Fibers of Maps to Totally Nonnegative Spaces

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Background on Totally Nonneg. Spaces

- A real matrix is totally nonnegative if all its minors are nonnegative
- Thm (Whitney): The TNN matrices of the form $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ are products of exponentiated Chevalley generators

$$x_i(t) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \leftarrow \text{row } i$$

$\uparrow$ column $i+1$

Some Past Work on Totally Nonnegative Spaces

- Lusztig (94): Studied totally nonneg. part of reductive groups; related to canonical bases
- Fomin-Shapiro (00): Combinatorics of Strata; Conj'd Regular CW Complex w/ Bruhat Order as Face Poset
- Hersh (14): Proof of Fomin-Shapiro Conjecture
- Galashin-Karp-Lam (22): Totally Nonneg parts of flag varieties are homeom. to closed balls

Map Whose Fibers We Study

$$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \to TNN(\underbrace{U_n}_{\text{unipotent radical}})$$

$$(t_1,\dots,t_d) \mapsto x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_d}(t_d)$$

Note: Preimage $\mathbb{R}^d_{\geq 0}$ stratified based on which coordinates are positive and which are 0.

Cell Stratification of Fibers: We prove for each $p \in TNN(U_n)$, the natural stratification for $\mathbb{R}^d_{\geq 0}$ induces a stratification of $f^{-1}_{(i_1\dots i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$ into cells (sets homeom. to open balls)

Baby Example of Fiber

$$\underbrace{x_1(t_1)}_{\begin{pmatrix}1 & t_1 & \\ & 1 & \\ & & 1\end{pmatrix}}\ \underbrace{x_1(t_2)}_{\begin{pmatrix}1 & t_2 & \\ & 1 & \\ & & 1\end{pmatrix}}\ \underbrace{x_1(t_3)}_{\begin{pmatrix}1 & t_3 & \\ & 1 & \\ & & 1\end{pmatrix}} = x_1(5) = \begin{pmatrix}1 & 5 & \\ & 1 & \\ & & 1\end{pmatrix}$$

$$= x_1(t_1+t_2+t_3) \rightsquigarrow t_1+t_2+t_3=5$$

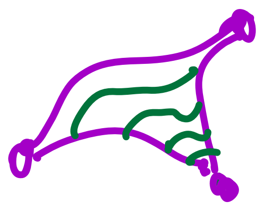
rightmost reduced word for s_1 in $(1,1,1)$

$$= f_{(1,1,1)}(t_1,t_2,t_3)$$

t_2^{max} continuous fn of t_1

$$f^{-1}_{(1,1,1)}(5) = \left\{(t_1,t_2,t_3)\in\mathbb{R}^3_{\geq 0} \;\middle|\; \begin{array}{l} 0\leq t_1\leq 5 \\ 0\leq t_2\leq \boxed{5-t_1} \\ t_3 = 5-t_1-t_2 \end{array}\right\}$$

"f_3" cont fn of t_1 & t_2



- A cell decomposition of topological space X is a decomposition into disjoint union of cells, namely pieces homeom. to $(0,1)^s$ for some s
- A cell stratification is a cell decomp. with $\sigma \cap \overline{\tau} \neq \phi \Rightarrow \sigma \subseteq \overline{\tau}$
- The face poset of a cell stratific. is partial order on cells with $\sigma < \tau \iff \sigma \subseteq \overline{\tau}$

Main Results for fibers

Combinatorial:

- stratification has same face poset as interior dual block complex of subword complex
- these interior dual block complexes are contractible

Topological

- each stratum is homeom. to $(0,1)^s$ for some s
- parametrizations for collections of strata homeom to $[0,1]^s$

Conjectural

- fibers contractible, implying another proof of Fomin-Shapiro Conj.

Running Example: Totally Nonnegative Part of a Space of Matrices

$$\bullet\quad \underset{\substack{\shortparallel \\ \exp(te_i)}}{x_i(t)} = I_n + tE_{i,i+1} = \begin{array}{c} \text{column } i+1 \\ \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{row } i \end{array}$$

(general finite type, expon'd Chevalley generator)

(type A)

$$\bullet\quad f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \in \mathbb{R}^{n^2}$$

reduced word

$$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

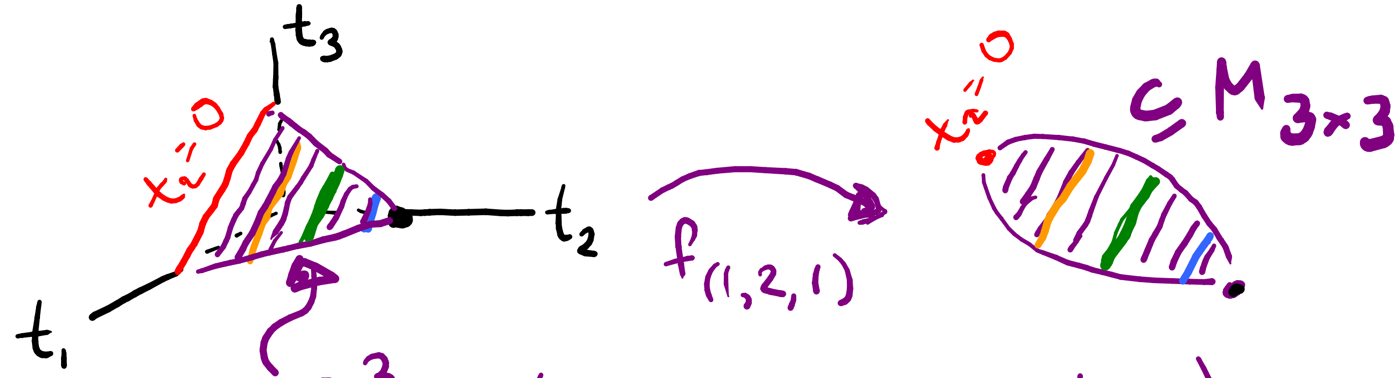
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

w_0 case:

$$\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{array}{l} \text{tot.} \\ \text{nonneg} \end{array} \right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\ldots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = \underbrace{K}_{\text{positive constant}} \text{ hyperplane})$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$t_2 = 0$

$$f_{(1,2,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)} \cdot \underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t > 0\} = \{x_1(t_1)x_1(t_2) \mid t_1, t_2 > 0\}$

Combinatorics of fibers via "Demazure Product"

- $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

  $x_i x_i \to x_i$ "modified nil move"

  $\rightsquigarrow \delta(s_i, s_i) = s_i$

- $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}(t_1')x_i(t_2')x_{i+1}(t_3')$

  for $t_1' = \frac{t_2 t_3}{t_1+t_3}$; $t_2' = t_1+t_3$; $t_3' = \frac{t_1 t_2}{t_1+t_3}$

  $x_i x_{i+1} x_i \to x_{i+1} x_i x_{i+1}$ "braid move"

  $\rightsquigarrow \delta(s_i, s_{i+1}, s_i) = \delta(s_{i+1}, s_i, s_{i+1})$

- $x_i(t_1)x_j(u) = x_j(u)x_i(t)$

  $\rightsquigarrow \delta(s_i, s_j) = \delta(s_j, s_i)$ for $|j-i|>1$

The **Demazure product** for Coxeter group W satisfies

$$\delta(s_{i_1}, s_{i_2}, \ldots, s_{i_d}) = \begin{cases} \delta(s_{i_2}, \ldots, s_{i_d}) & \text{if } \ell(u) < \ell(s_{i_1} u) \\ s_{i_1}\delta(s_{i_2}, \ldots, s_{i_d}) & \text{otherwise} \end{cases}$$

$u := \delta(s_{i_2}, \ldots, s_{i_d})$

e.g. $\delta(1,2,1,2,1) = \delta(2,1,2,1)$

$= \delta(1,2,1) = s_1\delta(2,1)$

$= s_1 s_2 \delta(1) = s_1 s_2 s_1$

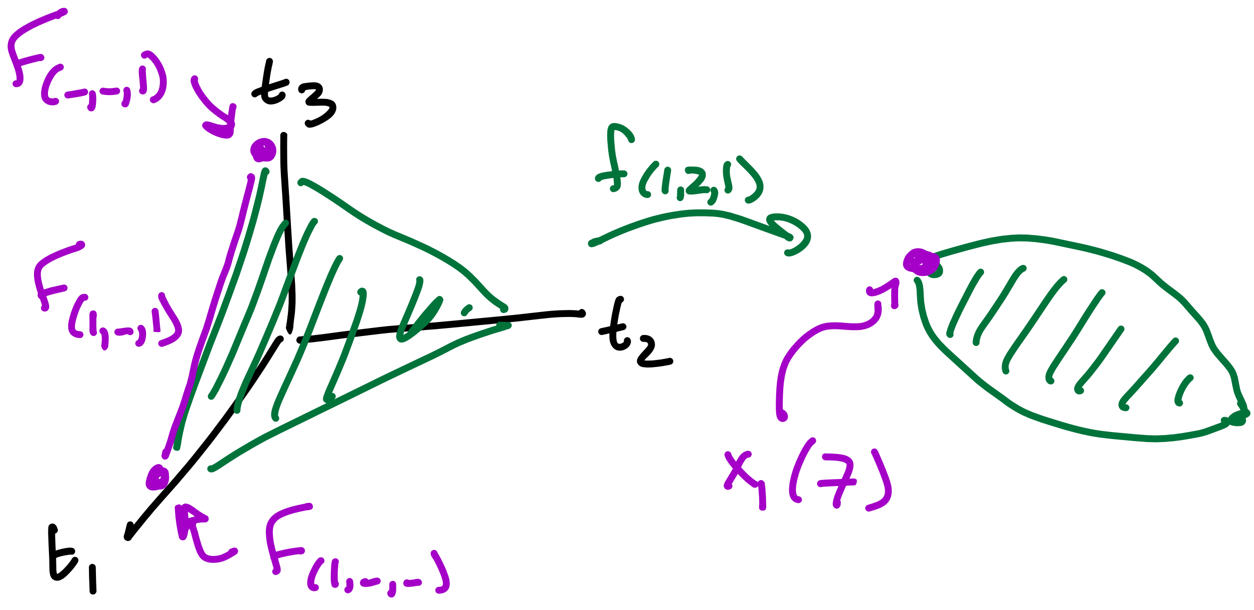
Fact: $f_{(i_1 \ldots i_d)}(\mathbb{R}^{Q}_{>0}) = f_{(i_1 \ldots i_d)}(\mathbb{R}^{Q'}_{>0})$

$\iff \delta(Q) = \delta(Q')$

fact: stratum $\mathbb{R}^{Q}_{>0} \cap f^{-1}_{(i_1,\dots,i_d)}(p)$ is nonempty $\iff$ $p \in U(\omega)$ for $\omega = \delta(Q)$

e.g. $f^{-1}_{(1,2,1)} \begin{pmatrix} \overbrace{1 \quad 7}^{x_1(7)} & 0 \\ 0 \quad 1 & 0 \\ 0 \quad 0 & 1 \end{pmatrix}$

has nonempty strata given by subwords $(1,-,-)$ & $(1,-,1)$ & $(-,-,1)$ of $(1,2,1)$ since $x_1(7) \in U(s_1)$



Thm (Lusztig): If $(i_1, \ldots, i_d)$ is reduced & $w = \delta(i_1, \ldots, i_d)$, then

$$f_{(i_1, \ldots, i_d)} : \mathbb{R}^d_{>0} \to U(w)$$

is homeomorphism

A Key Step in Cell Stratif. for Fibers:

Substantially generalize above,

e.g. show that map

$$(t_1, t_2, t_3, t_4) \mapsto x_4(1) x_2(5) x_{\underline{4}}(t_1) x_1(3) x_{\underline{2}}(t_2) x_{\underline{1}}(t_3) x_{\underline{2}}(t_4)$$

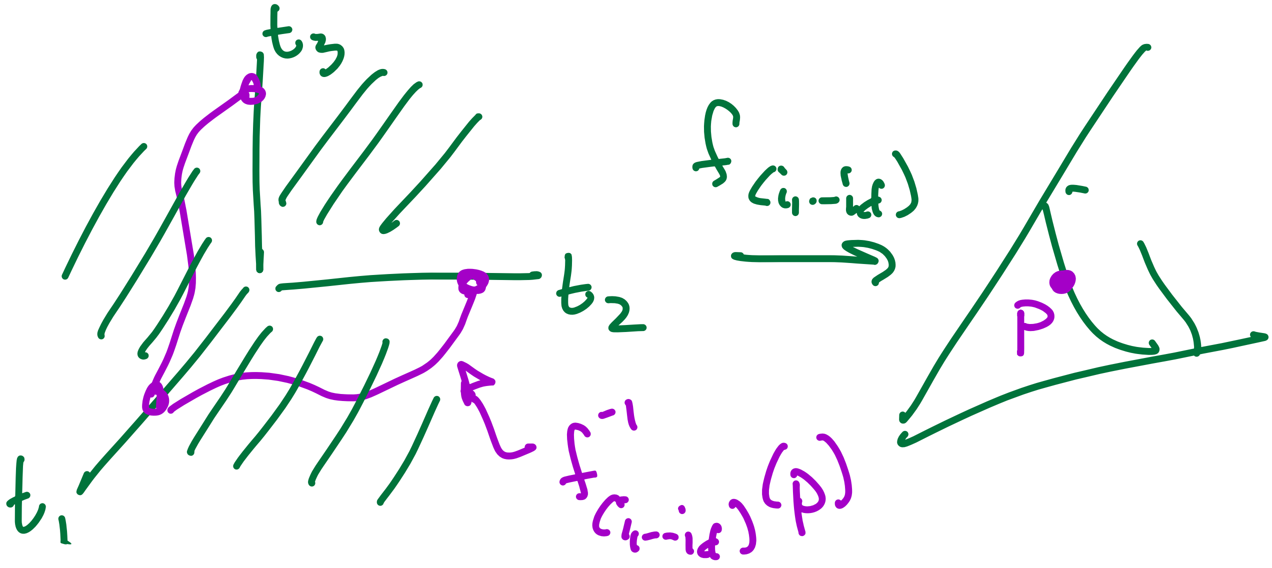
$\cap$

$\mathbb{R}^4_{>0}$

rightmost red. word for

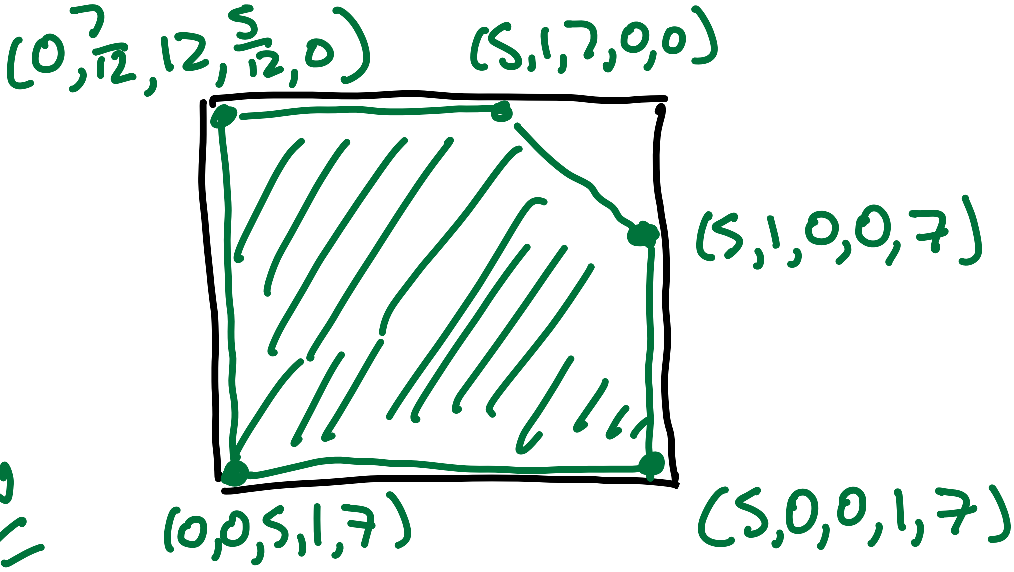
$s_4 s_2 s_1 s_2 = \delta(4,2,4,1,2,1,2)$

is homeomorphism from $\mathbb{R}^4_{>0}$ to its image, in particular is injective



Example of a Fiber

e.g.



$\cong$

$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = x_1(5)\,x_2(1)\,x_1(7)$

Subword Complexes & their Interior Dual Block Complexes

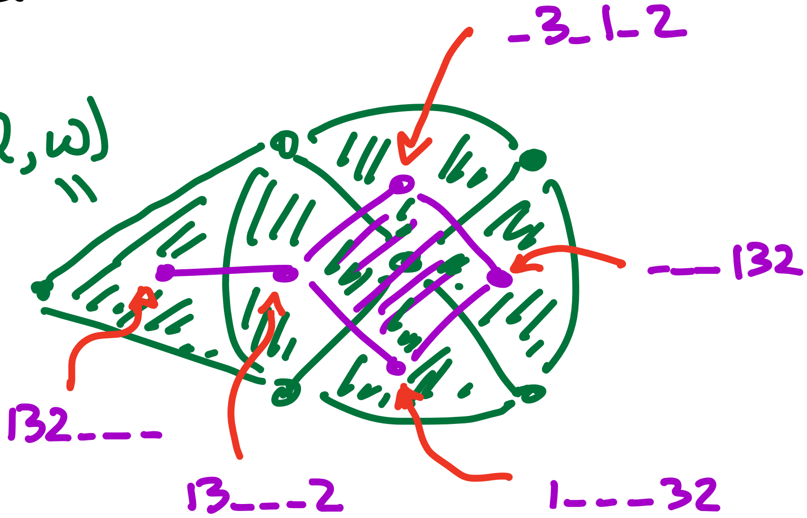
Defn (Knutson-Miller): The subword complex $\Delta(Q,w)$ has vertices the letters in word Q & facets the complements of subwords that are reduced words for w.

e.g. $\Delta(Q,w)$ =

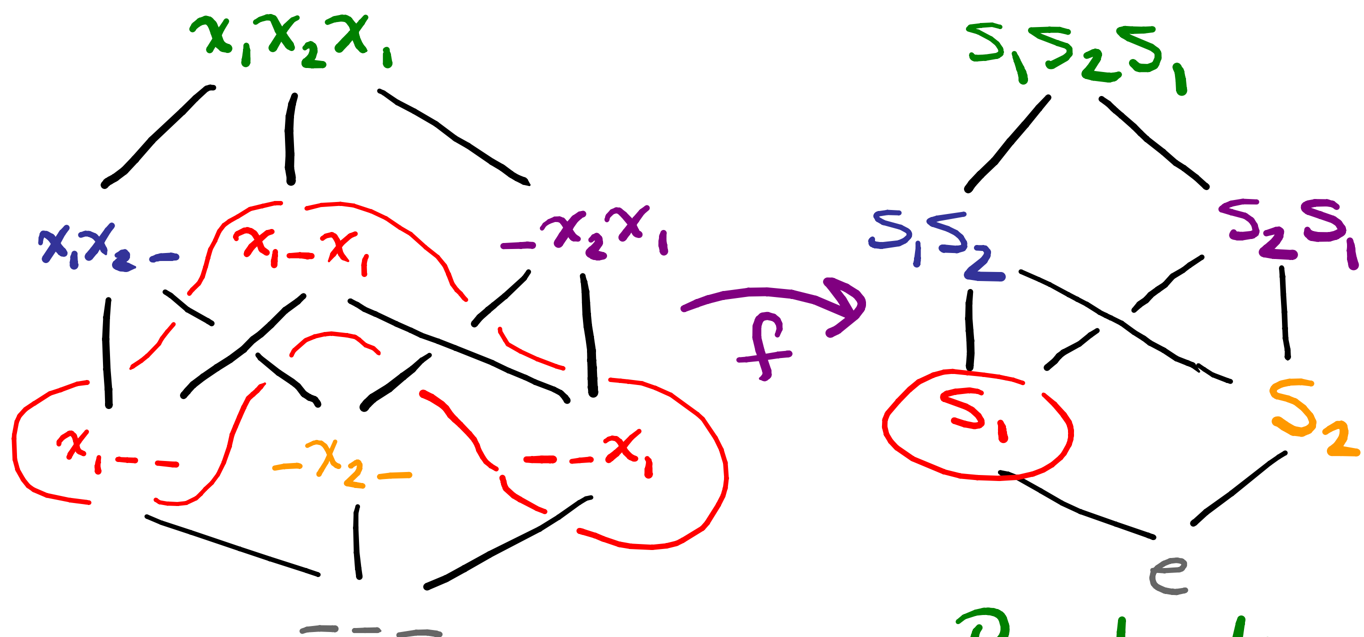
$Q=(1,3,2,1,3,2)$

$w=s_1s_3s_2$

with its interior dual block complex in purple



Map of Face Posets Induced by Map $f_{(i_1,\dots,i_d)}$ of Spaces



Boolean lattice B_n $=$ Face poset of simplex

Bruhat order $=$ $F(\mathrm{im}(f_{(i_1,\dots,i_d)}))$

$$f: x_{i_{j_1}}\cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}}\cdots x_{i_{j_r}})$$

Thm (DHM): For each $u \in W$, $f^{-1}(u) := \{x \in B_n \mid f(x) = u\}$ is face poset for interior dual block complex of the subword complex $\Delta((i_1,\dots,i_d),u)$. (a regular CW complex)

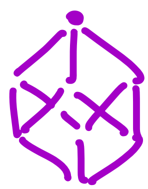
Thm (DHM): The interior dual block complex of every subword complex is contractible.

Remark: Strong evidence $f^{-1}_{(i_1,\dots,i_d)}(\varphi)$ is contractible.

e.g.

$f^{-1}_{(i_1,\dots,i_d)}(\varphi)$

Face Poset =



Consequence: Way to See:

$$f^{-1}_{(i_1\ldots i_d)}(p)\cap \mathbb{R}^d_{\geq 0} = f^{-1}_{(i_1\ldots i_d)}(p)\cap \underbrace{\Delta^{d-1}_K}$$

simplex in $\mathbb{R}^d_{\geq 0}$ with coord. sum K

for some $K>0$

- show change of coord's $(t_1,\ldots,t_d)\longmapsto(t_1',\ldots,t_d')$ given by braid & modified nil moves preserve sum of parameters
- use gallery-connectedness of subword complexes to show $f^{-1}_{(i_1\ldots i_d)}(p)\cap\mathbb{R}^d_{\geq 0}$ is connected by braid & modified nil-moves

Paramet. of (Unions of) Cells

- Consider rightmost subword Q of $(i_1, \ldots, i_d)$ that is reduced word for $\delta(i_1, \ldots, i_d)$

  e.g. $(3, 1, 2, 1, 2, 1, 2)$

- Parametrize cells w/ $t_1, t_5, t_6, t_7 > 0$ in $f^{-1}_{(3,1,2,1,2,1,2)}(p)$, $p \in U(s_3 s_2 s_1 s_2)$ using $[0,1)^3$

$$x_3(t_1)\, x_1(t_2)\, x_2(t_3)\, x_1(t_4)\, x_2(t_5)\, x_1(t_6)\, x_2(t_7)$$

choice of values $\cong [0,1)$

values determined by other params

Cell Stratification: Key Lemmas
(& Definitions)

Defn: Consider word $(i_1,\dots,i_d)$.
The letter i_ℓ is redundant in $(i_1,\dots,i_d)$ if $\delta(i_1,\dots,i_d)=\delta(i_1,\dots,\hat{i}_\ell,\dots,i_d)$ and nonredundant otherwise.

e.g. $\delta(\underbrace{1}_{\text{redundant}},2,\underbrace{1}_{\text{nonredundant}},2)=s_2s_1s_2$

Note:

(1) i_1 nonredundant $\iff$
$$\delta(i_1,i_2,\dots,i_d)=s_{i_1}\,\delta(i_2,\dots,i_d)$$

(2) $f_{(i_1,\dots,i_d)}(k_1,\dots,k_d)=p \iff f_{(i_2,\dots,i_d)}(k_2,\dots,k_d)=x_{i_1}(-k_1)p$

Lemma: If i_1 is nonredundant in $(i_1, \dots, i_d)$, then there is unique value k_1 for t_1 s.t.

$$f_{(i_1, \dots, i_d)}(k_1, t_2, \dots, t_d) = p$$

has a solution.

This has $x_{i_1}(-k_1)p \in \mathcal{U}(\delta(i_2, \dots, i_d))$

Lemma: If i_1 is redundant in $(i_1, \dots, i_d)$, then

$$f_{(i_1, \dots, i_d)}(k_1, t_2, \dots, t_d) = p$$

has solutions $\iff k_1 \in [0, t_1^{max}]$ for some $t_1^{max} > 0$. Moreover, $k_1 \in [0, t_1^{max}) \Rightarrow x_{i_1}(-k_1) \in \mathcal{U}(\delta(i_2, \dots, i_d))$

Lemma: Given i_j nonredundant in $(i_j, \dots, i_d)$ & given $k_1, \dots, k_{j-1} \geq 0$ s.t.

$$x_{i_{j-1}}(-k_{j-1}) \cdots x_{i_1}(-k_1) p \in \mathcal{U}(\delta(i_j, \dots, i_d))$$

then there is a unique value

$$\underbrace{f_j(k_1, \dots, k_{j-1})}_{\text{"}k_j\text{"}} \in \mathbb{R}_{\geq 0} \text{ for } t_j \text{ s.t.}$$

$$f_{(i_1, \dots, i_d)}(k_1, k_2, \dots, k_{j-1}, k_j, t_{j+1}, \dots, t_d) = p$$

has a solution with $t_{j+1}, \dots, t_d \in \mathbb{R}_{\geq 0}$.

e.g. $f^{-1}_{(1,2,1,2)}(x_1(5)x_2(7)x_1(3))$

$$\underbrace{x_1(5)x_2(7)x_1(3)}_{\begin{pmatrix} 1 & 8 & 35 \\ & 1 & 7 \\ & & 1 \end{pmatrix}}$$

has $f_2(k_1) = \frac{21}{8-k_1}$

$f_3(k_1, k_2) = 8 - k_1$

$f_4(k_1, k_2, k_3) = 7 - k_2$ & $t_1 \in [0, 5]$

Lemma: Given i_ℓ redundant in $(i_\ell, \ldots, i_d)$ & any $k_1, \ldots, k_{\ell-1} \geq 0$ s.t.

$$x_{i_{\ell-1}}(-k_{\ell-1}) \cdots x_{i_1}(-k_1)\,p \in U(\delta(i_\ell \cdots i_d))$$

then t_ℓ takes exactly the values in $[0, K]$ for some $K > 0$.

$$K = \text{"}t_\ell^{\max}(k_1, \ldots, k_{\ell-1})\text{"}$$

Example: $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & 14 \\ & & 1 \end{pmatrix}$ then

$$f^{-1}_{(1,2,1,2)}(M) = \{(t_1, t_2, t_3, t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$$

achieves every $t_1 \in [0, \frac{e}{14}]$

$\frac{e}{14}$ = "$t_1^{\max}$"

Useful Characterization of Parameters Whose Values are Uniquely Determined

Lemma: $S^c = \{j_1, \dots, j_s\} \subseteq \{1, \dots, d\}$ indexes rightmost subword of $(i_1, \dots, i_d)$ which is reduced word for $w = \delta(i_1, \dots, i_d)$

$$\Longleftrightarrow \quad S^c = \left\{ j \in [d] \;\middle|\; i_j \text{ is nonredundant in } (i_j, \dots, i_d) \right\}$$

e.g. $(1, 2, 1, 2, 4, 1, 5, 2, 4, 5)$ $\quad d = 10$

$$\begin{array}{cccccccccc} 1 & 2 & 3 & \underset{\sim}{4} & 5 & \underset{\sim}{6} & \underset{\sim}{7} & \underset{\sim}{8} & \underset{\sim}{9} & \underset{\sim}{10} \end{array}$$

S^c

$S^c = \{4, 6, 7, 8, 9, 10\}$ since $(2, 1, 5, 2, 4, 5)$ is rightmost reduced word for $\delta(1, 2, 1, \dots, 4, 5)$

$S = \{1, 2, 3, 5\}$ $\quad$ **Notation:** $S = \{j_1', \dots, j_{d-s}'\}$

Domain for Homeom. from $[0,1)^S$ to Union of Strata

$D^{<\max}$

$\parallel$

$$\left\{ (t_1,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p) \;\middle|\; \begin{array}{l} t_j < t_j^{\max}(t_1,\dots,t_{j-1}) \\ \forall j \in S \end{array} \right\}$$

Recall: S^c is index set $\{j_1,\dots,j_{\ell(\omega)}\}$ for rightmost reduced word $(i_{j_1},\dots,i_{j_{\ell(\omega)}})$ for ω in $(i_1,\dots,i_d)$.

Thm (DHM): $D^{<\max} = \bigcup\limits_{\text{strata } \sigma \text{ s.t. } v \subset \bar{\sigma}} \sigma$

where v = rightmost reduced word for $\delta(i_1,\dots,i_d)$

Generalization of Lusztig Result

Lemma: Given $S(i_1 \ldots i_d) = \omega$ & $D_{k_{j_1}, \ldots, k_{j_s}} =$

$$\left\{ (t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0} \;\middle|\; \begin{array}{l} t_{j'_\ell} > 0 \text{ for } j'_\ell \in S^c \\ t_{j_r} = k_{j_r} \text{ for } j_r \in S \end{array} \right\}$$

for any fixed $k_{j_1}, \ldots, k_{j_s} \geq 0$, then

$f_{(i_1, \ldots, i_d)}\big|_{D_{k_{j_1}, \ldots, k_{j_s}}}$ is a homeomorphism

to its image within $U(\omega)$.

e.g.

$(t_2, t_5, t_6) \mapsto x_1(3) x_4(t_2) x_2(7) x_1(5) x_2(t_5) x_1(t_6)$

Consequence: Within $D^{<\max}$ for fiber $f^{-1}_{(i_1 \ldots i_d)}(p)$, redundant parameters determine values uniquely for all nonredundant parameters.

Example of Parametrization, Revisited

$x_1(t_1)\,x_1(t_2)\,x_1(t_3) = x_1(5)$ fiber

$$\overset{\shortparallel}{f_{(1,1,1)}}(t_1,t_2,t_3)$$

$$0 \le t_1 < t_1^{max} = 5$$

$$0 \le t_2 < t_2^{max}(k_1) = 5-k_1$$

$$t_3 = 5-k_1-k_2 = f_3(k_1,k_2)$$

$$[0,1)^2 \cong \left\{ (\underset{\underset{[0,1)}{\sim}}{k_1},\underset{\underset{[0,1)}{\sim}}{k_2},k_3) \;\middle|\; \begin{array}{l} 0 \le k_1 < 5 \\ 0 \le k_2 < 5-k_1 \\ k_3 = 5-k_1-k_2 \end{array} \right\}$$

this requires continuity of t_2^{max} & f_3!

Thm (DHM): $[0,1)^{d-\ell(\omega)} \cong D^{<max}$ with restriction to $(0,1)^{d-\ell(\omega)} \cong$ single stratum

Continuity Lemmas

Recall: $f_\ell(k_1,\dots,k_{\ell-1})$ is the unique value for t_ℓ in $f^{-1}_{(i_1,\dots,i_d)}(p)$ s.t. $t_i = k_i$ for $i=1,\dots,\ell-1$

Lemma: Given i_ℓ non-redundant in $(i_\ell,\dots,i_d)$ and $k_1,\dots,k_{\ell-1} \geq 0$ s.t.
(1) $k_i < t_i^{max}(k_1,\dots,k_{i-1})\ \forall i \in S$ & (2) $\exists$ $(k_1,\dots,k_{\ell-1},t_\ell,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$,
then the function f_ℓ is continuous.

e.g. $f^{-1}_{(1,2,1,2)}(x_1(5)x_2(7)x_1(3))$

$t_1^{max} = 5$; $\quad f_3(k_1,k_2) = 8-k_1$

$f_2(k_1) = \frac{21}{8-k_1}$; $\quad f_4(k_1,k_2,k_3) = 7-k_2$

Idea: $x_{i_1}(k_1)\cdots x_{i_{\ell-1}}(k_{\ell-1})\, x_{i_\ell}(t_\ell)\cdots x_{i_d}(t_d) = p$

$$\Updownarrow$$

$x_{i_\ell}(t_\ell)\cdots x_{i_d}(t_d) = x_{i_{\ell-1}}(-k_{\ell-1})\cdots x_{i_1}(-k_1)\, p$

so $\{(t_\ell,\dots,t_d) \mid (k_1,\dots,k_{\ell-1},t_\ell,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)\}$

$$\|$$

$$f^{-1}_{(i_\ell,\dots,i_d)}\underbrace{(x_{\ell-1}(-k_{\ell-1})\cdots x_1(-k_1)\,p)}_{\text{continuous function of } k_1,\dots,k_{\ell-1} \text{ for fixed } p}$$

so f_ℓ is coordinate fn composed with $f^{-1}_{(i_\ell,Q)}$ where (i_ℓ,Q) is reduced subword of $(i_\ell,\dots,i_d)$

Lemma: If i_ℓ is redundant in $(i_\ell, \dots, i_d)$, then t_ℓ^{max} is continuous fn of $k_1, \dots, k_{\ell-1}$

Proof Idea:

$$\underbrace{t_\ell^{max}(k_1, \dots, k_{\ell-1})}_{\substack{\text{for word} \\ (i_1,\dots,i_d)\{ \\ f^{-1}_{(i_1 \dots i_d)}(p)}} = \underbrace{f_\ell(k_1, \dots, k_{\ell-1})}_{\substack{\text{for subword } (i_1 - i_\ell, Q) \\ \{ f^{-1}_{(i_1 - i_\ell, Q)}(p) \text{ where} \\ (i_\ell, Q) \text{ is reduced word}}}$$

so f_ℓ cont. $\Rightarrow$ t_ℓ^{max} continuous

Example: $f_{(1,1,1,2)}(t_1, t_2, t_3, t_4) = x_1(7)x_2(5)$

$t_2^{max}(k_1) = 7 - k_1$

$\|$

$f_2(k_1)$

$$f_{(1,1,-,2)}(t_1, t_2, -, t_4) = x_1(7)x_2(5)$$

with $x_1(7)x_2(5) = p = x_1(7)x_2(5)$

$(1,1,2)$ nonreduced

$(1,-,2)$ reduced $\quad Q = (2)$

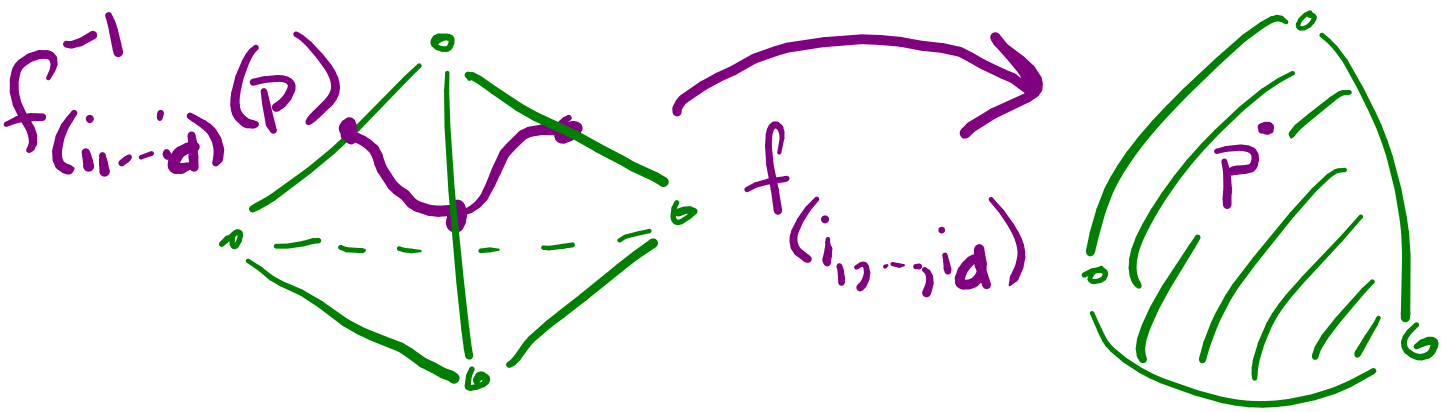
Remarks: We focus on $D^{<\max}$ where $t_\ell < k_\ell^{\max}$ for all $\ell \in S$ because $t_\ell = k_\ell^{\max}$ part much less well behaved.

Summary: $(0,1)^S \xrightarrow{\cong} \bigcup_{v \leq \bar\sigma \leq \bar\tau} \sigma$ which restricts to $(0,1)^S \xrightarrow{\cong} \tau$, showing each stratum is an open ball.

An Open Qu: Given word Q, a subword $(i_{j_1} \ldots i_{j_\ell})$ that is reduced word for $\delta(Q)$ and constants $\{k_j \geq 0 \mid j \notin \{j_1, \ldots j_\ell\}\}$, is $(t_{j_1}, \ldots, t_{j_\ell}) \mapsto f_Q(u_1, \ldots, u_d)$ for $u_j = \begin{cases} t_j & \text{if } j \in \{j_1, \ldots, j_\ell\} \\ k_j & \text{otherwise} \end{cases}$ homeom to image?

e.g. $(t_1, t_5) \longmapsto x_1(t_1) x_2(3) x_1(1) x_2(7) x_1(t_5)$

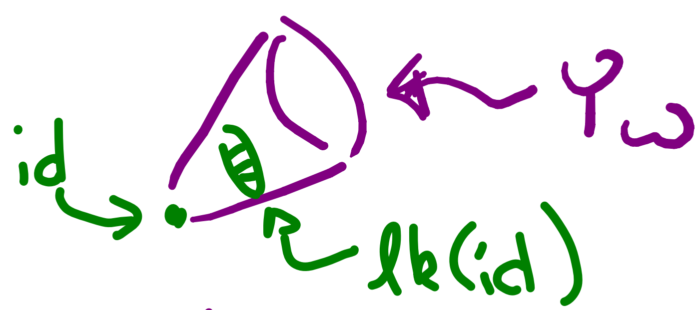
Conjecture (Davis-H-Miller): $f^{-1}_{(i_1,\dots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\dots,i_d),w)$ for $p \in Y_w^o$.



Thm (DHM): $f^{-1}_{(i_1,\dots,i_d)}(p)$ has cell decomposition & "correct" face poset.

Thm (DHM): Interior dual block complex of $\Delta((i_1,\dots,i_d),w)$ is contractible.

Fomin-Shapiro Conjecture: The Bruhat stratification of lk(id) in totally nonneg. real part of unipotent radical in Borel in algebraic group is regular CW complex homeomorphic to closed ball (w/ Bruhat order as face poset)



$$Y_w = \left[\overline{B^- w B^-} \cap \binom{\text{unipotent}}{\text{subgp of } B}\right]_{\geq 0}$$

lower triangular opposite Borel B^-

permutation w

upper triang. w/ 1's on diagonal

totally nonneg. part

Thm (DHM): DHM Conjecture $\Rightarrow$ A New Proof of Fomin-Shapiro Conjecture.

Idea: Use shellability of Bruhat order + the following topological fact:

If $g: B \to Z$ is continuous surjection from ball B to Hausdorff space Z s.t.

(1) $g|_{int(B)}$ is embedding

(2) $g(\partial B) \cong \partial B$

(3) $g(int B) \cap g(\partial B) = \emptyset$

(4) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

then $Z \cong B$.

Ingredients in this Relationship Between fibers & Image

- "CE-Approximation Theorem (Kirby-Siebenmann (all dim's); Quinn (dim 4); Armentrout (dim 3))
  - $g: \partial B \to \partial B$ as above may approximated by homeomorphisms

- Local Contractibility of $Homeo^+(S^n, S^n)$
  - any two homeomorphisms "close enough" to each other connected by path of homeomorphisms