Combinatorics & Topology of Totally Nonnegative Spaces

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(partly joint work with Jim Davis & Ezra Miller & with Rick Kenyon)

(slides available at: https://plhersh.math.ncsu.edu/ND18.pdf)

Topological Aspects of Total Positivity

- Lusztig, Fomin, ... initiated study of: totally nonnegative, real part of spaces of matrices, spaces of flags (i.e. GL_n/B)...
(i.e. having all minors nonnegative)
- Conjecturably / provably homeomorphic to closed balls
- Proving this by studying fibers (method used in H. & in DHM)
  - imposes restrictions on reln's amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases

Today:

1. Review: What is known about these spaces which all arise as images of "nice" maps
2. Fomin-Shapiro Conj (2014 Proof)
   $\text{im}(f_{(i_1,\dots,i_d)})$ is regular CW ball
3. New Result Regarding Fibers of $f_{(i_1,\dots,i_d)}$, including:
   (a) cell decomposition
   (b) combinatorial model

(Recall:

)

Background on Coxeter Groups

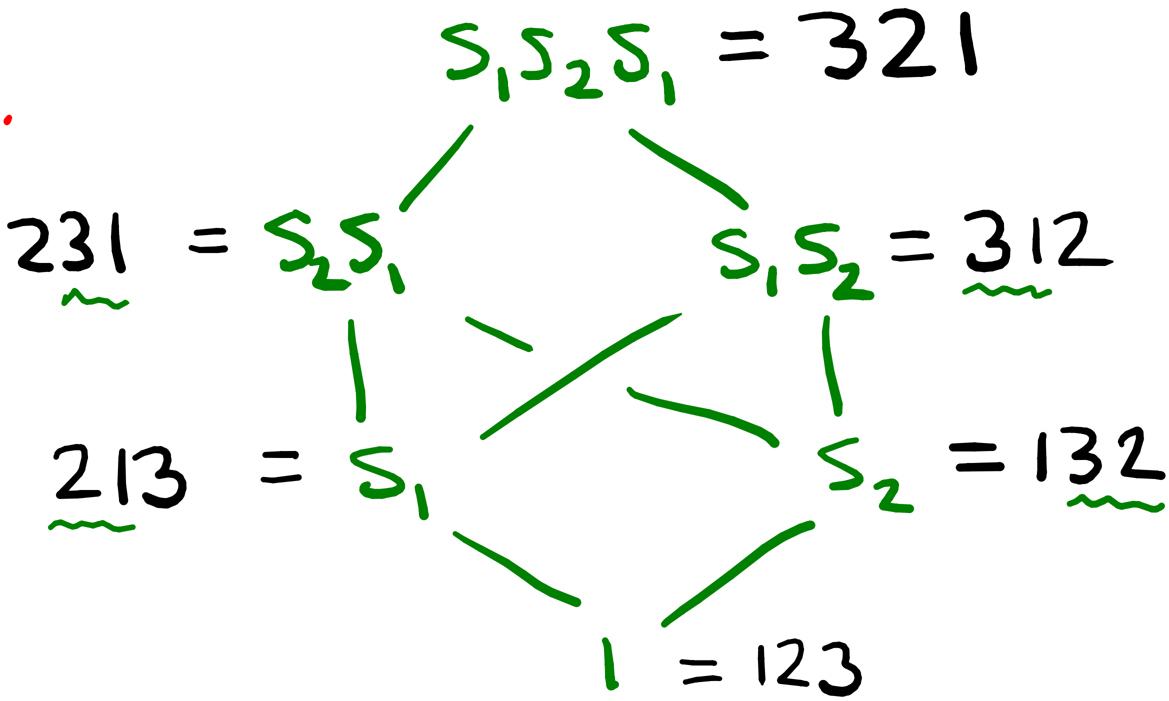
- $s_i := \underbrace{(i, i+1)}_{\text{type A (i.e. } W = S_n)}$ a simple reflection in W
- $s_{i_1} \cdots s_{i_d}$ is reduced expression for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for d as small as possible
- length of w, denoted $\ell(w)$, is this smallest d.

e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

$$s_1 s_2 s_1 = 321 \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123$$

- $(i_1,\ldots,i_d)$ is **reduced word**
- **Bruhat order**: partial order

e.g.



on W (e.g. S_n) with $u \le v$ for $u, v \in W \iff$ any reduced word for v has subword that is reduced word for u.

Running Example: Totally Nonnegative Part of a Space of Matrices

column $i+1$

- $x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ row i

$\quad \| \qquad\qquad\qquad\qquad$ (type A)

exp(te_i)

(general finite type, expon'd Chevalley generator)

- $f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$

reduced word

$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$

e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

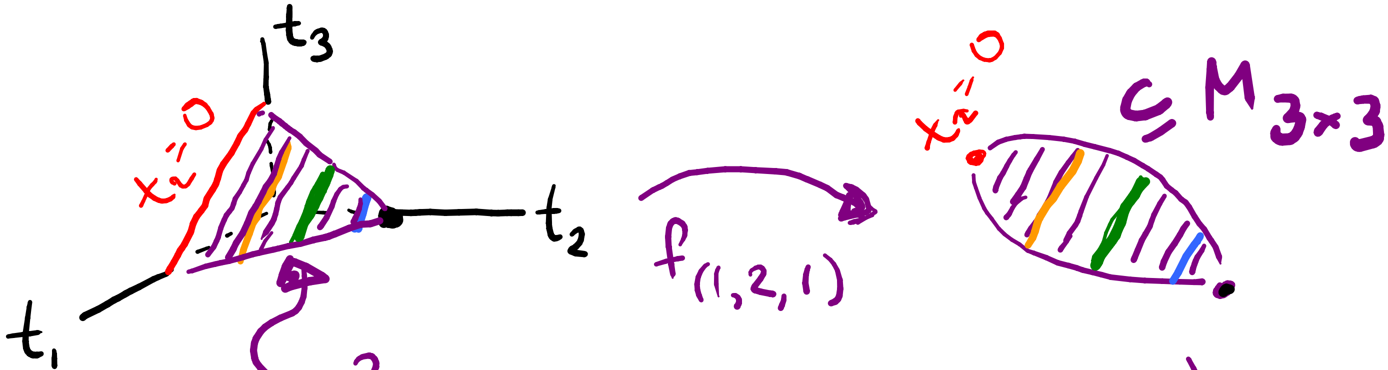
e.g.

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

w_0 case:

$$\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{matrix}\text{tot.} \\ \text{nonneg}\end{matrix} \right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\dots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = \underset{\text{positive constant}}{K} \text{ hyperplane})$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow t_2 = 0$$

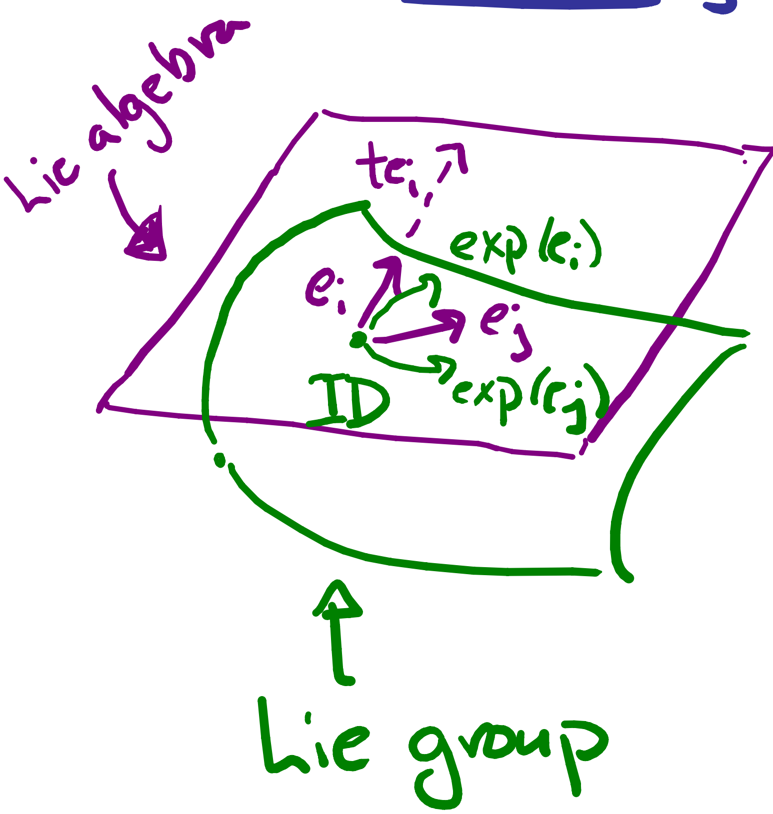
$$f_{(1,2,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)} \cdot \underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t>0\} = \{x_1(t_1)x_1(t_2) \mid t_1,t_2>0\}$

A Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators



$$te_1 = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \exp(\_)$$

$$\begin{pmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{pmatrix}$$

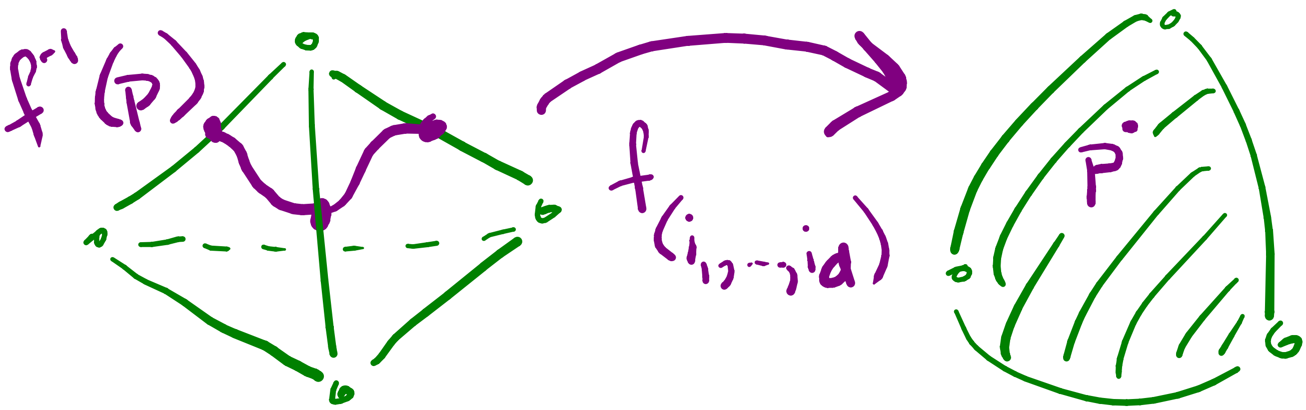
$$\exp(t_1 e_i)\exp(t_2 e_j)$$

$$f_{(i,j)}(t_1,t_2) = x_i(t_1)\, x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{t^2 \frac{e_i^2}{2}}_{0=} + \underbrace{t^3 \frac{e_i^3}{6}}_{0=} + \dots$$

Reln's $\Longleftrightarrow$ Elts in same fiber of $f_{(i_1 \dots i_d)}$

Conjecture (Davis-H-Miller):
$f^{-1}_{(i_1,\ldots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\ldots,i_d),w)$ for $p \in Y^{o}_{w}$.



Thm (H. 2014): $\mathrm{im}(f_{(i_1,\ldots,i_d)})$ is regular CW complex $\cong$ closed ball.

Thm (DHM): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ has cell decomposition & correct face poset.

Subword Complexes

(introduced by Allan Knutson & Ezra Miller)

$\Delta(\underset{\sim}{Q}, \underset{\sim}{w})$ = (abstract) simplicial complex whose faces are subwords Q' of Q whose complement $Q \setminus Q'$ contains a reduced word for w

Q: reduced or nonreduced word

w: Coxeter group element

Aside:

1st arose as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.

Thm (Knutson-Miller): $\Delta(Q,\omega)$ is homeomorphic to ball or sphere.

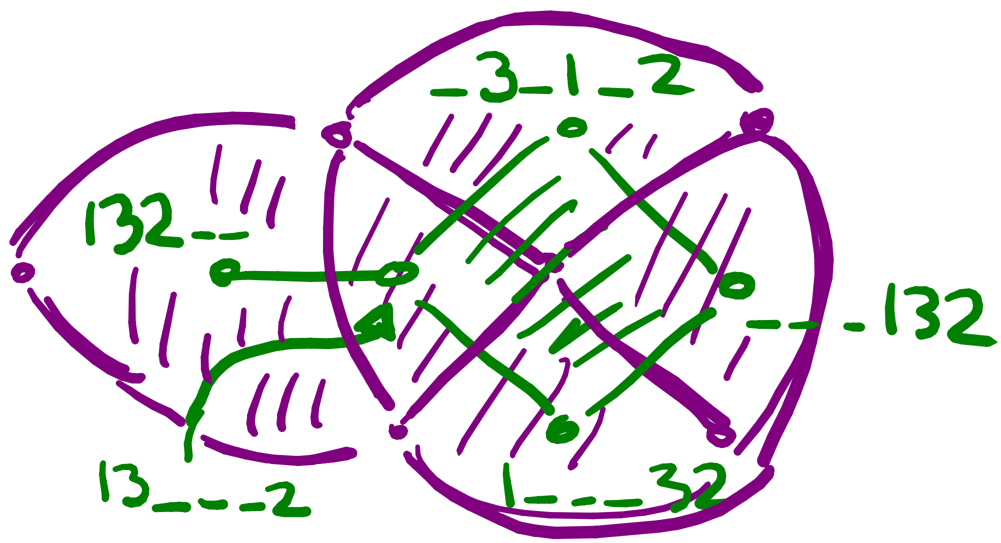
Example of Subword Complex & its Interior Dual Block Complex

$\Delta(Q,\omega) =$

$Q = 132132$

$\omega = s_1 s_3 s_2$

& its



Interior dual block complex

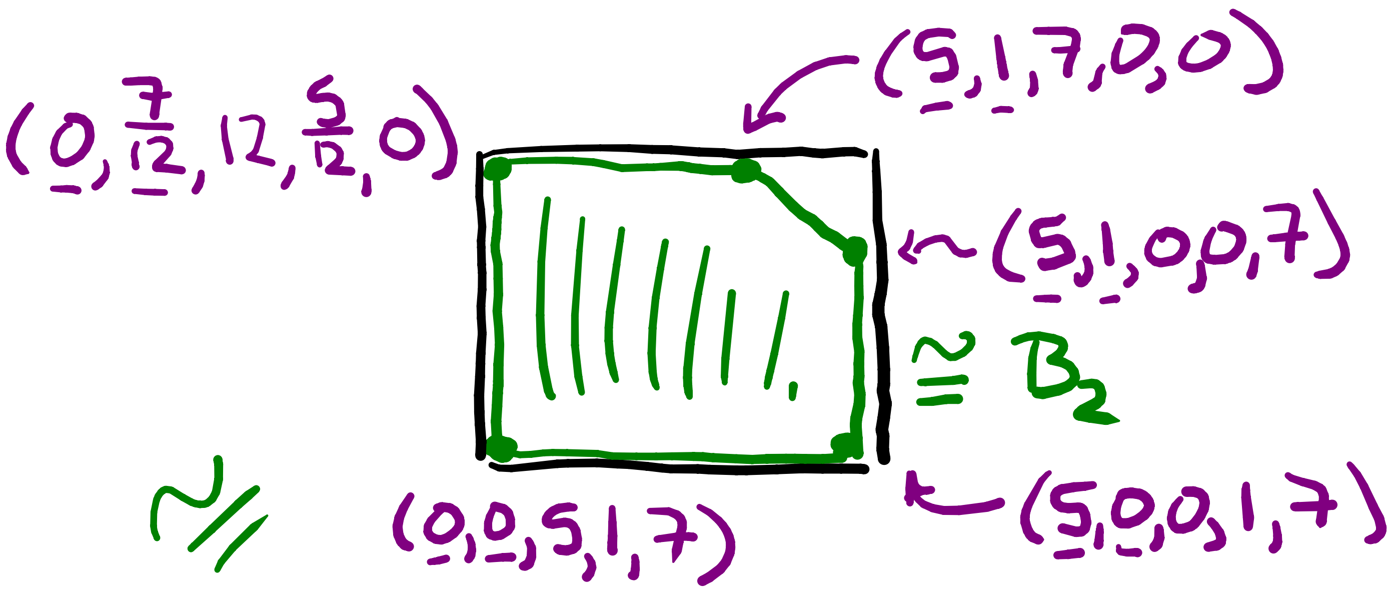
$$\parallel$$

$f^{-1}_{(1,3,2,1,3,2)}(M)$ for $M \in Y^{\circ}_{(1,3,2)}$

Examples of Fibers:

(with realizations as suggested by various results towards proof of DHM conjecture)

e.g.

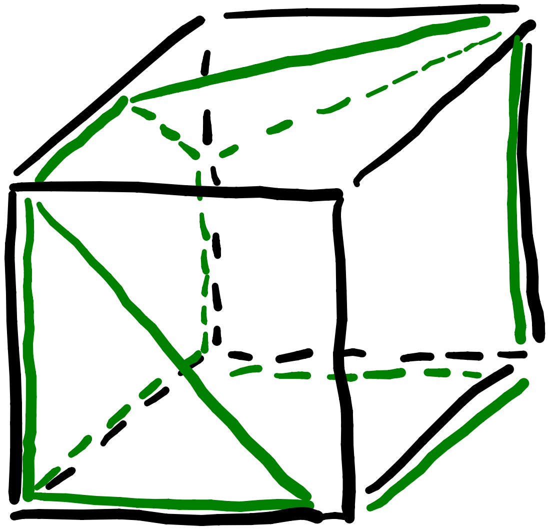


$\cong$

$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = x_1(5)x_2(1)x_1(7)$

$\cap$

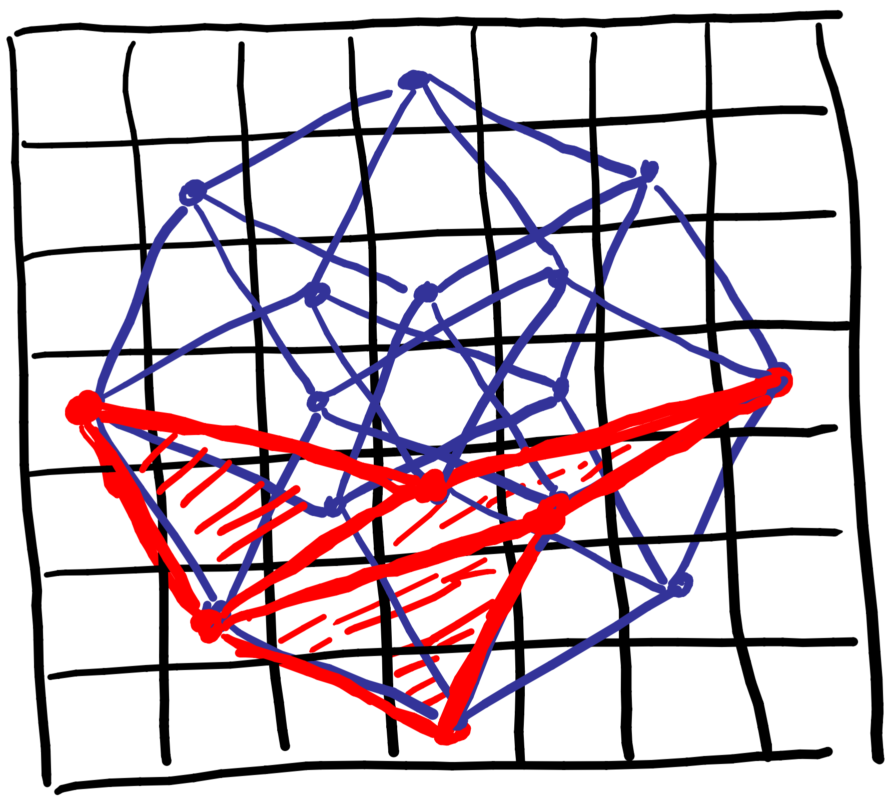
$U^{0}_{s_1 s_2 s_1}$



$\cong B_3$

$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{s_1 s_2 s_1}$

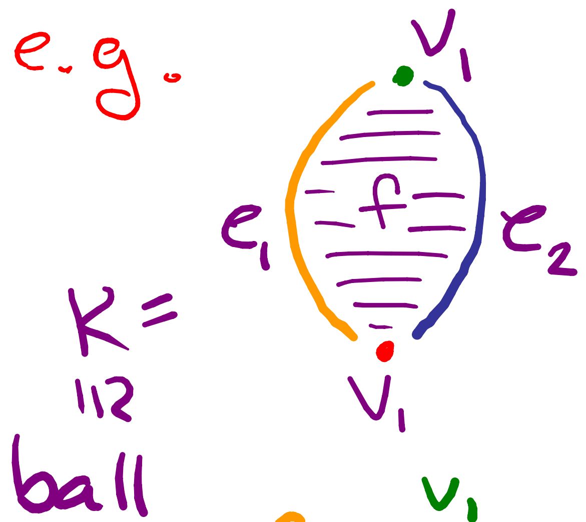


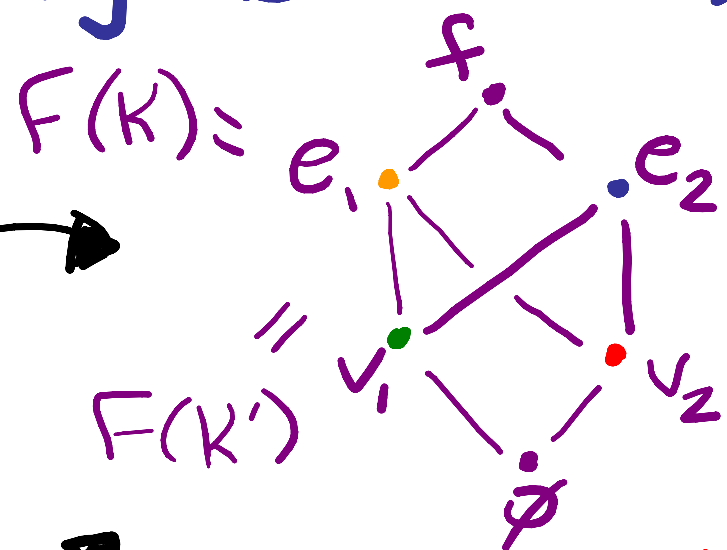
$\cong$

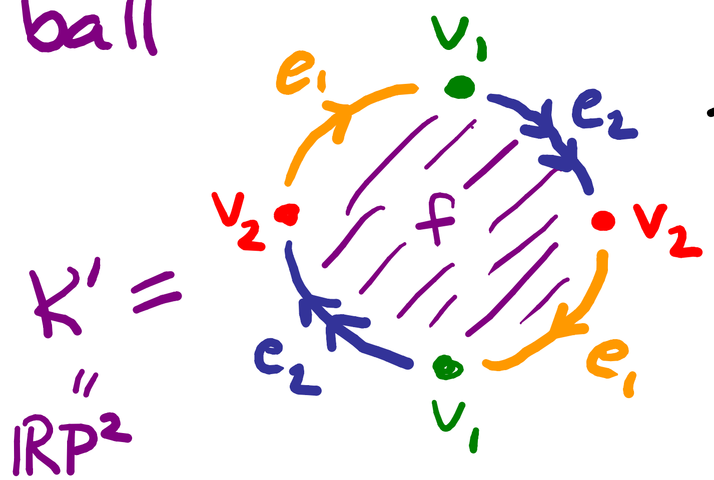
$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{(1,2)}$

CW Complexes & their Face Posets (Partially Ordered Sets)

e.g.







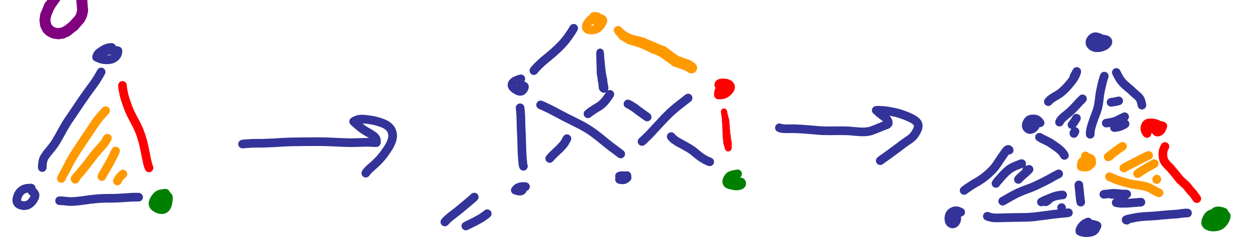
"face poset"

$(u \le v \iff u \le \bar{v})$

Recall: A **CW complex**: cells $e_\alpha \cong \mathbb{R}^{\dim(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \subseteq \bar{\alpha}} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

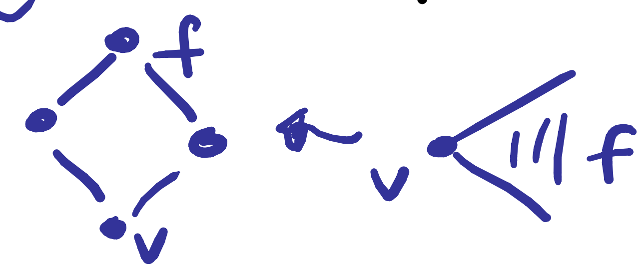
Recall: CW complex is regular if each f_α is homeomorphism.

- $\Delta(P)$ = "nerve" or "order complex" of P
- K regular $\Rightarrow K \cong \Delta(F(K) \setminus \emptyset) = sd(K)$

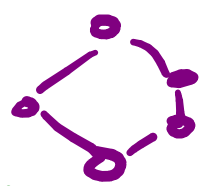


K = $F(K) \setminus \emptyset$ = $\underbrace{\Delta}_{\text{"nerve"}}(F(K) \setminus \emptyset)$

- a CW poset is any face poset of regular CW complex
- "Shellable" + "Graded" + "thin" $\Rightarrow$ CW poset



e.g. not



e.g. Bruhat order (Björner-Wachs; Matthew Dyer)

Totally Nonnegative Spaces w/ Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $Gr_{\geq 0}(k,n) = (GL_n/P)_{\geq 0}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $Gr_{\geq 0}(k,n)$ + theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galashin-Karp-Lam 2017 preprint: $Gr_{\geq}(k,n)$ is homeom. to closed ball.
(still open qn for cell closures)

2. Totally nonnegative real part of flag variety $\mathcal{F}\ell_{\geq 0} = (GL/B)_{\geq 0}$

Rietsch: poset of closure rel'ns. Cells $R^{\circ}_{u,v}$ given by $u \leq v$ in Bruhat order

Marsh-Rietsch: parametrization for $R^{\circ}_{u,v}$

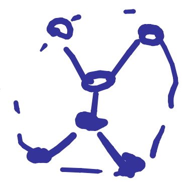
Williams: poset is CW poset

Rietsch-Williams: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

Galashin-Karp-Lam: Homeomorphism type for closure of "big cell"

3. Stratified Spaces of Electrical Networks

(Curtis-Ingerman-Morrow, Kenyon-Wilson, ...)

• Image of map Resp :  $\mapsto$ response matrix

Lam: (1) $E_n \hookrightarrow Gr_{\geq 0}(n-1,2n)$

(2) $F(E_n)$ is "Eulerian" ("i.e correct Euler char")

H-Kenyon: $F(E_n)$ shelling & $F(E_n)$ is CW poset (uses Dyer Bruhat shelling)

H-K Cor: shelling for each $[u,v]$ in face poset for edge-product space of phylogenetic trees (so CW poset)

Galashin-Karp-Lam: Homeom. type for closure of "big cell" for graph analogue of w_0, "well-connected graph"

4. (Deconed) Totally nonneg. real part unipotent radical of Borel in semisimple simply connected algebraic group

(a) Vertex "link" (nbhd) in $(\mathcal{F}l_n)_{\geq 0}$ via Marsh-Rietsch parameters → 0

(b) Image$(f_{(i_1,\dots,i_d)})$ for $w_0 = w(i_1,\dots,i_d)$

(c) Bruhat order as poset of closure rel'ns

(d) Each cell closure homeom. to closed ball.

- known as Fomin-Shapiro Conjecture
- proved by H., 2014, Invent.
- special case of w_0 in type A (shorter proof) GKL, 2017

More Motivation for Nonnegative Real Part of Unipotent Radical

- Given quantized env. alg. $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$ of Kac-Moody alg. (e.g. affine Lie alg.), then canonical basis is a basis B for U^- such that highest weight module with highest weight vector v_λ has basis $\{v_\lambda b \mid b \in B, v_\lambda b \neq 0\}$ for each λ.

- $f^{-1}_{(i_1,\ldots,i_d)}(p) \to f^{-1}_{(j_1,\ldots,j_d)}(p)$ coordinate change

$$(t_1, t_2, t_3) \longmapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3}\right)$$

tropicalizes to coordinate change:

$$(a, b, c) \mapsto (b + c - \min(a,c), \min(a,c), a + b - \min(a,c))$$

for canonical bases w/ same braid move

Role of 0-Hecke Algebra in Stratification for $\mathrm{im}(f_{(i_1,\ldots,i_d)})$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

$\rightsquigarrow$ suppress parameters

$x_i x_i = x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$ for $t_1,t_2,t_3>0$

$\rightsquigarrow$ (type A)

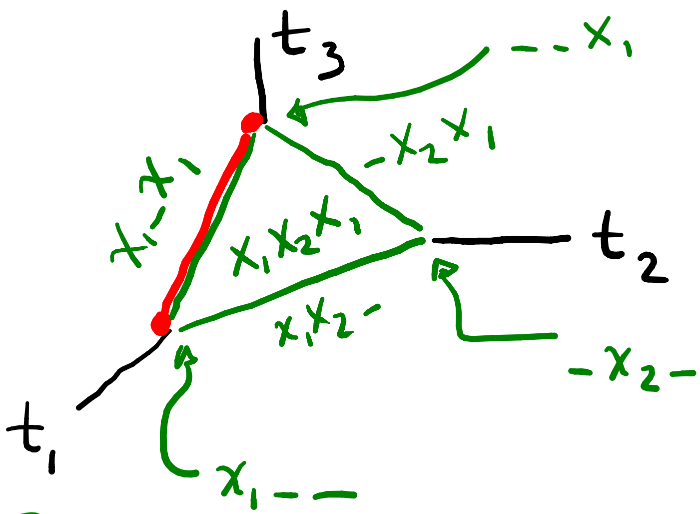
$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$

(& analogous relations outside type A)

Upshot: $\mathrm{im}(f_1) = \mathrm{im}(f_2) \Leftrightarrow \underbrace{x(f_1) = x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$

Thm (Lusztig): If $(i_1,\ldots,i_d)$ is reduced, then $f_{(i_1,\ldots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra



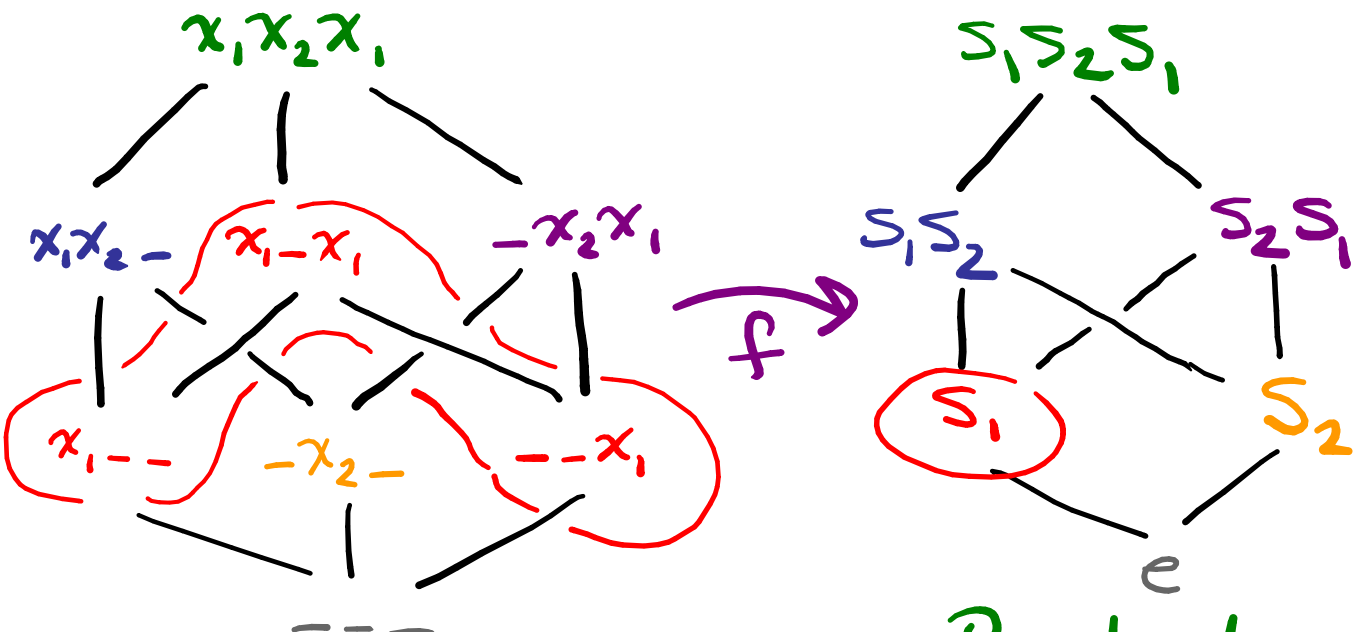
- let Y^o_w = open cell in $\mathrm{im}(f_{(i_1 \ldots i_d)})$ indexed by $w \in W$

- let $\delta(x_{i_1} \ldots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1 x_2 x_1}_{x_2x_1x_2} x_2 x_1) = \delta(x_2 x_1 x_2 x_1) = s_1 s_2 s_1$

since $x_2 x_2 = x_2$

Map of Face Posets Induced by Map $f_{(i_1,\dots,i_d)}$ of Spaces



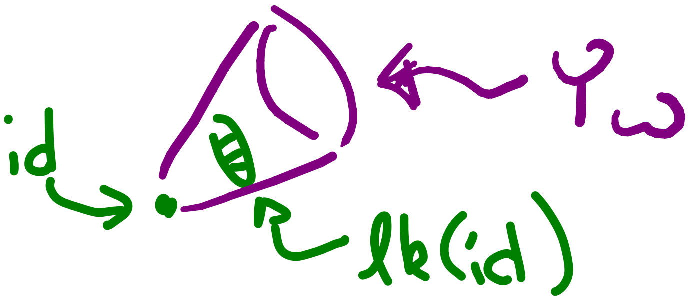
Boolean lattice B_n $\quad$ Bruhat order

$\quad\;\;\,\|\qquad\qquad\qquad\qquad\qquad\|$

Face poset of simplex $\quad$ $F(\mathrm{im}(f_{(i_1,\dots,i_d)}))$

$$f : x_{i_{j_1}}\cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}}\cdots x_{i_{j_r}})$$

Fomin-Shapiro Conjecture: The Bruhat stratification of $lk(id)$ in totally nonneg. real part of unipotent radical in Borel in algebraic group (e.g. $im(f_{(i_1,\dots,i_d)})$) is regular CW complex homeom. to closed ball (with Bruhat order as face poset).



$$Y_w = \left[\overline{B^- w B^-} \cap \binom{\text{unipotent}}{\text{subgp of } B}\right]_{\geq 0}$$

lower triangular opposite Borel B^-

permutation w

upper triang. w/ 1's on diagonal

totally nonneg. part

Proof of Fomin-Shapiro Conjecture
(H., 2014, Invent.)

Set-up: Surjective fn $f_{(i_1,\dots,i_d)}: \Delta_{d-1} \to Y$ s.t. $f_{(i_1,\dots,i_d)}\big|_{\mathrm{int}(\Delta_{d-1})}$ homeom. to $\mathrm{int}(Y)$.

Step 1: Perform "cell collapses" on $\partial(\Delta_{d-1})$ designed to preserve homeom. type & regularity via cont. surjective fns $g_i: \Delta_{d-1} \to \Delta_{d-1}$ yielding $(\Delta_{d-1})/\sim$ s.t.

$$x_1 \sim x_2 \implies f_{(i_1,\dots,i_d)}(x_1) = f_{(i_1,\dots,i_d)}(x_2)$$

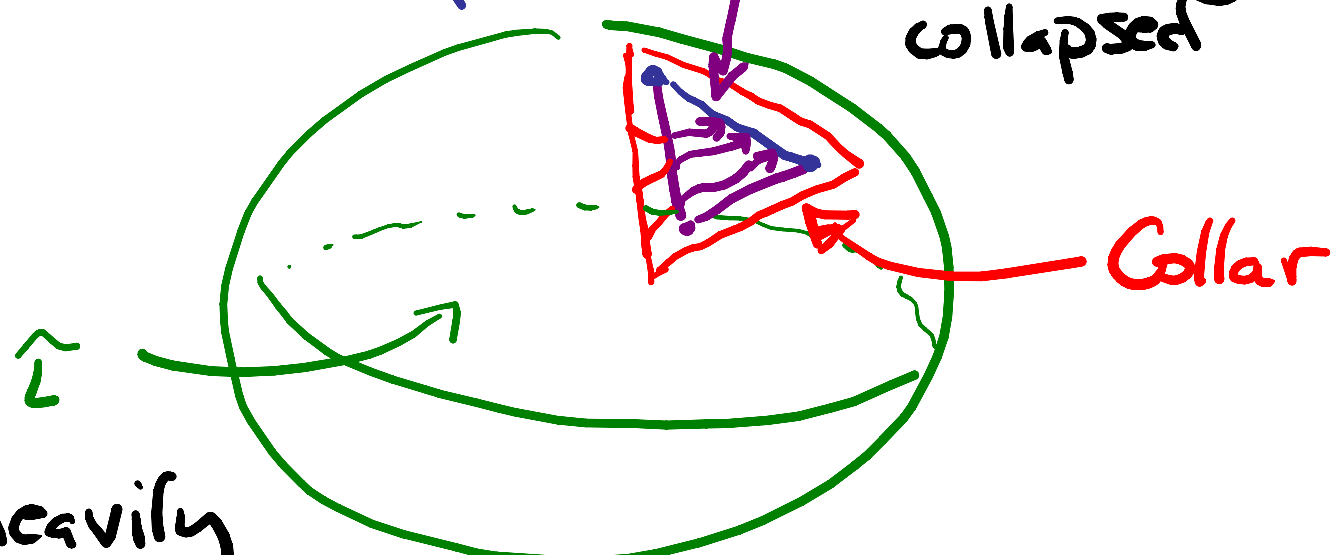
(eliminating faces given by nonreduced words)

Step 2: Prove $\overline{f}_{(i_1,\dots,i_d)}: (\Delta_{d-1})/\sim \;\to Y$ is a homeomorphism via new regularity criterion for finite CW complexes.

Corollary of Proof: Contractibility of fibers

<u>Step 1</u>: Cell Collapses Preserving Homeom. Type, Regularity & Topol. Manifold

- Cover each non-reduced σ with "nice" curves each in single fiber of $f_{(i_1,\ldots,i_d)}$
- Collapse $\bar{\sigma}$ onto $\bar{\rho} \subseteq \partial\sigma$ by stretching curve extensions in "collar" for $\partial Z \setminus \sigma$ to cover $\bar{\sigma} \setminus \bar{\rho}$



✱ heavily uses combinatorics of reduced & nonreduced words in 0-Hecke algebra

Step 2 via: New Regularity Criterion

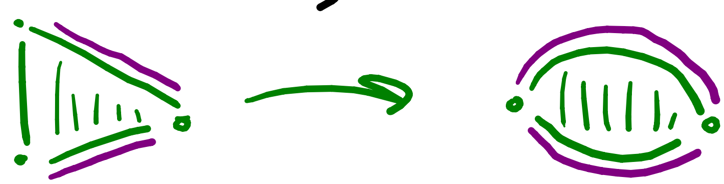
Preparatory Lemma (H.): Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose:

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

Insightful feedback (Quinn): Next theorem "spreads around" injectivity requirement.

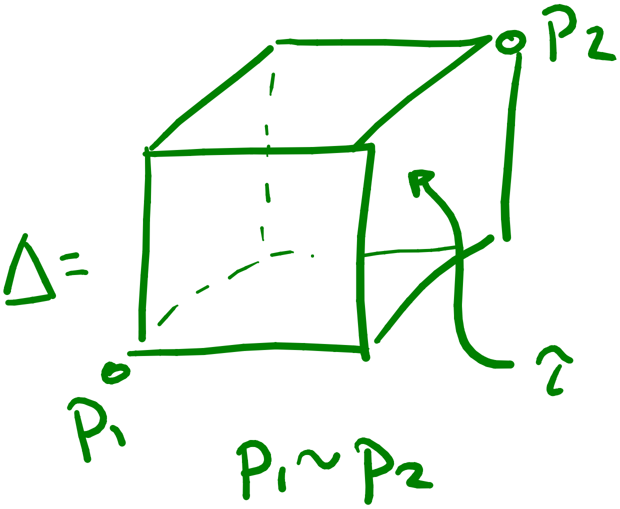
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ $\Leftrightarrow$

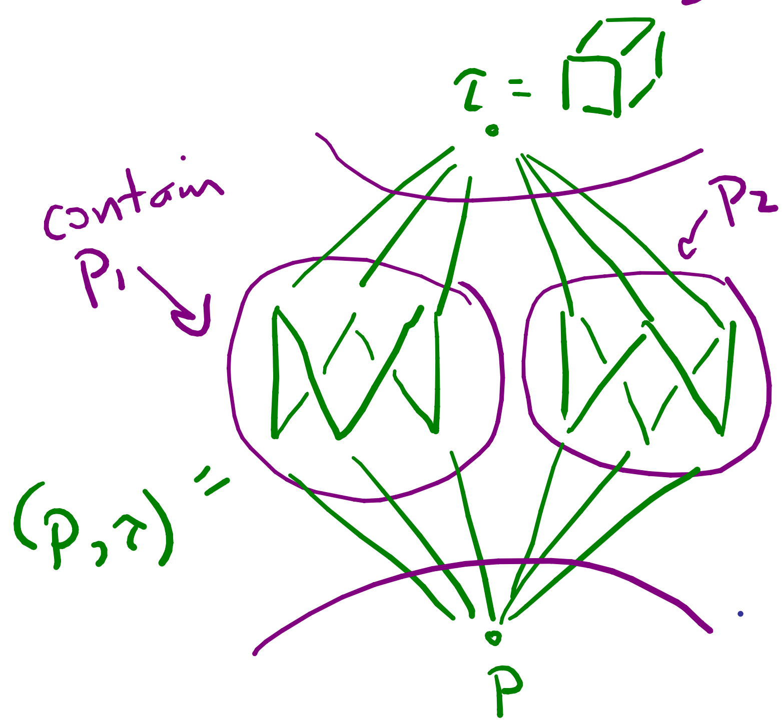
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





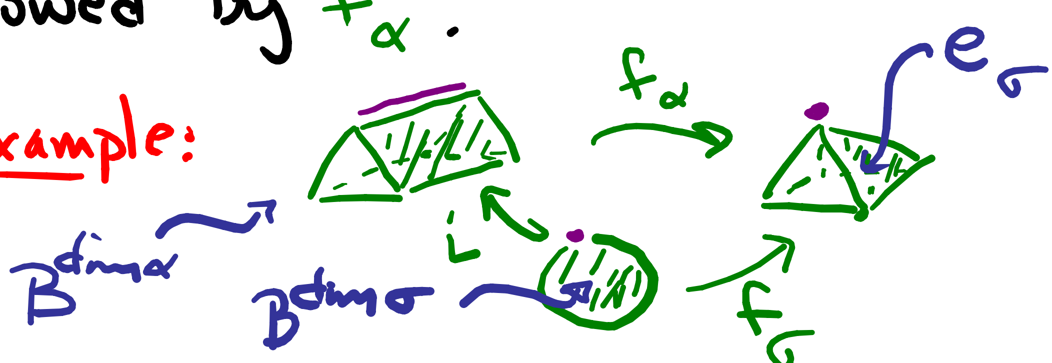
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $\iota : B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example:



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

Proof: Induction on difference in dim.

Codimension One Injectivity via Exchange Axiom for Coxeter Groups

$$s_1 s_2 s_1 s_3$$

$$- s_2 s_1 s_3 \qquad \cancel{s_1 - s_1 s_3} \qquad s_1 s_2 - s_3 \qquad s_1 s_2 s_1 -$$

$$\| \qquad \text{non-reduced} \qquad \| \qquad \|$$

$$3142 \neq 2341 \neq 3214$$

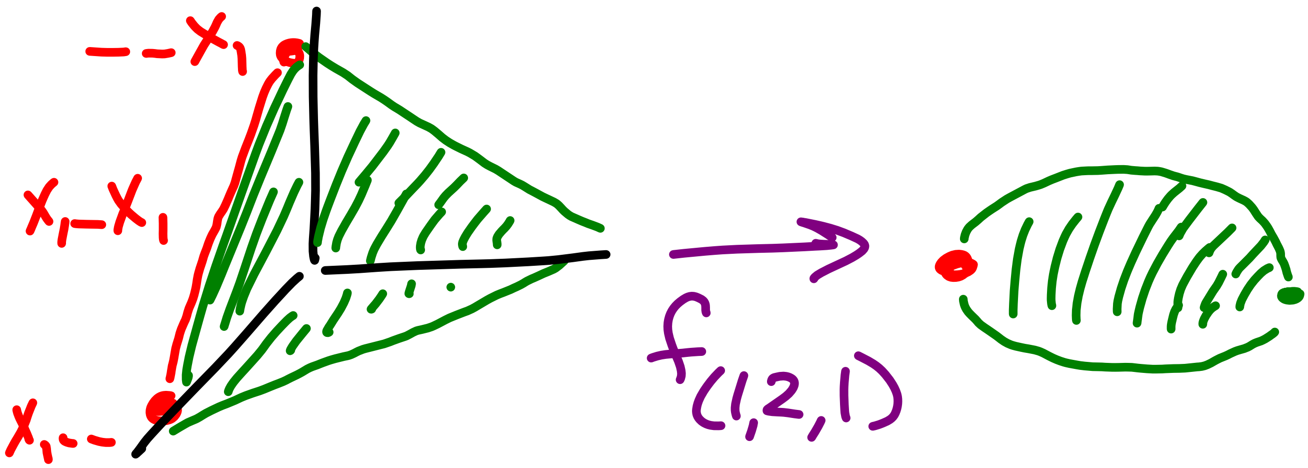
reduced subexpressions of a reduced expression obtained by deleting one letter must give distinct $u \in W$

This argument needs codim one!

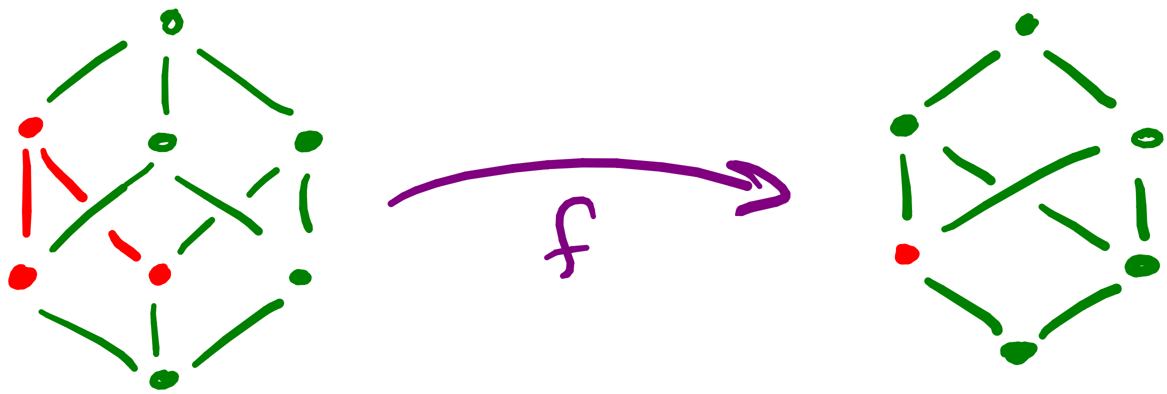
e.g. $s_1 s_2 s_1 s_3$

$s_1 - - -$ $\qquad$ $- - s_1 -$

Combinatorics of Fibers



Induced map of face posets:



Thm (Armstrong-H., 2011): For each $u \in W$, $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1, \ldots, i_\ell), u)$.

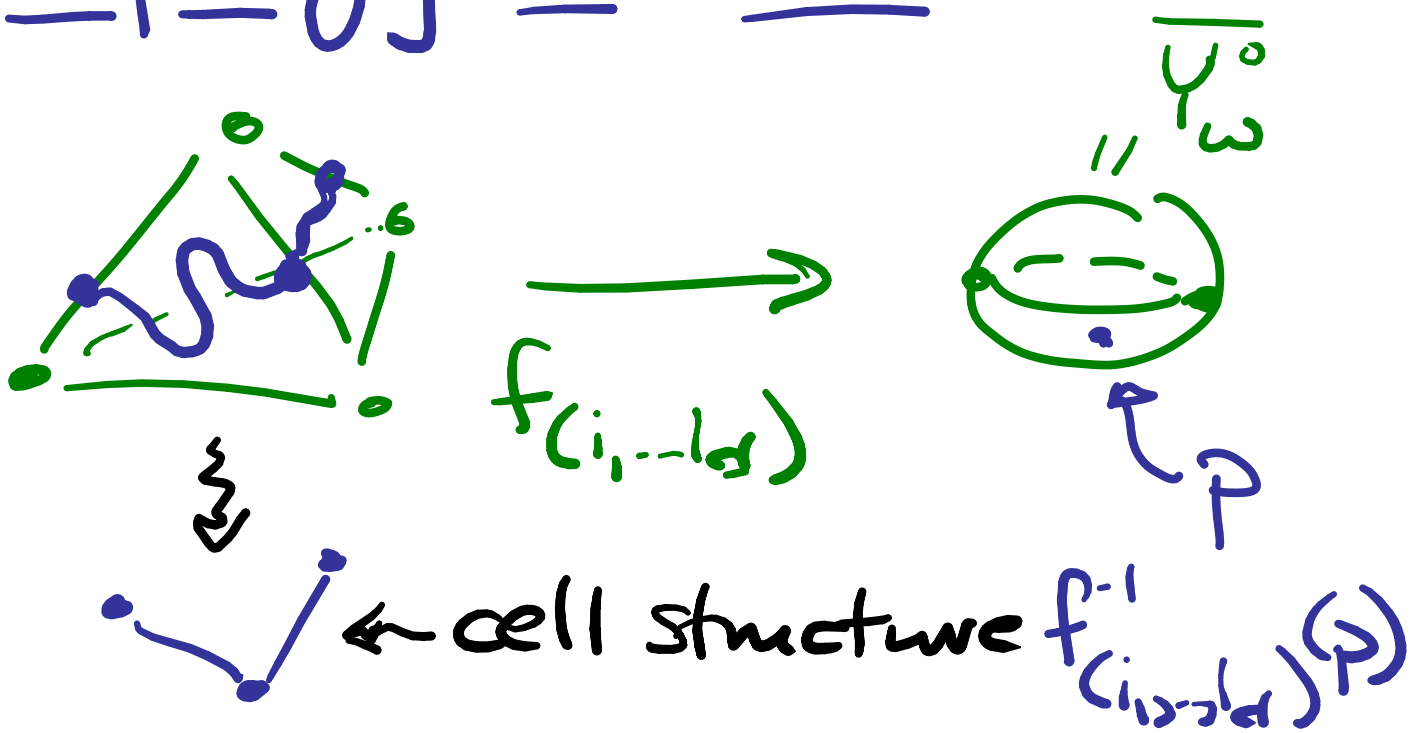
Thm (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\dots,i_d),u)$

Thm (DHM, 2018): Interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ is contractible.

Pf: Discrete Morse theory

Combining: DHM Conjecture would imply $f^{-1}_{(i_1,\dots,i_d)}(p) \cong$ interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ for $p \in Y_u^o$, hence $f^{-1}_{(i_1,\dots,i_d)}(p)$ contractible.

Topology of Fibers



Thm (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex Δ_{d-1}.

Topology of Fibers: Key Lemmas

Defn (Davis-H-Miller): The letter x_{i_1} is **redundant** in $x_{i_1} \cdots x_{i_d}$ if $\delta(i_1, \ldots, i_d) = \delta(i_2, \ldots, i_d)$

Lemma (Davis-H-Miller): x_{i_1} is non-redundant in $x_{i_1} \cdots x_{i_d}$

$\iff$ $f^{-1}_{(i_1 \ldots i_d)}(p)$ for $p \in Y^{\circ}_{\delta(i_1 \ldots i_d)}$

$$\{(t_1, \ldots, t_d) \mid x_{i_1}(t_1) \cdots x_{i_d}(t_d) = p\}$$

has unique value k_1 for t_1.

$\iff$ $f^{-1}_{(i_1 \ldots i_d)}(p) \cong f^{-1}_{(i_2 \ldots i_d)}(x_{i_1}(-k_1)p)$

__Lemma__ (Davis-H-Miller): Given $(t_1,\dots,t_d)\in f^{-1}_{(i_1,\dots,i_d)}(p)$ with $t_1>0$ and x_{i_1} redundant, then $\exists\,(t_1',\dots,t_d')\in f^{-1}_{(i_1,\dots,i_d)}(p)$ for every $t_1'\in[0,t_1]$.

e.g. $M=\begin{pmatrix}1&\pi&e\\&1&14\\&&1\end{pmatrix}$ then $f^{-1}_{(1,2,1,2)}(M)$

$\mathbin{/\!/}$

$\{(t_1,t_2,t_3,t_4)\,|\,x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4)=M\}$

achieves every $t_1\in[0,\frac{e}{14}]$.

Thm (DHM): DHM Conjecture $\Rightarrow$ (2nd proof of) Fomin-Shapiro Conjecture

- by Shellability of Bruhat Order + (Well Known) Topological Relationship

Fibers to Image: Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z whose restriction to $\mathrm{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B = S^n$

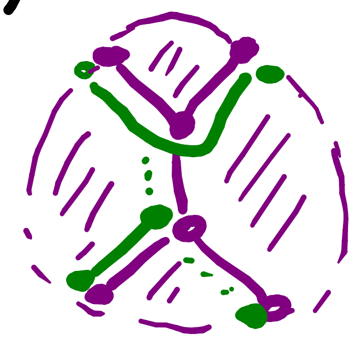
(2) $g(\partial B) \cap g(\mathrm{int}(B)) = \emptyset$

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

Then $Z \cong B$.

Some Further Questions

1. Subword complex analogues for fibers of other maps?
   e.g. H-Kenyon: $\Delta(G, \mathrm{elec}(H))$ for planar electrical networks (for H a minor of G)



2. DHM Conjecture? Analogues?
3. Regular CW decomps for cell closures & fibers in cases of graphs, $Gr(k,n)_{\geq 0}$, $(\mathcal{F}\ell_n)_{\geq 0}$ & $(GL_n/P)_{\geq 0}$?

Thanks!