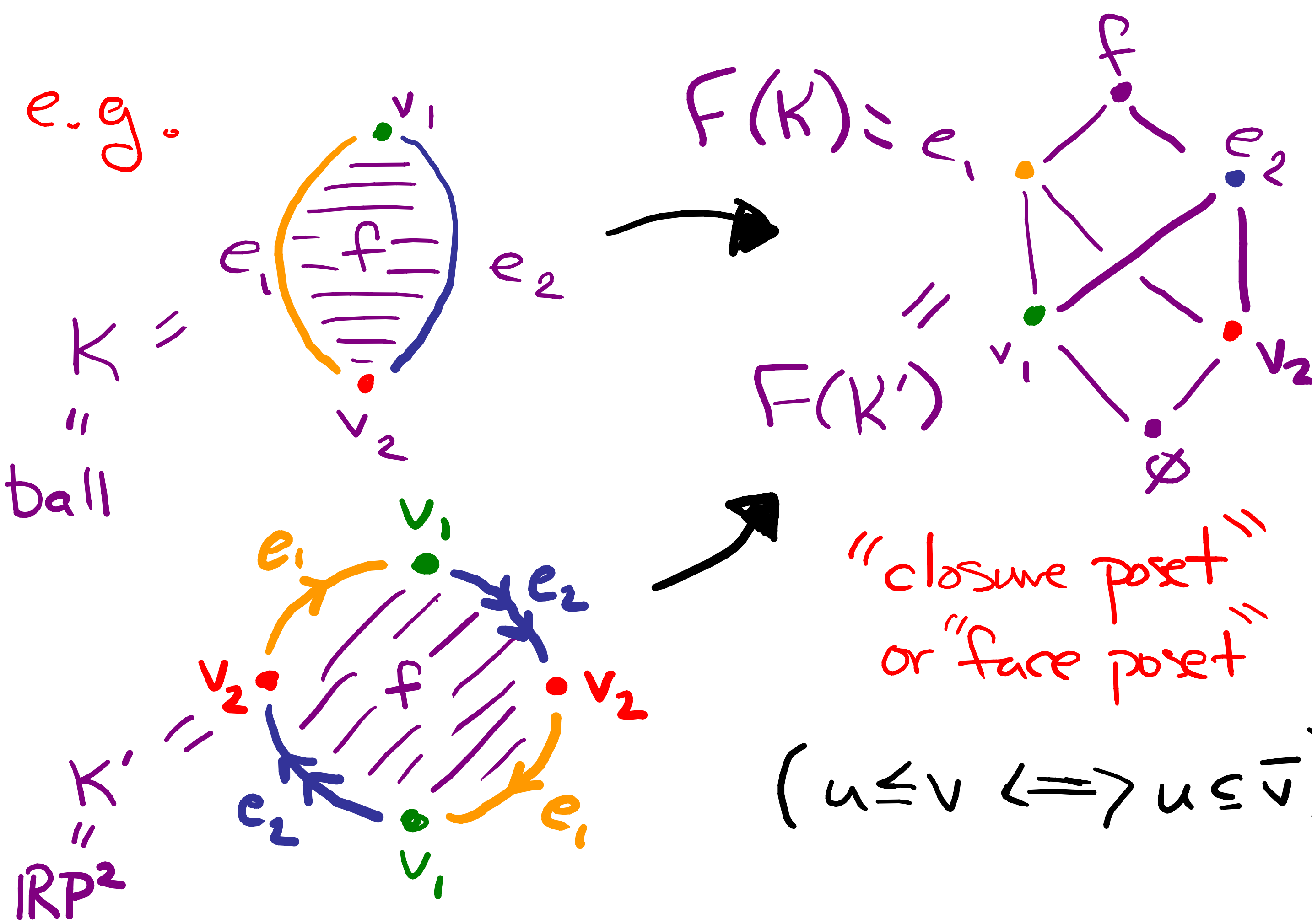
Uncrossing Posets & the Topological Combinatorics of Stratified Spaces of Planar Electrical Networks

Patricia Hersh, University of Oregon

- joint works with Rick Kenyon
& with Grace Stadnyk

CW Complexes & their Face Posets



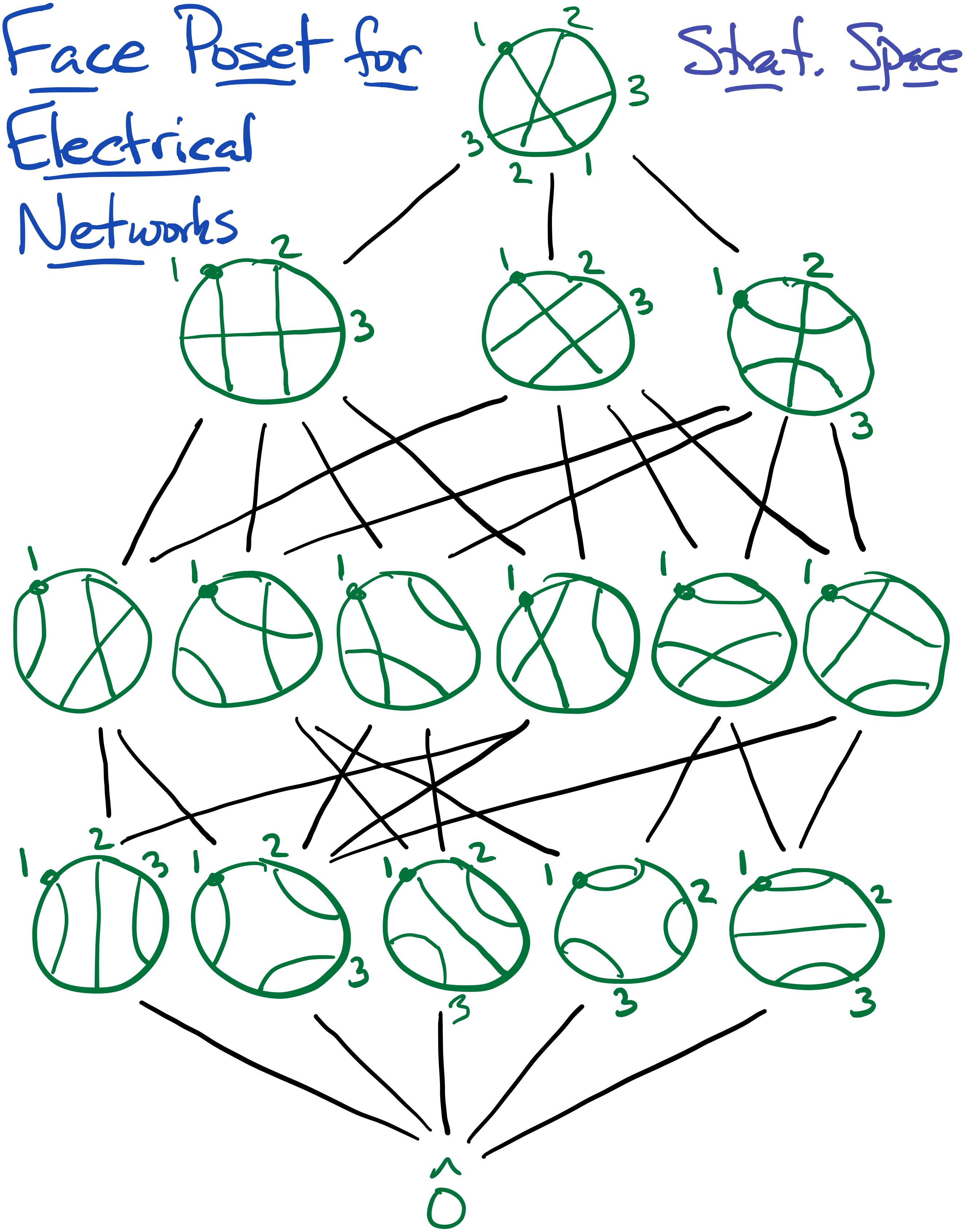
"closure poset" or "face poset"

$(u \leq v \iff u \leq \bar{v})$

Recall: A CW complex is comprised of open cells each homeomorphic to an open ball. A regular CW complex further has cell closures homeomorphic to closed balls. e.g. simplicial complexes

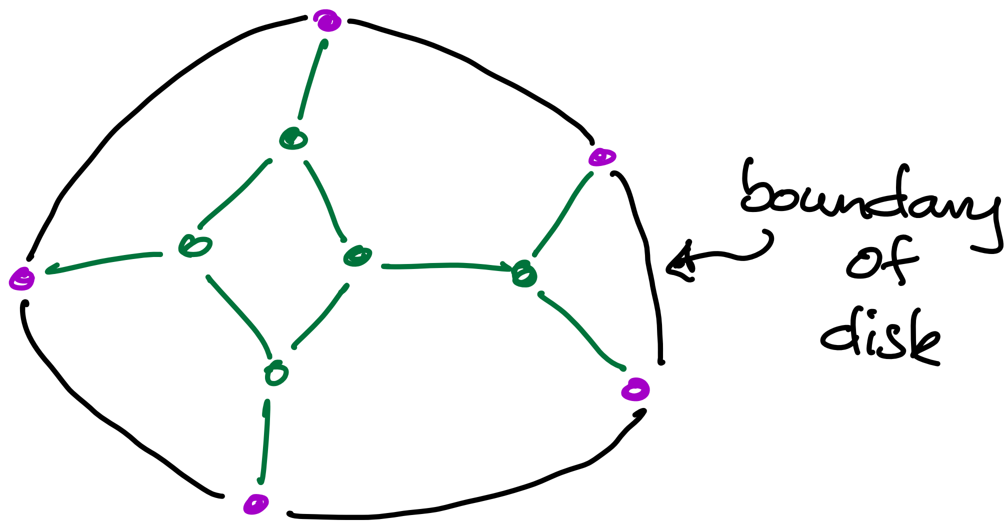
"Uncrossing Poset", Namely

Face Poset for Strat. Space of Electrical Networks



Space of "Response Matrices" of a Planar Electrical Network

e.g.



I = interior nodes

N = boundary nodes

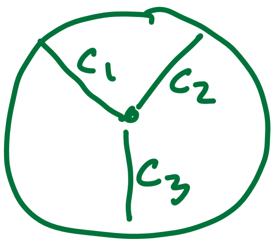
$$\underbrace{\Delta}_{\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}} \underbrace{\begin{pmatrix} V_N \\ V_I \end{pmatrix}}_{\text{vector of voltages}} = \underbrace{\begin{pmatrix} C_N \\ 0 \end{pmatrix}}_{\text{vector of currents}}$$

$$\boxed{(A - BC^{-1}B^T)}\, V_N = C_N$$

"response matrix" of the network (after negation)

• entries of response matrix are functions of the edge conductances

e.g.



$$\longrightarrow \begin{pmatrix} \frac{-c_1c_2-c_1c_3}{c_1+c_2+c_3} & \frac{c_1c_2}{c_1+c_2+c_3} & \frac{c_1c_3}{c_1+c_2+c_3} \\ \frac{c_2c_1}{c_1+c_2+c_3} & \frac{-c_2c_1-c_2c_3}{c_1+c_2+c_3} & \frac{c_2c_3}{c_1+c_2+c_3} \\ \frac{c_3c_1}{c_1+c_2+c_3} & \frac{c_3c_2}{c_1+c_2+c_3} & \frac{-c_3c_1-c_3c_2}{c_1+c_2+c_3} \end{pmatrix}$$

Goal: Study stratified space of response matrices, starting by studying its face poset.

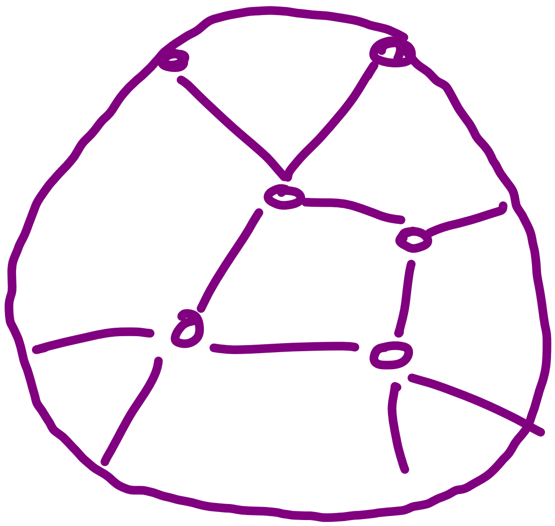
- letting edge conductances vary within $\mathbb{R}_{\geq 0} \cup \infty$ gives stratified space of response matrices of planar graph G

  - conductance $\to 0$ (resistance $\to \infty$) $\longleftrightarrow$ delete edge
  - conductance $\to \infty$ (resistance $\to 0$) $\longleftrightarrow$ contract edge

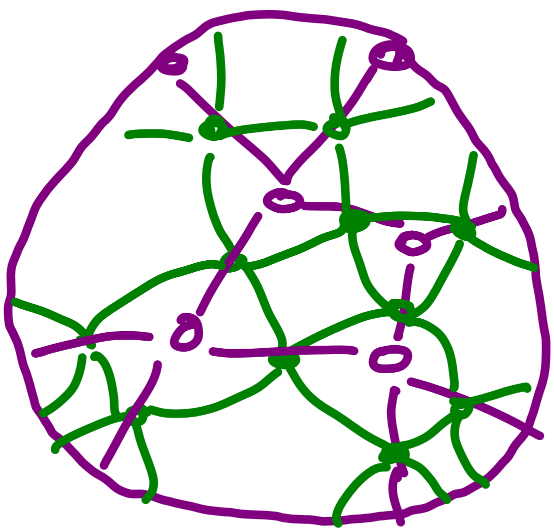
Correspondence: From Graphs to Uncrossing Wire Diagrams

"Medial Graphs"

$\Gamma =$

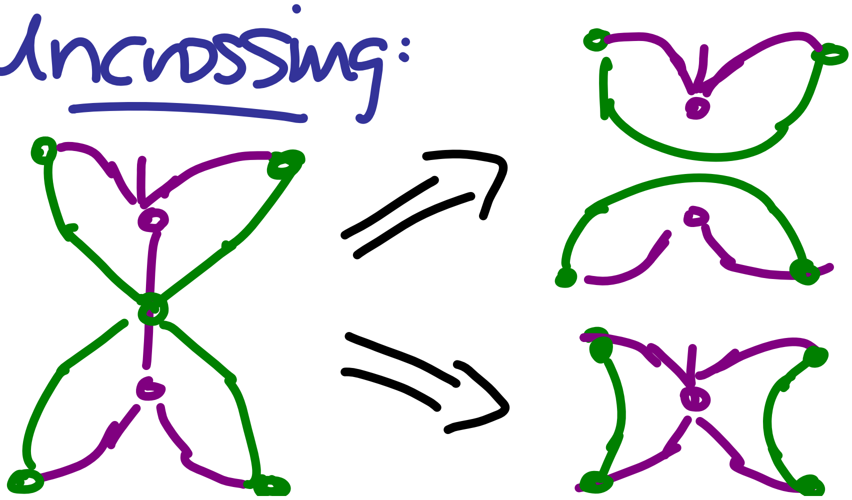


$G(\Gamma) =$



= wire diagram

Uncrossing:



deletion

contraction

A Conjecture of Thomas Lam

Thm (Lam): The uncrossing poset is Eulerian.

Conjecture (Lam): The uncrossing poset is lexicographically shellable.

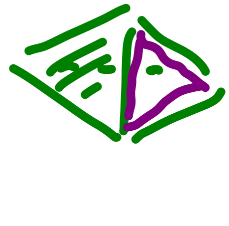
Thm (H.-Kenyon): Uncrossing posets are dual EC-shellable.

Cor: They are "CW posets", i.e. face posets of regular CW complexes.

Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \ldots, F_k$, a **shelling**, on facets s.t. $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$ is pure, codimension one subcomplex of $\overline{F_j}$ for each $j>1$ (hence is $\partial \overline{F_j}$ or has a cone point).

OR

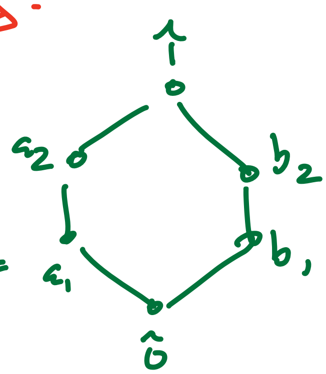


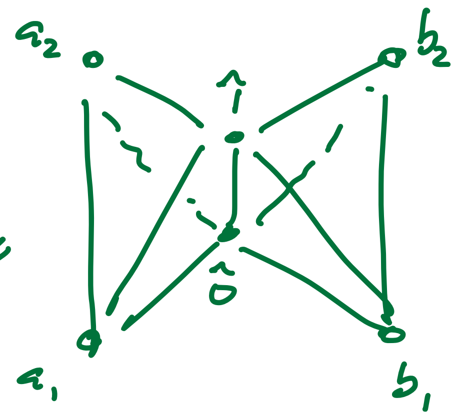
- Each facet attachment preserves homotopy type or closes off a new sphere

Def'n: A poset is shellable if its "order complex" is shellable.

Def'n: The order complex, $\Delta(P)$, of a poset P is the abstract simplicial complex whose i-dimensional faces are the chains $v_0 < v_1 < \dots < v_i$ of $i+1$ comparable poset elements.

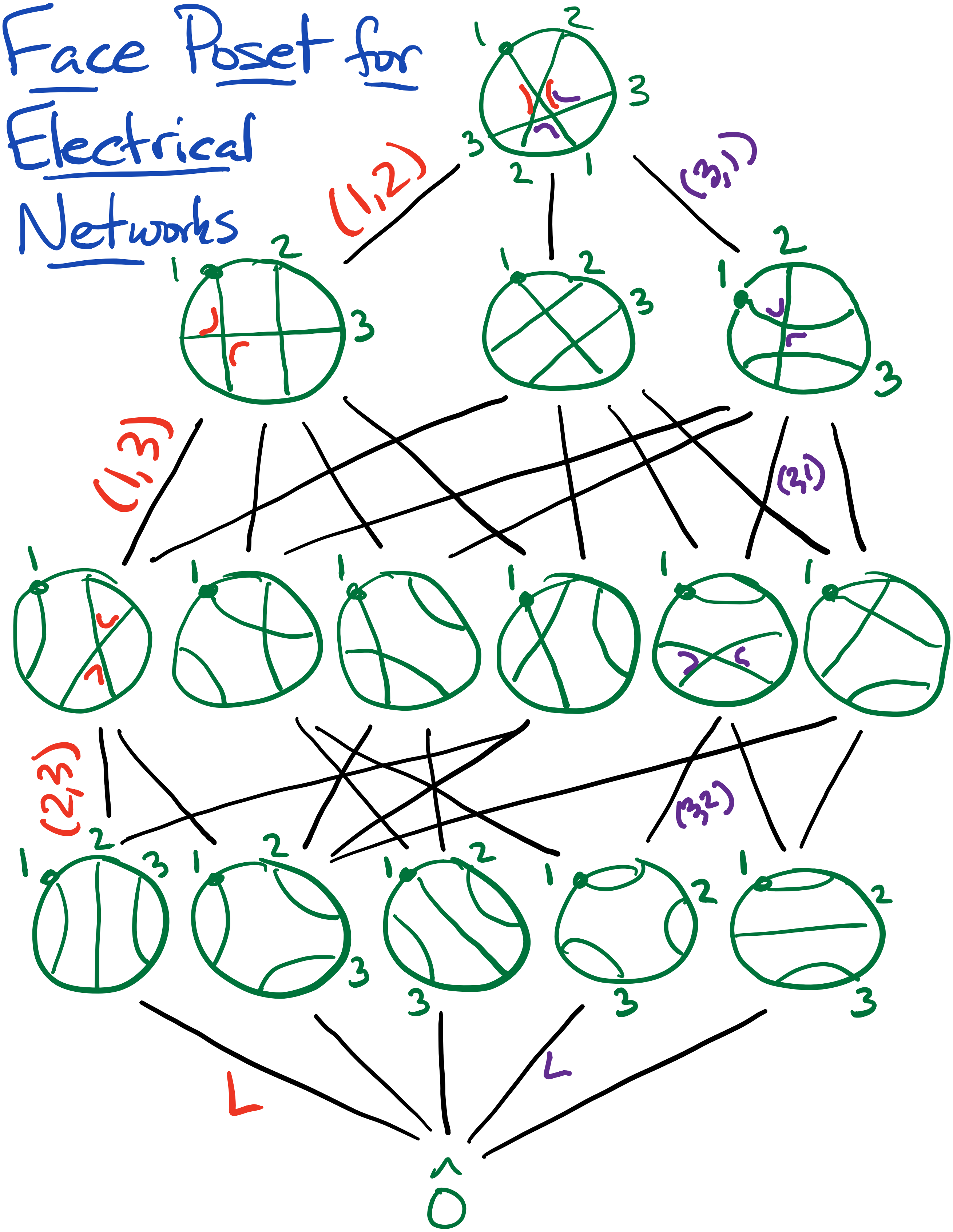
e.g.

$P =$ 

$\Delta(P) =$ 

Fact: The order complex of CW poset is barycentric subdivision of regular CW complex for which poset is face poset

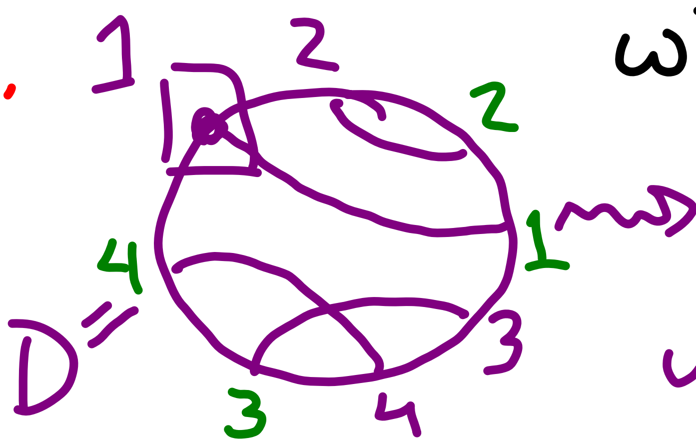
Face Poset for Electrical Networks



Edge Labeling (for proving shellab.)

Step 1: Define **word** of wire diagram D, denoted $w(D)$, as sequence of $2n$ wire endpts encountered clockwise starting with "root" 1.

e.g.

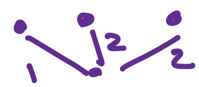


$= 12213434 = w(D)$

Step 2: Label $D \lessdot D'$ for $i<j$ as

- (i,j) if $ijij$ in $w(D')$ becomes $ijji$ in $w(D)$
- (j,i) if $ijij$ becomes $iijj$
- Label $\hat{0} < D$ with label L

Recent Work with Grace Stadnyk

- **H.-Stadnyk 2024:** Algorithm to convert "dual CC-labeling" with "UE property" to "dual CL-labeling" 

- **Consequence:** Uncrossing posets are dual CL-shellable
- **Key idea:** Generalize notion of "recursive atom ordering" (RAO). Give algorithm to transform any generalized recursive atom ordering to RAO.

Thm (H.-Stadnyk): Let P be a finite bounded poset. Then the following are equivalent:

(1) P admits recursive atom ordering

(2) P admits "generalized recursive atom ordering"

(3) P admits CL-labeling

(4) P admits CC-labeling with "UE property"

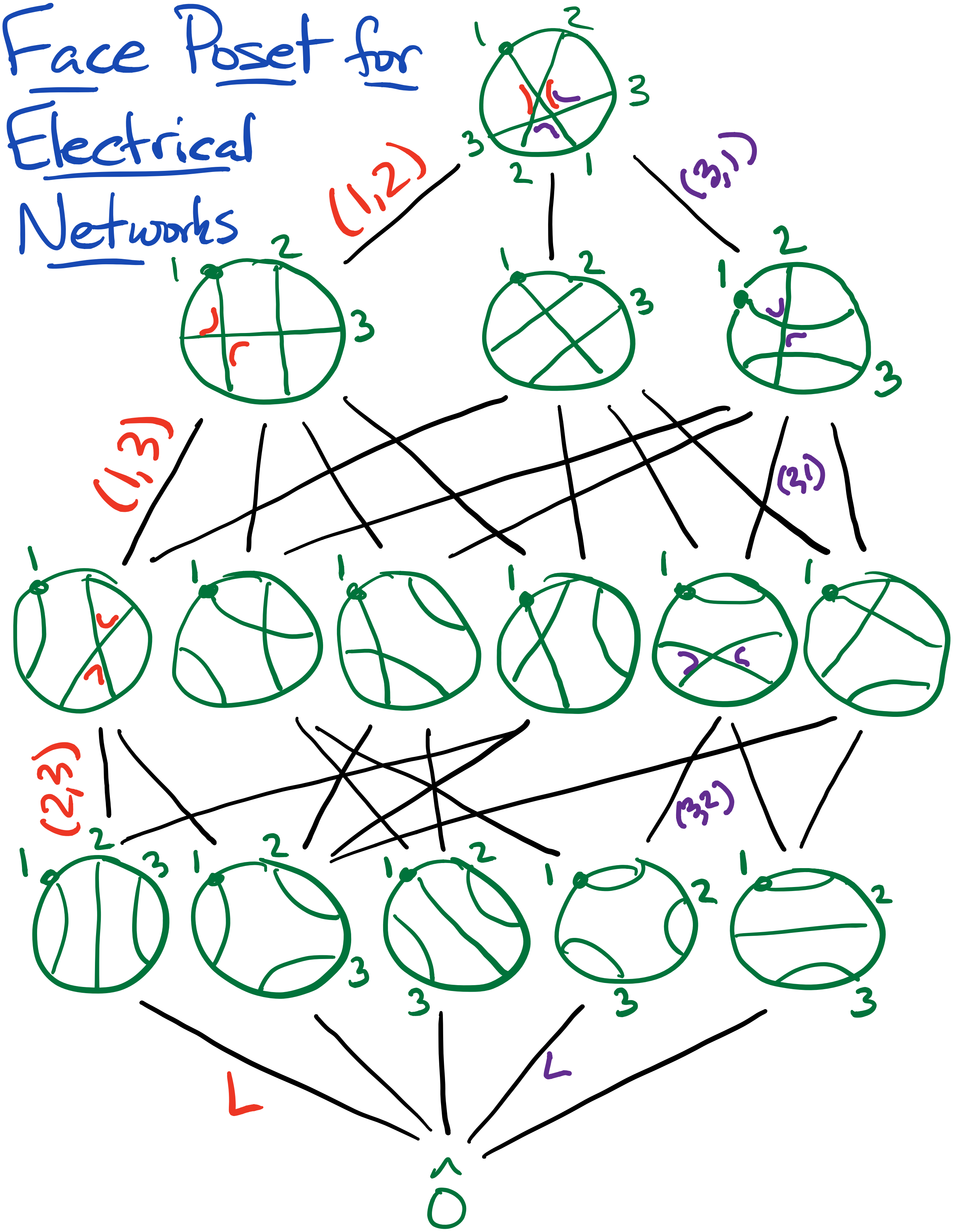
(5) P admits CL-labeling with "UE property"

(6) P admits TCL-labeling with "UE property"

Moreover, all of these implications are obtained constructively.

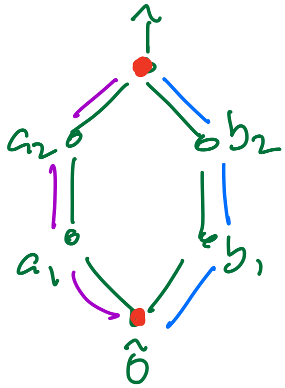
Application: Uncrossing order is dual CL-shellable by virtue of being dual CC-shellable via labeling with UE property

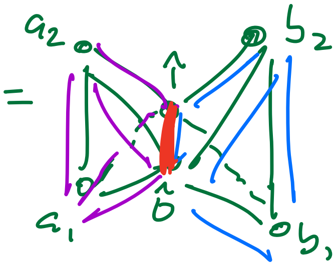
Face Poset for Electrical Networks



Thank you for your attention!

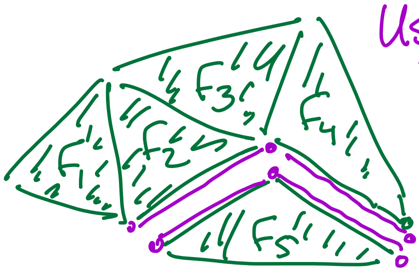
Def'n: The order complex, $\Delta(P)$, of a finite poset P is simplicial complex whose i-dimensional faces are the chains $v_0 < v_1 < \cdots < v_i$ of $i+1$ comparable poset elements.

e.g. $P =$ 

$\Delta(P) =$ 

- pure means all max faces of same dimension

Def'n: A shelling of (not necessarily pure) simplicial complex Δ is total order $F_1, F_2, \ldots, F_k$ on maximal faces of Δ s.t. $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i}\right)$ is pure codimension one subcomplex of $\overline{F_j}$ for $j = 2, 3, \ldots, k$.

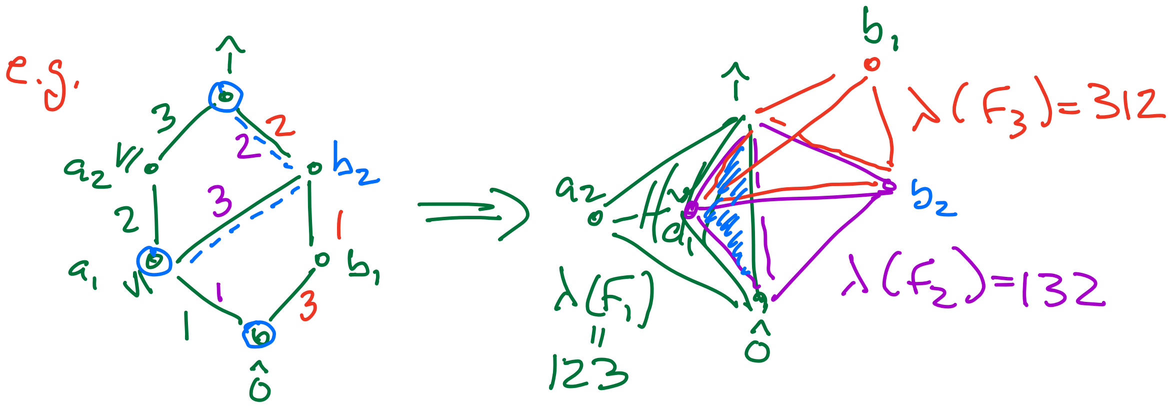


Usefulness:
- Implies homotopy equiv. to wedge of spheres or contractible,

- Calculates Möbius function of P via $\mu_P(u,v) = \tilde{\chi}(\Delta_P(u,v))$

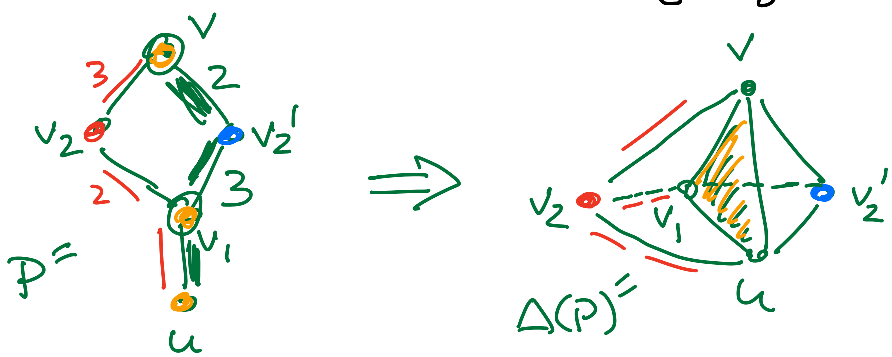
Thm (Björner, Björner–Wachs): EL-labeling of $P \Rightarrow$ Shelling for $\Delta(P)$ by ordering label sequences lexicographically, breaking ties in any manner.

e.g.

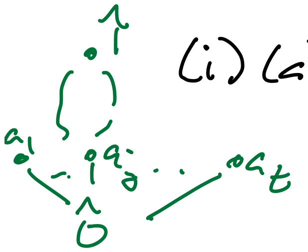


$123 <_{lex} 132 <_{lex} 312$

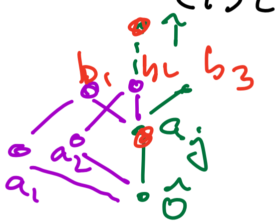
Idea: descent in consec. labels of chain $\leftrightarrow$ codim. one face shared with earlier facet (by replacing descent with lex. earlier ascending sequence)



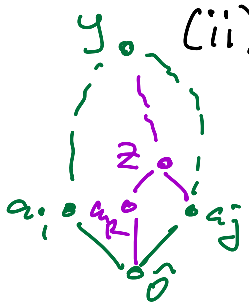
Defn (Björner-Wachs): A recursive atom ordering (RAO) on a finite bounded poset P of length > 1 is a total order $a_1, a_2, \dots, a_t$ on the atoms of P s.t.:



(i)(a) For all $j = 1, \dots, t$, the closed interval $[a_j, \hat{1}]$ admits an RAO $b_1, b_2, \dots, b_s$



(i)(b) For $j \neq 1$, the first atoms $b_1, \dots, b_i$ in RAO $b_1, \dots, b_s$ for $[a_j, \hat{1}]$ are those b_ℓ s.t. $a_i < b_\ell$ for some atom a_i with $i < j$.



(ii) For each $i < j$ and for each $y \in P$ s.t. $a_i < y$ & $a_j < y$, there exists $k < j$ & $z \in P$ s.t. $a_j \lessdot z \leq y$ and $a_k < z \leq y$.

Any P of length 1 has an RAO. 

Thm (Björner & Wachs): A finite bounded poset P is CL-shellable $\iff$ P admits RAO.

A Goal: Given a graph G, study the space of response matrices as image of

$$f: \left\{\begin{matrix}\text{conductance}\\ \text{vectors}\end{matrix}\right\} \to \left\{\begin{matrix}\text{response}\\ \text{matrices}\end{matrix}\right\}$$

$$\left\{\text{conductance vectors}\right\} = \left(\mathbb{R}_{\geq 0} \cup \{\infty\}\right)^{|E|}$$

Note:

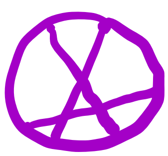
contracting an edge $\rightsquigarrow$ sending conductance to ∞ (i.e. resistance to 0)

deleting an edge $\rightsquigarrow$ sending conductance to 0 (resistance to ∞)

Secondary Goal: Study fibers of f

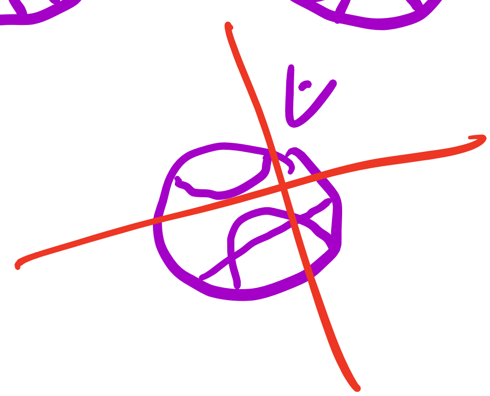
The Uncrossing Poset (Face Poset for Electrical Networks)

- $\hat{1}$:= wire diagram w/ all $\binom{n}{2}$ crossings of n wires



- $u \lessdot v$ if u obtained from v uncrossing pair of wires without introducing double crossing

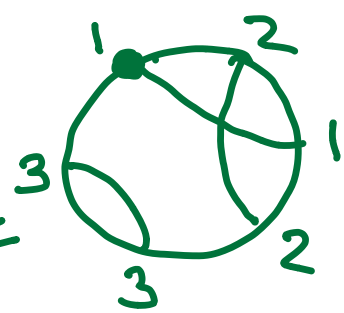
- $\hat{0}$ adjoined below Catalan many atoms



"Start Sets" and Connection to Type A Bruhat Order

The start set of D, denoted $S(D)$, is the subset of $\{1,2,\ldots,2n\}$ of positions in $w(D)$ where 1st copies of letters occur.

e.g.

$D =$ 

$w(D) = \underline{1}\,\underline{2}\,1\,2\,\underline{3}\,3$

$S(D) = \{1,2,5\}$

Prop'n: If $D' < D$ and $S(D') = S(D)$, then $[D', D] \cong [\underbrace{\pi(D'), \pi(D)}_{\text{subwords of } w(D') \,\&\, w(D)}]_{\text{Bruhat}}$

- uncrossing order labeling coincides here w/ Dyer's Bruhat order labeling

Label Ordering

- labels $\{(i,j) \mid 1 \le i < j \le n\} \cup \{L\}$
  $\cup \{(j,i) \mid 1 \le i < j \le n\}$

- $(i,j) < L < (r,s)$ for all $i<j$
  and all $r>s$

- $(1,2) < (1,3) < (1,4) < \ldots < (1,n) < (2,3)$
  $< (2,4) < \ldots < (2,n) < (3,4) < \ldots < (3,n)$
  $< \ldots < (n-1,n)$

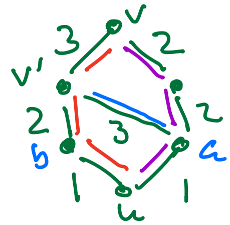
- $(n,n-1) < (n,n-2) < (n-1,n-2) < (n,n-3)$
  $< (n-1,n-3) < (n-2,n-3) < \ldots$
  $< (2,1)$

Rk: finite type A "reflection order" $\{(i,j)\}$ then L, then reversal for $\{(j,i)\}$

"CC-Analogue" of CL-shellable $\iff$ RAO

Def'n (H.-Stadnyk): A chain-labeling for P is **self-consistent** if: for a the atom in lex 1st saturated chain of $[u,v]_r$ & b any other atom of $[u,v]_r$, then for $v' > u$ s.t. $a, b \in [u,v']$ all saturated chains of $[u,v']_r$ containing a have lex smaller label sequence than all saturated chains of $[u,v']_r$ containing b.

e.g. not

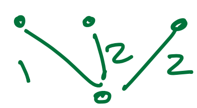


u to v': $12 <_{lex} 13$

u to v: $122 <_{lex} 123$

Def'n (H.-Stadnyk): A chain-labeling has **UE property** if each $[u,v]_r$ has smallest label that appears on any $u \lessdot a$ only appears on one such cover relation.

e.g.



Prop'n: UE property $\Rightarrow$ self-consistent

Prop'n: All CL-labelings have UE property