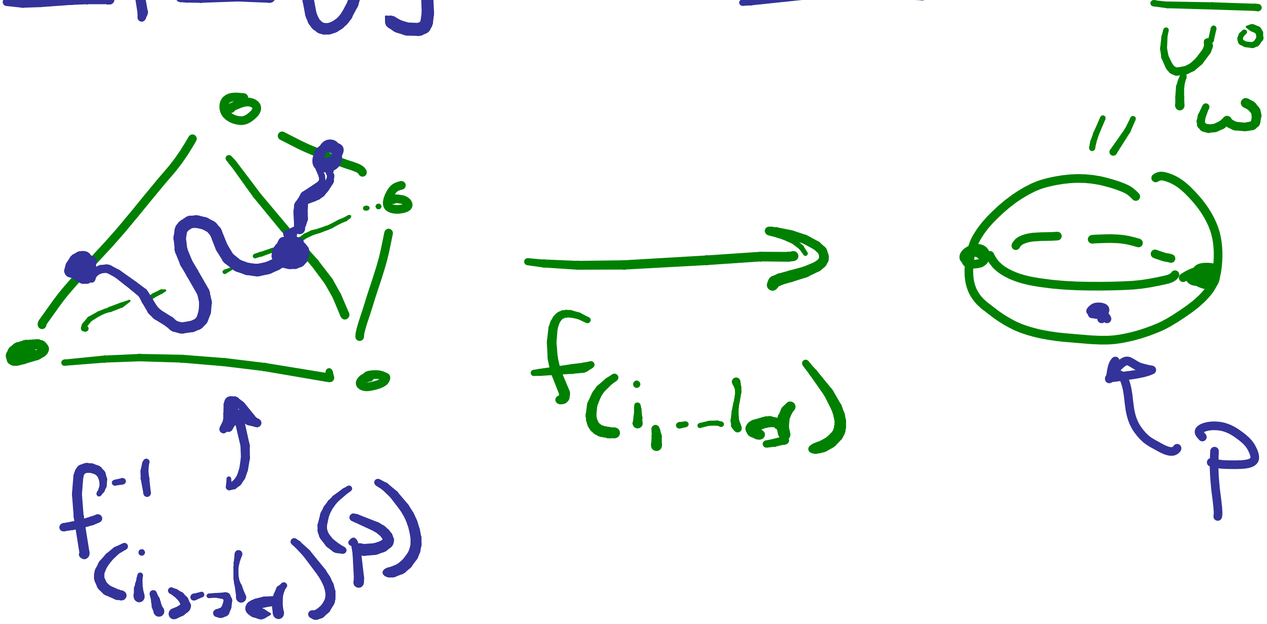
$\Delta(Q,\omega) =$

$Q = 132132$

$\omega = s_1 s_3 s_2$



Topology of Fibers



Thm (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex Δ_{d-1}.

Proof: Parametrization + continuity lemmas

Parametrization of Open Cells

e.g. $\delta(3,1,2,1,2,1,2,1) = s_3 s_1 s_2 s_1 = \omega$

$$f^{-1}_{(3,1,2,1,2,1,2,1)}(P) \supseteq Y^\circ_\omega$$

rightmost subword for ω

$$x_3(t_1)x_1(t_2)x_2(t_3)x_1(t_4)x_2(t_5)x_1(t_6)x_2(t_7)x_1(t_8)$$

"free" parameters "dependant" parameters

Thm (Davis-H-Miller):

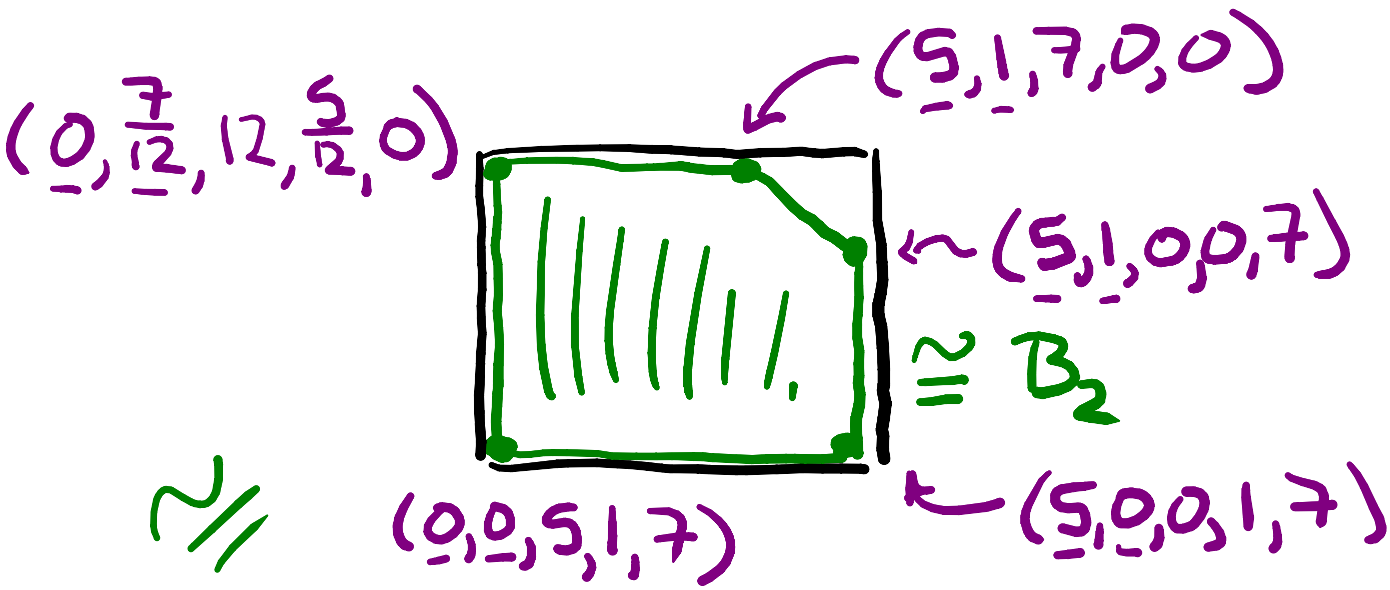
$$[0,1)^{\dim(\sigma)} \cong \bigcup_{v \le \bar\tau \le \bar\sigma} \tau$$

v source vertex of $\bar\sigma$

Examples of Fibers:

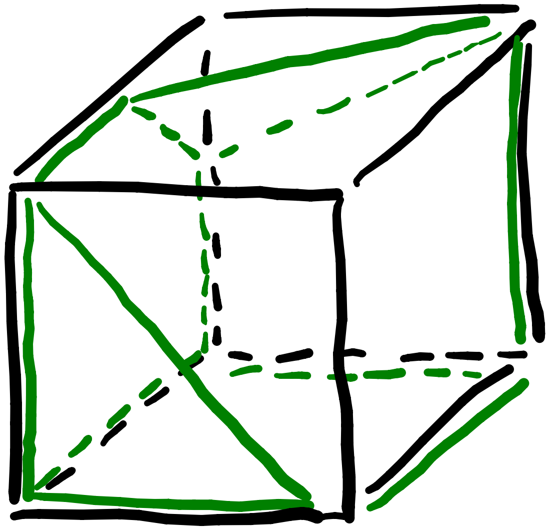
(from vantage point of cell parametrization)

e.g.



$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = x_1(5)x_2(1)x_1(7)$

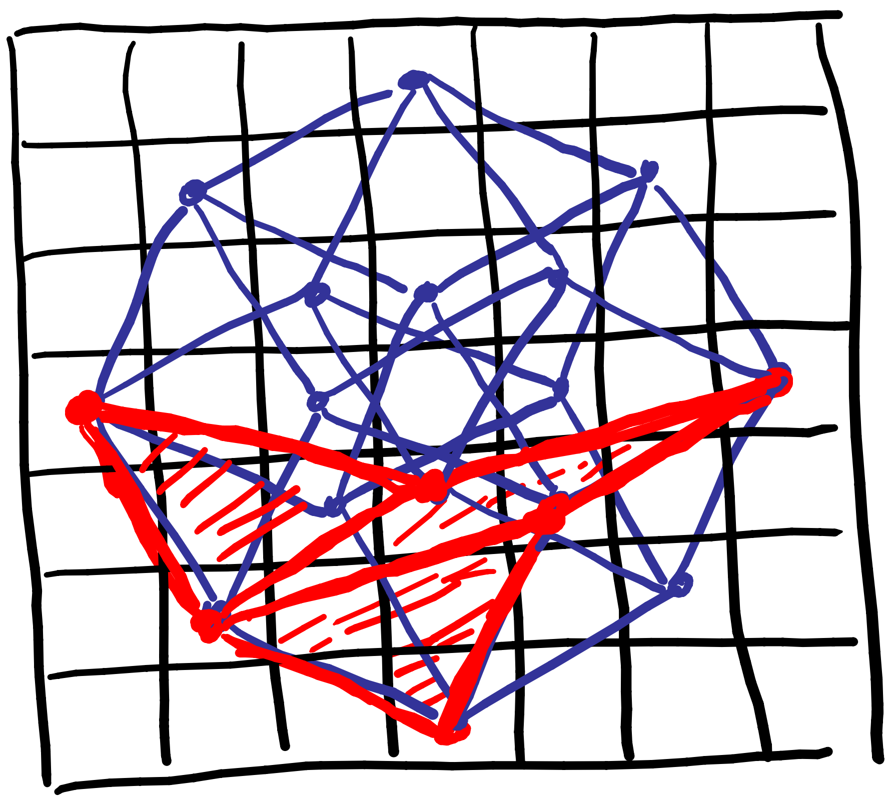
$\in U^{0}_{s_1 s_2 s_1}$



$\cong B_3$

$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{s_1 s_2 s_1}$



$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{(1,2)}$

Thm (DHM, 2018): Interior dual block complex of $\Delta((i_1, \ldots, i_d), u)$ is collapsible, hence contractible.

Pf: Discrete Morse theory

Observation: DHM Conjecture $\Rightarrow$ $f^{-1}_{(i_1, \ldots, i_d)}(M)$ is contractible.

Idea: DHM Conjecture says $f^{-1}_{(i_1, \ldots, i_d)}(p) \cong$ interior dual block complex of $\Delta((i_1, \ldots, i_d), u)$ for $p \in Y_u^\circ$, which is contractible.

Thm (DHM): DHM Conjecture would imply new proof of FS-Conjecture.

Idea: Use Topological Relationship Fibers to Image: Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z whose restriction to $\mathrm{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B \cong S^n$

(2) $g(\partial B) \cap g(\mathrm{int}(B)) = \emptyset$

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

Then $Z \cong B$.

- Shellability of Bruhat order $\Rightarrow$ $\partial B \cong S^n$ requirement by induction on dimension.

Maps/Spaces with Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $Gr_{\geq 0}(k,n) = (GL_n/P)_{\geq 0}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $Gr_{\geq 0}(k,n)$ + theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galashin-Karp-Lam 2017 preprint: $Gr_{\geq}(k,n)$ is homeom. to closed ball

2. Totally nonneg. real part of Flag variety: $\mathcal{F}\ell_{\geq 0} = (GL/B)_{\geq 0}$

Rietsch: poset of closure reln's. Cells $R^{\circ}_{u,v}$ given by $u \leq v$ in Bruhat order

Marsh-Rietsch: parametrization for $R^{\circ}_{u,v}$

Williams: poset is CW poset

Rietsch-Williams: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

Galashin-Karp-Lam 2018 preprint:
Homeomorphism type for total space.

3. Map to Stratified Space E_n of Electrical Networks

(Curtis-Ingerman-Morrow, Kenyon-Wilson, ...)

- arises as image of:

Resp :

$\longmapsto$ response matrix

Lam:

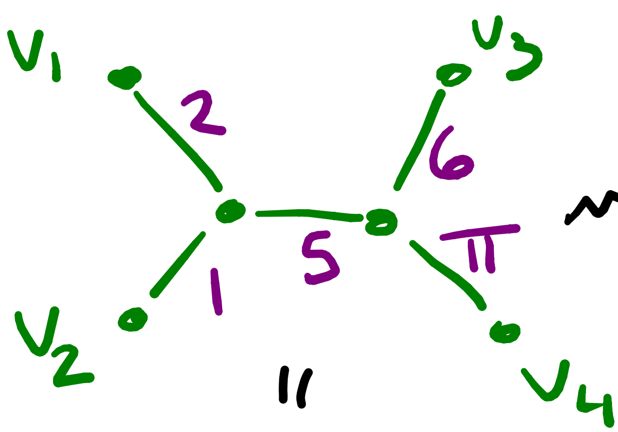
- $F(E_n)$ is "Eulerian", i.e. $\mu(u,v) = (-1)^{rk(v)-rk(u)} = \tilde{\chi}(u,v)$

H-Kenyon:

- $F(E_n)$ shelling & hence $F(E_n)$ is a CW poset (i.e. face poset of regular CW complex)

H-Kenyon Corollary: shelling for each $[u,v]$ in face poset for "edge product space of phylogenetic trees", hence face poset is CW poset.

(builds upon Gill-Linusson-Moulton-Steel)



$\rightsquigarrow \left(\underbrace{3}_{d_{12}}, \underbrace{\overset{2+5+6}{13}}_{d_{13}}, \ldots \right)$

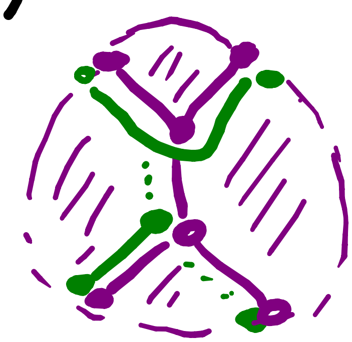
tree with edge lengths

vector of leaf pair distances

Some Further Questions

1. Subword complex analogues for fibers of other maps?

   e.g. H-Kenyon: $\Delta(G, \mathrm{elec}(H))$ for planar electrical networks (for H a minor of G)



2. Fiber Structure for: electrical networks, $(GL_n)_{\geq 0}$, $(\mathcal{F}\ell_n)_{\geq 0}$, $(GL_n/P)_{\geq 0}$, ...?

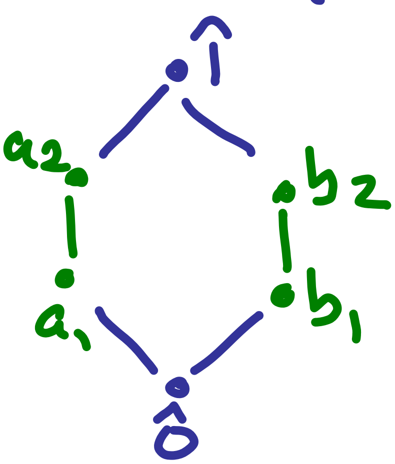
3. Proof of DHM Conjecture?

Thanks!

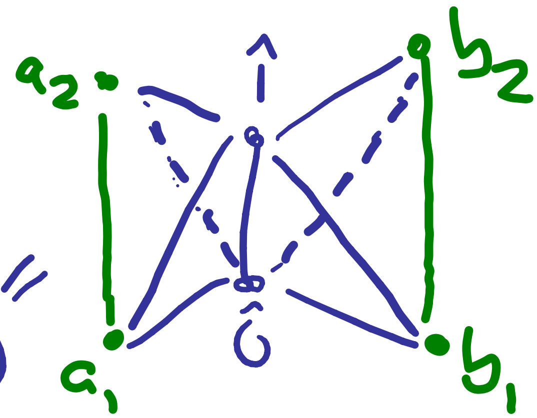
Appendix: Further Remarks & Details

Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \ldots < v_i$ in P.

e.g. $P =$



$\Delta(P) =$



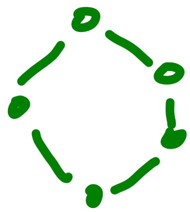
Thm (Hall; Popularized by Rota):

$$\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{(u,v)}_{\text{subposet } \{z \in P \mid u < z < v\}}))$$

- μ generalizes # theory Möbius fn
- a starting point for topol. combinatorics

Necessary Combinatorial Condition for Regular CW Complex

- CW poset is "graded" poset w/ $\hat{0}$ s.t. $\Delta(\hat{0},u) \cong S^{rk(u)-2}$ for all u

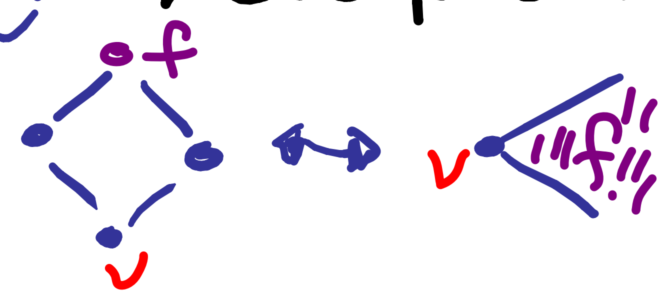


not graded

Thm (Björner): CW poset $\iff$ face poset of regular CW complex

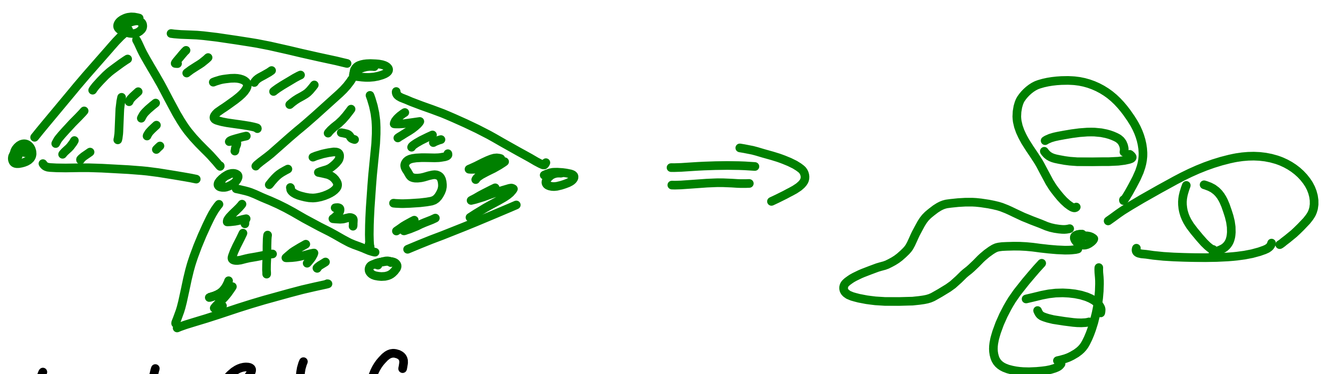
Thm (essentially Danaraj & Klee):

"Shellable" + "thin" $\Rightarrow$ CW poset
+
"Graded"



e.g. Bruhat order (Björner-Wachs; Matthew Dyer)

- Shellability is condition to build simplicial complex that is homotopy equiv. to wedge of spheres & count the spheres



- helpful for calculating poset "Möbius function" μ for counting by inclusion/exclusion via $\mu(u,v) = \tilde{\chi}(\Delta(u,v)) = \pm \#$ spheres
- generalizes μ from # theory
- a starting point for topological combinatorics, particularly "poset topology"

Bruhat Order as Face Poset

- L = Schubert cell decomposition of flag variety $\mathcal{F}l_n = GL_n/B$ & cell closures, i.e. "Schubert varieties" (over $\mathbb{C}$) & more general G/B
- $F(L)$ = Bruhat order

e.g. $\overline{\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}} \supseteq \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\}$

$\mathbb{R}^6 = \mathbb{C}^3$ cell indexed by $1432 \in S_4$

1342-indexed cell $\cong \mathbb{C}^2 = \mathbb{R}^4$

flags $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ * \\ * \\ 1 \end{pmatrix} \right\rangle \subseteq \ldots$

Remark: not a regular CW complex

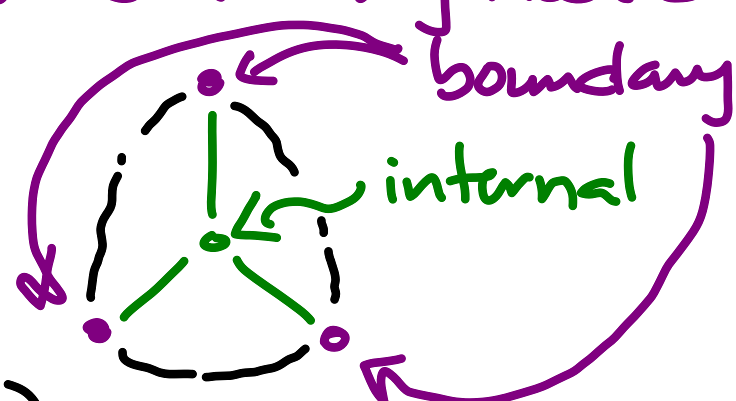
Maps Arising from Electrical Networks (& Face Poset for Map Image)

$$\underbrace{\Delta}_{\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}} \begin{pmatrix} v_N \\ v_I \end{pmatrix} = \begin{pmatrix} c_N \\ 0 \end{pmatrix}$$

↑ voltages ↑ currents

I = internal nodes

N = boundary nodes



$$\underbrace{(A - BC^{-1}B^T)}\, v_N = c_N$$

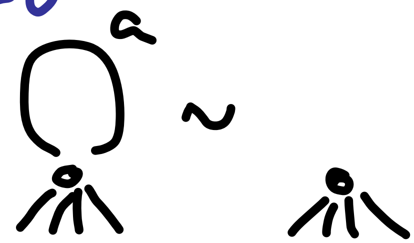
"response matrix" whose entries are nonneg. rat'l fns of conductances

(: input bdry voltages & output bdry currents)

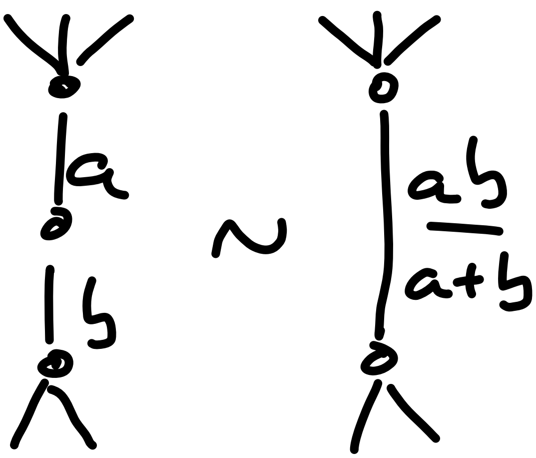
Fibers of "Resp" Map via "Electrical Equivalence"

(1)

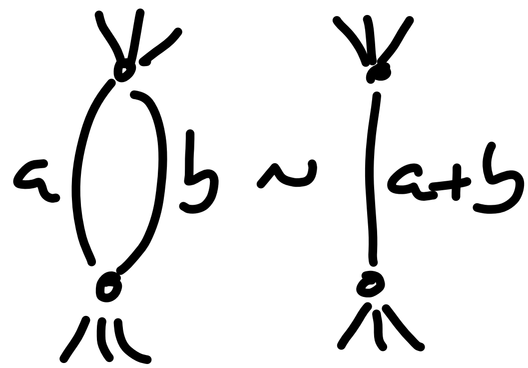
(2)



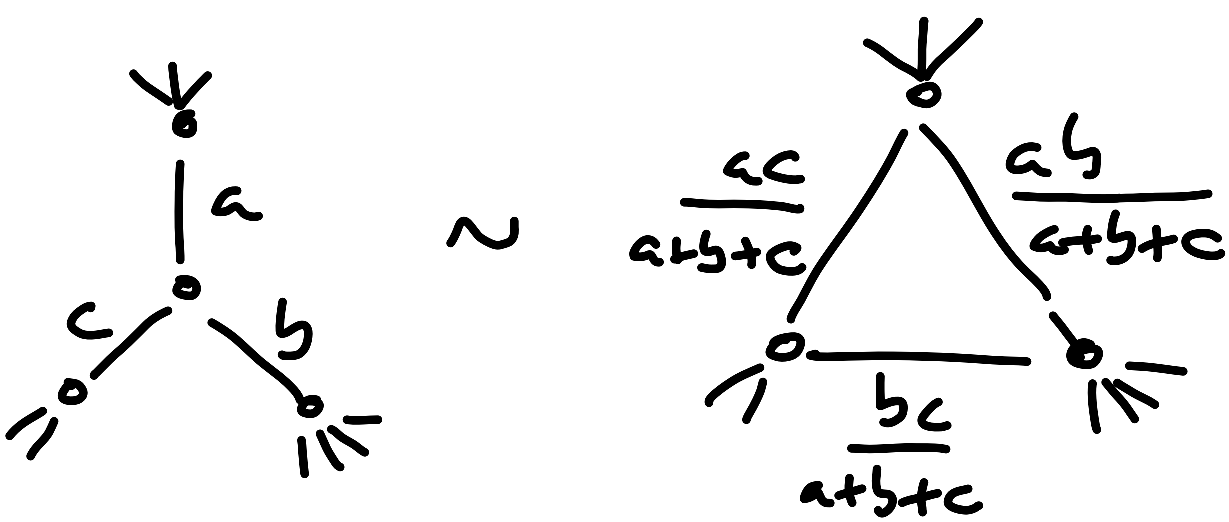
(3)



(4)



(5) "Y-Δ moves"



An Ongoing Goal: Given a graph G, study the space of response matrices as image of

$$f : \left\{ \begin{matrix} \text{conductance} \\ \text{vectors} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{response} \\ \text{matrices} \end{matrix} \right\}$$

$$\overset{\parallel}{\left(\mathbb{R}_{\geq 0} \cup \{\infty\}\right)^{|E|}}$$

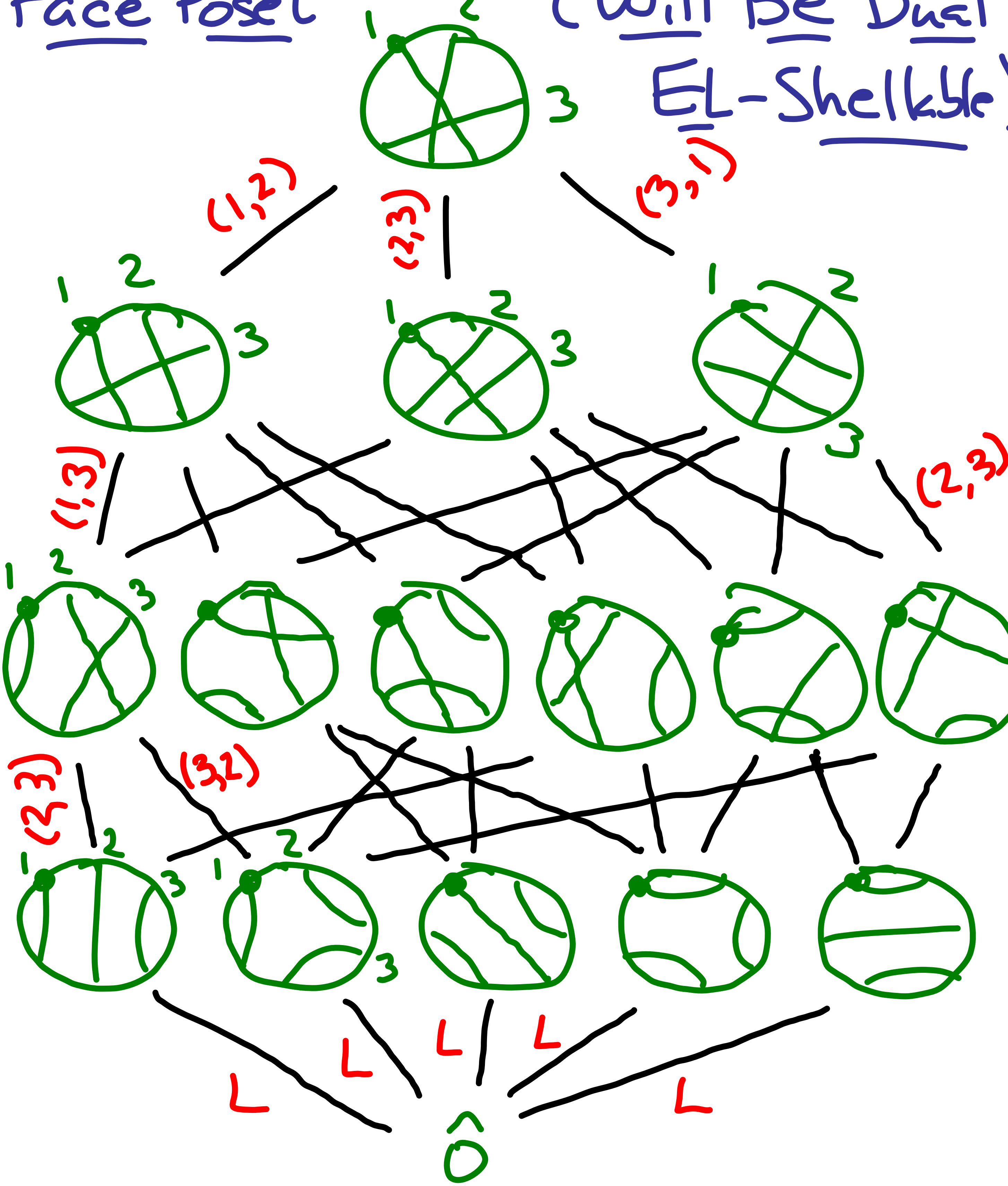
Note:

contracting an edge $\rightsquigarrow$ sending conductance to ∞ (i.e. resistance to 0)

deleting an edge $\rightsquigarrow$ sending conductance to 0 (resistance to ∞)

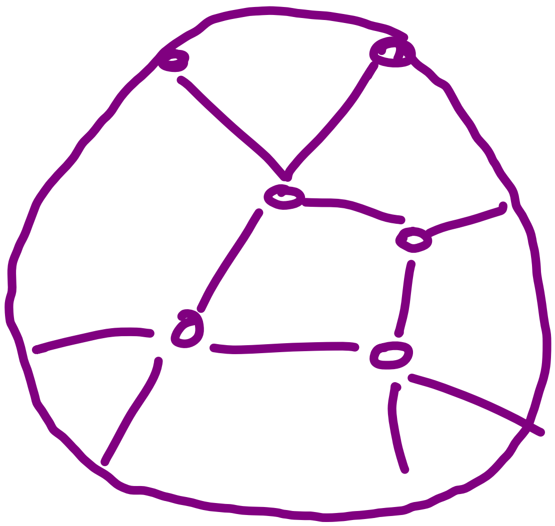
A Related Goal: Study fibers of f

Face Poset (Will Be Dual EL-Shellable)

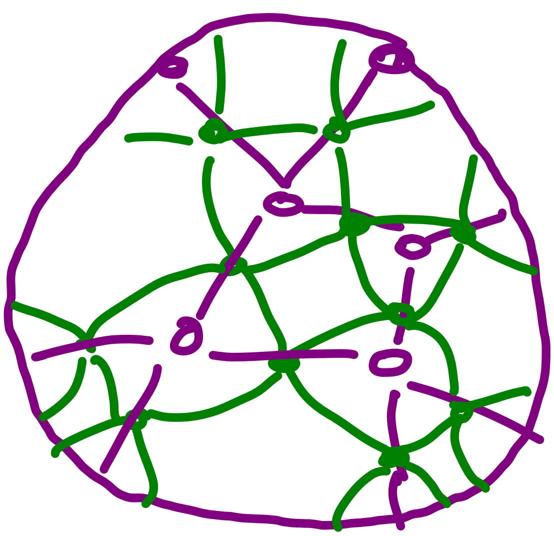


Correspondence: From Graphs to Uncrossing Wire Diagrams "Medial Graphs"

$\Gamma =$

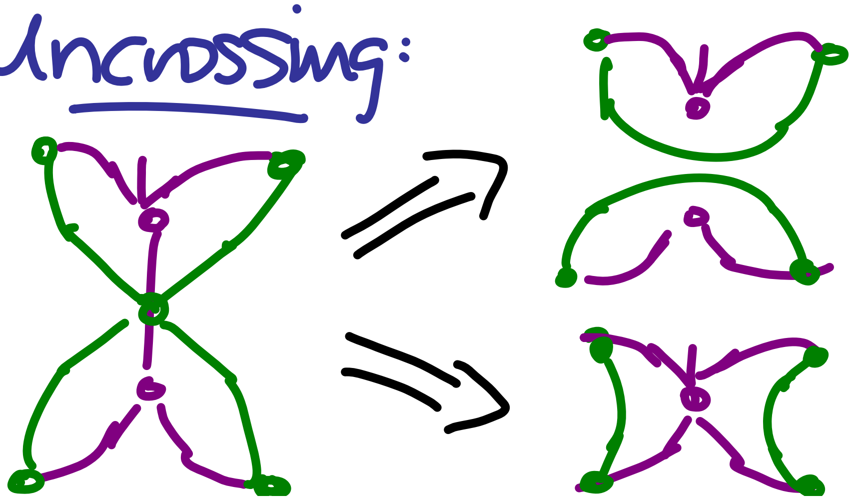


$G(\Gamma) =$



= wire diagram

Uncrossing:

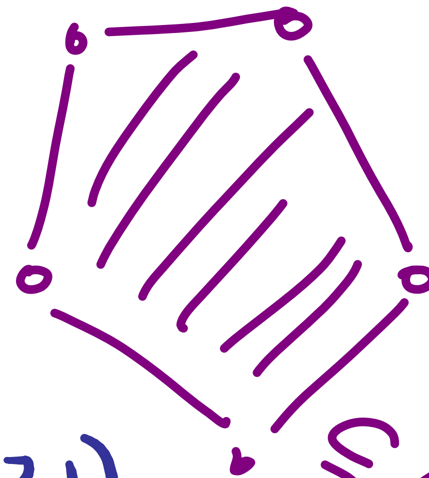


deletion

contraction

Interesting Examples of Fibers

$$f^{-1}_{(1,2,1,2,1)}\begin{pmatrix}1 & 2/3 & 1/18\\ 0 & 1 & 1/3\\ 0 & 0 & 1\end{pmatrix} \cong$$



$$\subseteq \Delta_4$$

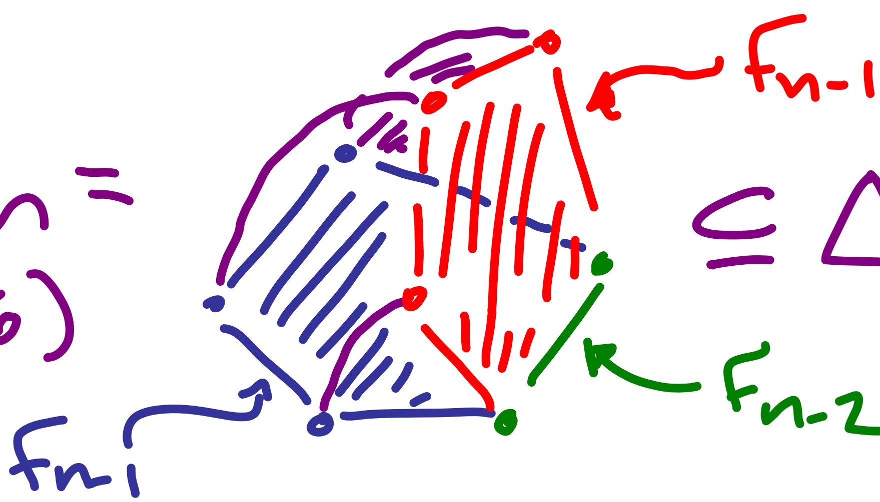
$$\cong$$

$$f^{-1}_{(1,2,1,2,1)}(M) \quad \forall M \in Y^{\circ}_{(1,2,1)}$$

$$f^{-1}_{(1,\underline{\overline{2}},\underline{\overline{1}},\underline{\overline{2}},\underline{\overline{1}},\overline{2})}\begin{pmatrix}1 & 2/3 & 1/18\\ 0 & 1 & 1/3\\ 0 & 0 & 1\end{pmatrix} \cong f^{-1}_{(1,2,1,2,1,2)}(M)$$

$$\cong \qquad \forall M \in Y^{\circ}_{(1,2,1)}$$

$f_n =$ (n=6)



$$\subseteq \Delta_5$$

A Motivation Stated by Fomin & M. Shapiro

"It should be mentioned that one of our 'hidden motivations' has been the desire to better understand the combinatorics of Kazhdan-Lusztig polynomials".

Their Results: face poset, homological results, fascinating projection map & much more!

4. $\mathrm{im}(f_{(i_1,\ldots,i_d)})$ from this Viewpoint

(Deconed) Totally nonneg. real part unipotent radical of Borel in semisimple simply connected algebraic group

(a) Vertex "link" (nbhd) in $(\mathcal{F}\ell_n)_{\geq 0}$ via Marsh-Rietsch parameters $\to 0$

(b) Image$(f_{(i_1,\ldots,i_d)})$ for $w_0 = w(i_1,\ldots,i_d)$

(c) Bruhat order as poset of closure rel'ns

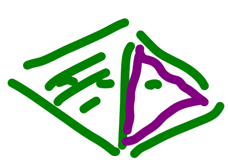
(d) Each cell closure homeom. to closed ball.

– "Fomin-Shapiro Conjecture", proved in H., 2014, Invent.

Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \dots, F_k$, a **shelling**, on facets s.t. $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$ is pure, codimension one subcomplex of $\overline{F_j}$ for each $j>1$ (hence is $\partial \overline{F_j}$ or has a cone point).

OR



- Each facet attachment preserves homotopy type or closes off a new sphere

Ingredients in this Relationship Between fibers & Image

- "CE-Approximation Theorem (Kirby-Siebenmann (all dim's); Quinn (dim 4); Armentrout (dim 3)
  - $g: \partial B \to \partial B$ as above may approximated by homeomorphisms

- Local Contractibility of $\mathrm{Homeo}^+(S^n, S^n)$
- any two homeomorphisms "close enough" to each other connected by path of homeomorphisms

Topology of Fibers: Key Lemmas

Defn (Davis-H-Miller): The letter x_{i_1} is **redundant** in $x_{i_1}\cdots x_{i_d}$ if $\delta(i_1,\ldots,i_d)=\delta(i_2,\ldots,i_d)$

Lemma (Davis-H-Miller): x_{i_1} is non-redundant in $x_{i_1}\cdots x_{i_d}$

$\Leftrightarrow$ $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for $p\in Y^{o}_{\delta(i_1,\ldots,i_d)}$

$$= \{(t_1,\ldots,t_d)\mid x_{i_1}(t_1)\cdots x_{i_d}(t_d)=p\}$$

has unique value k_1 for t_1.

$\Leftrightarrow$ $f^{-1}_{(i_1,\ldots,i_d)}(p)\cong f^{-1}_{(i_2,\ldots,i_d)}(x_{i_1}(-k_1)p)$

Lemma (Davis-H-Miller): Given $(t_1,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)$ with $t_1 > 0$ and x_{i_1} redundant, then $\exists (t_1',\dots,t_d') \in f^{-1}_{(i_1,\dots,i_d)}(p)$ for every $t_1' \in [0, t_1]$.

e.g. $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & 14 \\ & & 1 \end{pmatrix}$ then $f^{-1}_{(1,2,1,2)}(M)$

$$= \{(t_1,t_2,t_3,t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$$

achieves every $t_1 \in [0, \frac{e}{14}]$.

Proof of Fomin-Shapiro Conjecture

(H., 2014, Invent.)

Set-up: Surjective fn $f_{(i_1,\dots,i_d)}: \Delta_{d-1} \to Y$ s.t. $f_{(i_1,\dots,i_d)}\big|_{\mathrm{int}(\Delta_{d-1})}$ homeom. to $\mathrm{int}(Y)$.

Step 1: Perform "cell collapses" on $\partial(\Delta_{d-1})$ designed to preserve homeom. type & regularity yielding $(\Delta_{d-1})/\sim$ s.t.

$$x_1 \sim x_2 \implies f_{(i_1,\dots,i_d)}(x_1) = f_{(i_1,\dots,i_d)}(x_2)$$

(eliminating faces given by nonreduced words)

Step 2: Prove $\bar{f}_{(i_1,\dots,i_d)}: (\Delta_{d-1})/\sim \;\to Y$ is a homeomorphism via new regularity criterion for finite CW complexes.

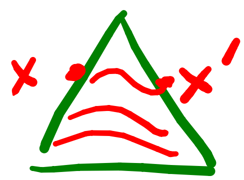
Corollary of Proof: Contractibility of Fibers

(Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

$$K_0 \xrightarrow{g_1} K_1 \xrightarrow{g_2} K_2 \to \cdots \to K_i$$

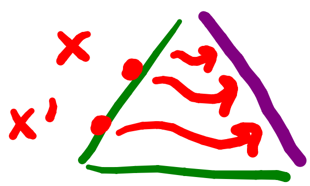
$K_0 = \Delta$ (polytope!), $\quad K_1 \cong \Delta/\sim_1$ "$=\Delta$", $\quad K_2 \cong \Delta/\sim_2$

- collapse face in K_i across images of parallel line segments in K_0 satisfying:
  - Distinct endpoints condition (DE):



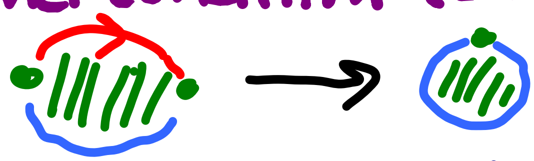
$x \neq x'$

  - Distinct initial points condition (DIP):



$x \neq x'$

  - Least upper bound condition (LUB)

to preclude 

(conditions checkable via 0-Hecke algebra)

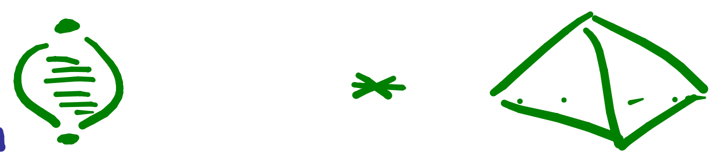
Additional Key Challenges

1. 0-Hecke algebra lacks inverses & lacks cancellation law.

   Key Idea: find ways to transfer properties from Coxeter gp

2. Need change-of-coords for braid moves as homeom's on closed cells

   Key Idea: induction by embedding smaller instance



$$(\underbrace{i_1, i_2, i_3, \ldots\ldots i_{d'-1}}_{\text{all possible collapses done}}, \underbrace{i_{d'}, \ldots, i_d}_{\text{not in any collapses yet}})$$

$\mathrm{im}(f_{(i_1 \ldots i_{d'-1})})$

3. Need maps to extend to full complex

4. Tricky combinatorics to verify LUB at each collapsing step.