

Fibers of Maps to Totally Nonnegative Spaces

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- joint work with Tim Davis
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- slides at: <https://pages.uoregon.edu/plhersh/ICERM-Sept25.pdf>

Defn: A real matrix is **totally nonnegative (TNN)** if all its minors are nonnegative.

Thm (Whitney in type A; Lusztig for semisimple simply conn. alg groups)

The unipotent TNN matrices

$\left\{ \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & t \\ & 0 & & \ddots \end{pmatrix} \right\}$ are products of exponentiated Chevalley generators

how these look in type A

$$x_i(t) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & t \\ & 0 & & \ddots \end{pmatrix} \text{ for } t \in \mathbb{R}_{\geq 0}$$

type A

$\leftarrow$ row i

$\uparrow$ column $i+1$

Today: When are products of $x_i(t)$'s equal?

Map Whose Fibers We Study:

$$f_{(i_1, \dots, i_d)} : \mathbb{R}_{\geq 0}^d \rightarrow TNN(\underbrace{U_n}_{\text{unipotent radical}})$$

$$(t_1, \dots, t_d) \mapsto x_{i_1}(t_1) x_{i_2}(t_2) \dots x_{i_d}(t_d)$$

e.g. $f_{(1,2,1)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 + t_3 & t_1 t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix}}_{x_1(t_1)} \underbrace{\begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix}}_{x_2(t_2)} \underbrace{\begin{pmatrix} 1 & t_3 \\ & 1 \end{pmatrix}}_{x_1(t_3)}$$

Example of fiber $C \times \mathbb{R}_{\geq 0}$

$$p^{-1}_{(1,2,1,2,1)} \begin{pmatrix} 1 & 12 & 5 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \cong \begin{array}{c} (0, \frac{7}{12}, 12, \frac{5}{12}, 0) \\ \bullet \text{-----} \bullet (5, 1, 7, 0, 0) \\ | \text{ \diagup \diagdown \diagup \diagdown } | \\ \bullet \text{-----} \bullet (5, 1, 0, 0, 7) \\ | \text{ \diagup \diagdown \diagup \diagdown } | \\ \bullet \text{-----} \bullet (5, 0, 0, 1, 7) \\ (0, 0, 5, 1, 7) \end{array}$$

Some Related Past Work

- Lusztig (94): Studied totally nonneg. part in reductive groups (as image of map $f_{(i_1, \dots, i_k)}$) & connected this to canonical bases
- Fomin-Shapiro (00): Results on Bruhat stratification of $\text{im}(f_{(i_1, \dots, i_k)})$ & conjectured it is regular cw ball.
- H. (14): Proof of Fomin-Shapiro Conj.
- Gukshin-Karp-Lam (22): Totally Nonneg part of any flag variety is regular cw ball & F-S Conj via Poincaré Conjecture.
- Loosely related: Positroid varieties, cluster algebras, braid varieties...

Some Motivations:

1. fibers of $f_{(i_1, \dots, i_k)}$ encode nonneg. real relations among exponentiated Chevalley generators in Lie theory
2. "braid relations" among x_i 's:

$$x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}\left(\frac{bc}{a+c}\right)x_i(b+c)x_{i+1}\left(\frac{ab}{a+c}\right)$$

tropicalize to change of coords

$$(a, b, c) \mapsto \left(b+c, \min(a, c), a+b, -\min(a, c) \right)$$

for Lusztig's dual canonical bases.

Stratf. we use: for each $p \in \text{TNN}(U_n)$ the stratification on $\mathbb{R}_{\geq 0}^d$ based on which coords. are positive vs. 0 induces stratf. for $f_{(i_1, \dots, i_k)}^{-1}(p) \cap \mathbb{R}_{\geq 0}^d$

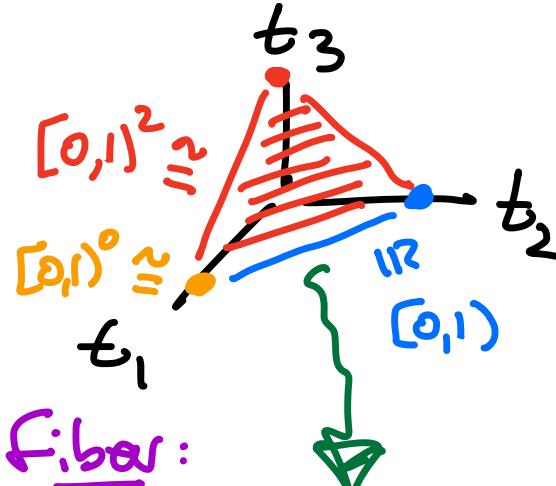
Baby Example of Fiber ($\neq$ how we Think About It)

$$\underbrace{x_1(t_1)}_{\begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix}} \underbrace{x_1(t_2)}_{\begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix}} \underbrace{x_1(t_3)}_{\begin{pmatrix} 1 & t_3 \\ & 1 \end{pmatrix}} = \underbrace{x_1(5)}_{\begin{pmatrix} 1 & 5 \\ & 1 \end{pmatrix}}$$

$$\parallel$$

$$x_1(t_1 + t_2 + t_3) = \begin{pmatrix} 1 & t_1 + t_2 + t_3 & 0 \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\parallel$$

$$f_{(1,1,1)}(t_1, t_2, t_3)$$


Natural Description of Fiber:

$$f_{(1,1,1)}^{-1}(x_1(5)) = \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 \mid \sum_{i=1}^3 t_i = 5 \right\}$$

More Useful Description (for param. & cell decomposition)

$$f_{(1,1,1)}^{-1} \left(\underbrace{\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}}_{(x_1(s))} \right)$$

" $t_2^{\max}$ " continuous fn of t_1

$$\{(t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 \mid \begin{cases} 0 \leq t_1 \leq 5 \\ 0 \leq t_2 \leq \boxed{5 - t_1} \\ t_3 = \boxed{5 - t_1 - t_2} \end{cases} \}$$

Sample Stratum:

$$0 < t_1 < 5$$

$$0 < t_2 < 5 - t_1$$

$$t_3 = 5 - t_1 - t_2$$

" f_3 " cont fn of t_1 & t_2

Observation: t_3 uniquely determined by t_1 & t_2 because rightmost s_i in $s_1 s_1 s_1$ is in rightmost reduced word for $s_1 = s(s_1, s_1, s_1)$ in nonreduced $s_1 s_1 s_1$.

- A **cell decomposition** of topol. space X is decomp. into disjoint union of cells, namely pieces homeom. to $(0,1)^S \cong \mathbb{R}^S$ for various $S \geq 0$

e.g. 
 $\mathbb{R}^0 \text{ --- } \mathbb{R}^1 \text{ --- } \mathbb{R}^0$

- A **cell stratification** is cell decomp. with $\sigma \cap \bar{\tau} \neq \emptyset \Rightarrow \sigma \subseteq \bar{\tau}$
- The **face poset** of a cell stratific. is partial order on cells with $\sigma \leq \tau \iff \sigma \subseteq \bar{\tau}$

e.g. 
 $\sigma_1 \text{ --- } \tau \text{ --- } \sigma_2 \Rightarrow \sigma_1 \text{ --- } \tau \text{ --- } \sigma_2$

Main Results for fibers

Combinatorial:

- stratification has same face poset as interior dual block complex of subword complex
- these interior dual block complexes are contractible

Topological

- each stratum is homeomorphic to $(0,1)^S$ for some $S \geq 0$.
- parametrizations for collections of strata using $(0,1)^S$

Conjectural

- $f_{(q-id)}^{-1}(p)$ is contractible regular CW complex.

How we Decide Whether Products have same Minors Positive

$$x_i(t) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & t \\ & & & \ddots \\ & & & & 1 \end{pmatrix} = I_n + t E_{i,i+1}$$

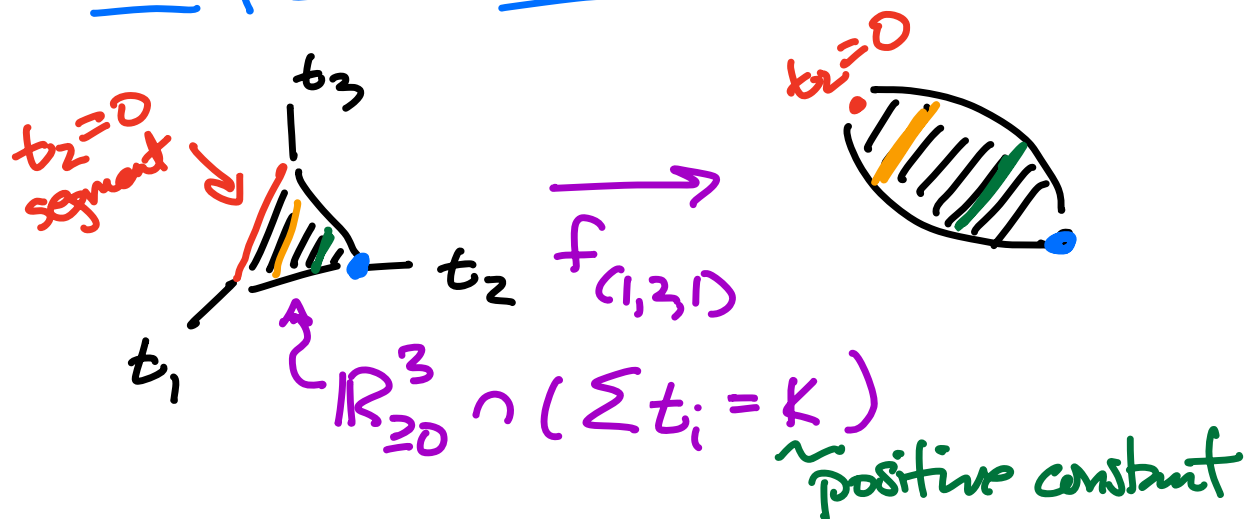
(type A)

$$f_{(\underbrace{i_1, \dots, i_d}_{\text{reduced or nonreduced word}})}(t_1, \dots, t_d) = x_{i_1}(t_1) \cdots x_{i_d}(t_d)$$

e.g.

$$\begin{aligned} f_{(1,2,1)}(t_1, t_2, t_3) &= \begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t_1+t_3 & t_1 t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix} \end{aligned}$$

Example Continued



$$f_{(1,2,1)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ & 1 \end{pmatrix}$$

$t_2=0$

$$f_{(1,-,1)}(t_1, 0, t_3) = \begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 \\ & 1 \end{pmatrix} \} x_1(t_1+t_3)$$

$$\{x_1(t) | t > 0\} = \{x_1(t_1)x_1(t_3) | t_1, t_3 > 0\}$$

$$\Downarrow$$

$x_i = x_i^2$ (sets map to same station)

Demazure Product δ (Equivalently
Unsigned O-Hecke Product) Governs
Which Minors are Positive

$$\bullet x_i(t_1) x_i(t_2) = x_i(t_1 + t_2)$$

$$x_i x_i \rightarrow x_i$$

"modified
nil move"

$$\leadsto \delta(s_i, s_i) = s_i$$

$$\bullet x_i(t_1) x_{i+1}(t_2) x_i(t_3) = x_{i+1}(t'_1) x_i(t'_2) x_{i+1}(t'_3)$$

$$\text{for } t'_1 = \frac{t_2 t_3}{t_1 + t_3}; \quad t'_2 = t_1 + t_3; \quad t'_3 = \frac{t_1 t_2}{t_1 + t_3}$$

$$x_i x_{i+1} x_i \rightarrow x_{i+1} x_i x_{i+1} \quad \text{"braid move"}$$

$$\leadsto \delta(s_i, s_{i+1}, s_i) = \delta(s_{i+1}, s_i, s_{i+1})$$

$$\bullet x_i(t) x_j(u) = x_j(u) x_i(t)$$

$$\leadsto \delta(s_i, s_j) = \delta(s_j, s_i) \text{ for } |j-i| > 1$$

The **Demazure product** for
Coxeter group W satisfies

$$\delta(s_{i_1}, s_{i_2}, \dots, s_{i_d}) = \begin{cases} \delta(s_{i_2}, \dots, s_{i_d}) & \text{if} \\ l(u) < l(s_{i_1}u) & \\ s_{i_1} \delta(s_{i_2}, \dots, s_{i_d}) & \text{otherwise} \end{cases}$$

$u := \delta(s_{i_2}, \dots, s_{i_d})$

e.g. $\delta(1, 2, 1, 2, 1) = \delta(2, 1, 2, 1)$
 $= \delta(1, 2, 1) = s_1 \delta(2, 1)$
 $= s_1 s_2 \delta(1) = s_1 s_2 s_1$

Fact: $f_{(i, -u)}(\mathbb{R}_{>0}^Q) = f_{(i, -id)}(\mathbb{R}_{>0}^{Q'})$

$$\Leftrightarrow \delta(Q) = \delta(Q')$$

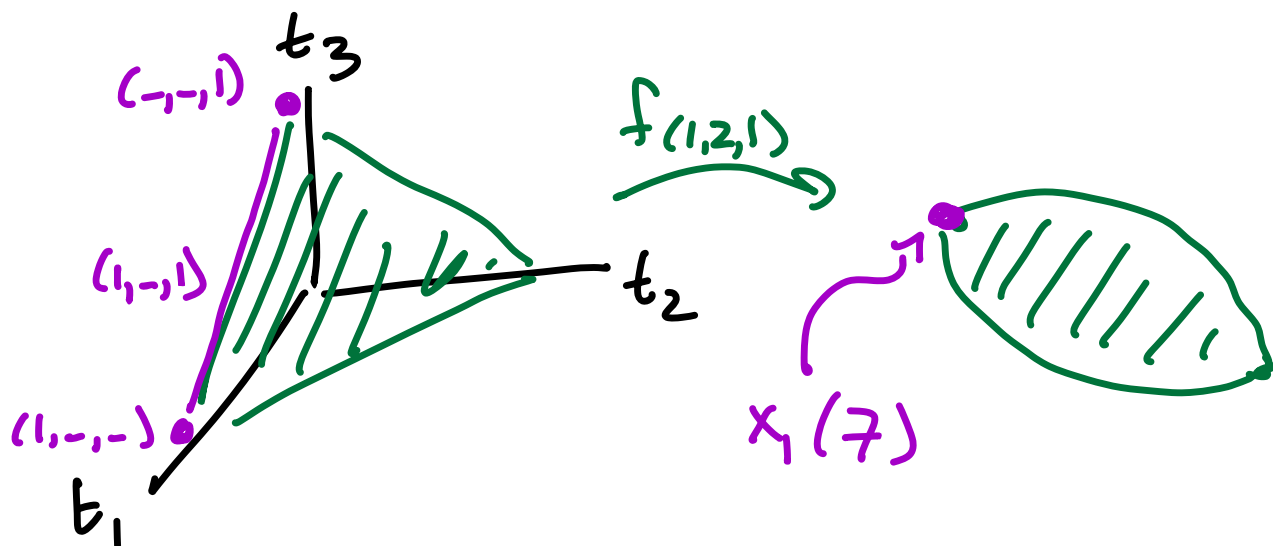
Notation: $U(\omega) = f_{(i_1, \dots, i_d)}(\mathbb{R}_{>0}^Q)$

for $\omega = \delta(Q)$.

$[B\omega B \cap \text{unipotent subgp of } B]_{\geq 0}$

e.g. $f_{(1,2,1)}^{-1} \left(\begin{matrix} 1 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right)$

has nonempty strata given by
subwords $(1, -, -) \frac{1}{2} (1, -, 1) \frac{1}{2} (-, -, 1)$
of $(1, 2, 1)$ since $x_1(7) \in U(s_1)$



Thm (Lusztig): (a) For $(i_1, \dots, i_d)$ reduced
 $\nexists \omega = \delta(i_1, \dots, i_d)$, $f_{(i_1, \dots, i_d)}: \mathbb{R}_{>0}^d \rightarrow U(\omega)$
 is homeomorphism.

(b) $U(\omega) \cap U(\omega') = \emptyset$ for $\omega \neq \omega'$.

A Key Step in Cell Stratif. for Fibers:

Substantially generalize (a) above,

e.g. show that map

$$(t_1, t_2, t_3, t_4) \mapsto x_4(1) x_2(5) \underline{x_1(t_1)} x_1(3) \underline{x_2(t_2)} x_1(t_3) \underline{x_2(t_4)}$$

$\cap$
 $\mathbb{R}_{>0}^4$

rightmost red. word for

$\delta(4, 2, 4, 1, 2, 1, 2) = s_4 s_2 s_1 s_2$ in $(4, 2, \underline{4}, \underline{1}, \underline{2}, \underline{1}, \underline{2})$

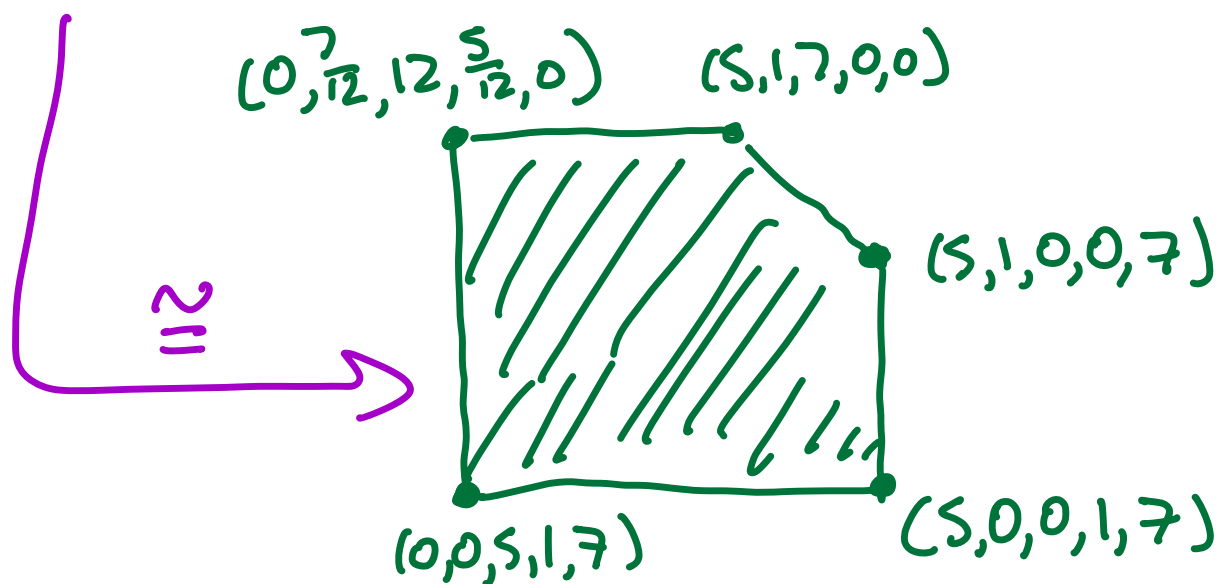
is homeomorphism from $\mathbb{R}_{>0}^4$ to its image,
 in particular is injective

Example of Fiber (Revisited):

$$f_{(1,2,1,2,1)}(t_1, t_2, \dots, t_5) = \begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} \dots$$

$$= \begin{pmatrix} 1 & t_1 + t_3 + t_5 & t_1 t_2 + t_1 t_4 + t_3 t_4 \\ & 1 & t_2 + t_4 \\ & & 1 \end{pmatrix}$$

$$f_{(1,2,1,2,1)}^{-1}(M) \text{ for } M = \begin{pmatrix} 1 & s_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & s_3 \\ & 1 \end{pmatrix}$$



vertices $\leftrightarrow$ reduced subwords
for s, s_2, s_1 in $(1, 2, 1, 2, 1)$

Subword Complexes & their Interior Dual Block Complexes

Defn (Knutson-Miller): The subword complex $\Delta(Q, w)$ has vertices given by letters in word Q & facets the complements of subwords that are reduced words for w .

e.g.

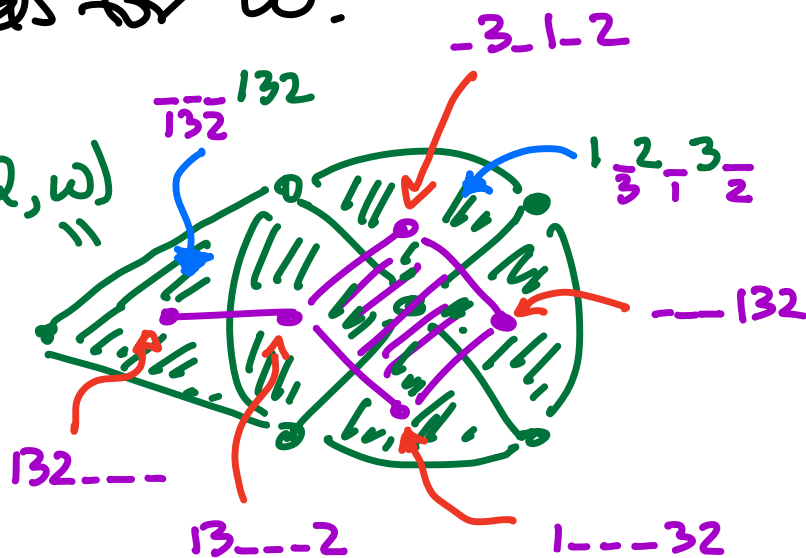
$$Q = (1, 3, 2, 1, 3, 2)$$

$$w = s_1 s_3 s_2$$

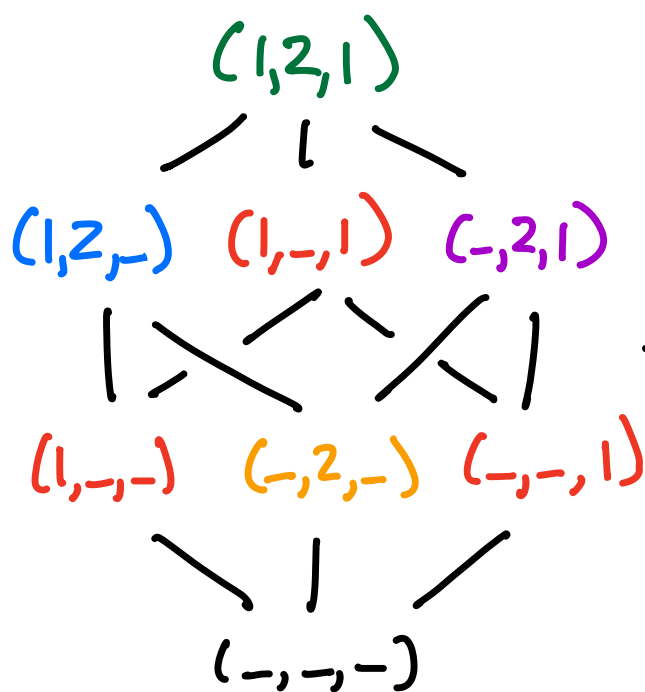
with

its interior

dual block complex in purple



Map f of Face Posets Induced by Map $f_{(i_1 \rightarrow i_e)}$ of Topol. Spaces



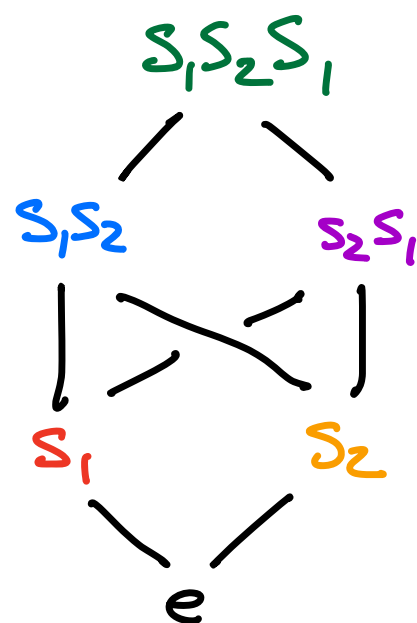
Boolean lattice

face poset of simplex

"Downward product"

$$f_{(i_1 \rightarrow i_e)} = \delta(s_{i_1, \dots, i_e})$$

e.g. $f(1, -, 1) = \delta(s_1, s_1) = s_1$



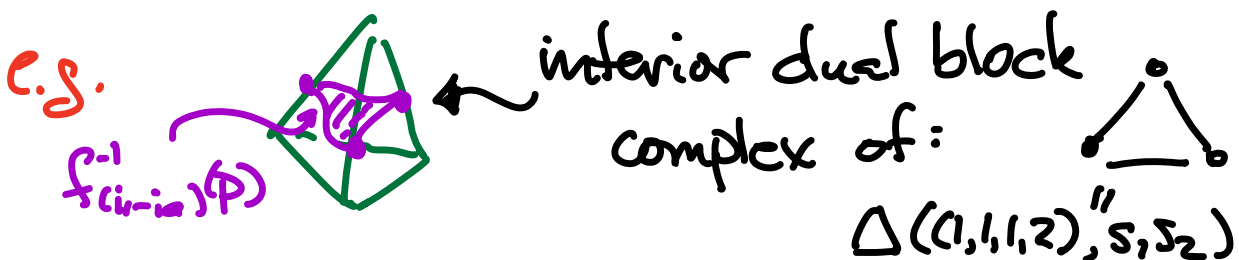
Brhat order
face poset of
 $TNN(U_n)$



Thm (DHM): For each $u \in W$,
 $f^{-1}(u) := \{x \in B_n \mid f(x) = u\}$ is
 face poset for interior dual block
 complex of $\Delta((i_1, \dots, i_d), u)$.

Thm (DHM): The interior dual
 block complex (IDBC) of every
 submanifold complex is regular CW
 complex & is contractible.

Technical Detail: If sphere, then
 include dual to empty face in I.D.B.C.



Properties of Subword

Complexes We'll Use

Knutson-Miller: Each subword complex is "vertex decomposable", hence "shellable" $\neq$ pure (all max faces same dim.) $\neq$ homeomorphic to ball or sphere.

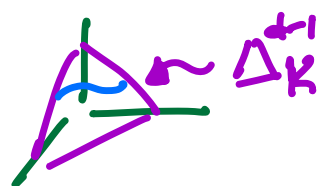
Corollary: They are gallery connected, i.e. max faces connected thru codimension one faces.



Application:

$$f_{(i, -i_d)}^{-1}(p) \cap R_{\geq 0}^d = f_{(i, -i_d)}^{-1}(p) \cap \Delta_K^{d-1}$$

for some $K > 0$



coord.
Sum K
Part of
 $R_{\geq 0}^d$

Proof:

- show change of coord's

$$(t_1, \dots, t_d) \mapsto (t'_1, \dots, t'_d) \text{ for}$$

braid & modified nil moves preserve
sum of parameters

e.g. $x_1(t_1)x_1(t_2) \mapsto x_1(t_1+t_2) \quad K = t_1+t_2$

- use gallery-connectedness of
subword complexes to show
state connected by braid &
modified nil-moves

Idea for Parametrizing Cell Collections

- Consider rightmost subword Q of $(i_1, \dots, i_d)$ that is reduced word for $\delta(i_1, \dots, i_d)$

c.g. $(3, 1, 2, 1, 2, 1, 2)$

- Parametrize the points in those strata satisfying $t_1, t_3, t_6, t_7 > 0$

in $f^{-1}_{(3,1,2,1,2,1,2)}(p) \subset U(S_3 S_2 S_1 S_2)$

using the set $[0, 1)^3 \cong [0, A) \times [0, B) \times [0, C)$
for $A, B, C > 0$

- can do this since

$t_1, t_5, t_6, t_7 > 0 \Leftrightarrow t_2, t_3, t_4$ satisfy:

$$t_2 \in [0, t_2^{\max})$$

$$t_3 \in [0, t_3^{\max}(t_2))$$

$$t_4 \in [0, \underbrace{t_4^{\max}}_{\text{continuous function to } \mathbb{R}_{>0}}(t_2, t_3))$$

e.g.,

continuous
function to $\mathbb{R}_{>0}$

$x_3(t_1) x_1(t_2) x_2(t_3) x_1(t_4) x_2(t_5) x_1(t_6) x_2(t_7)$

choice of
(nonmax.) values $\cong [0, 1)$

values determined
by other params

Cell Stratification: Key Lemmas (+ Definitions)

Def'n: Consider word $(i_1, \dots, i_d)$.

The letter i_ℓ is **redundant** in $(i_1, \dots, i_d)$ if $\delta(i_1, \dots, i_d) = \delta(i_1, \dots, \hat{i}_\ell, \dots, i_d)$ and **nonredundant** otherwise.

e.g. $\delta(\underbrace{1, 2}_{\text{redundant}}, \underbrace{1, 2}_{\text{nonredundant}}) = s_2 s_1 s_2 = \delta(\underbrace{2, 1, 2}_{\text{nonredundant}})$

Note:

(1) i_1 nonredundant $\Leftrightarrow$

$$\delta(i_1, i_2, \dots, i_d) = s_{i_1} \delta(i_2, \dots, i_d)$$

(2) $f_{(i_1, \dots, i_d)}(k_1, \dots, k_d) = p \Leftrightarrow f_{(i_2, \dots, i_d)}(k_2, \dots, k_d) = x_{i_1}(k_1) p$

Lemma: If i_1 is nonredundant in $(i_1, \dots, i_d)$, then there is unique value k_1 for t_1 s.t.

$$f_{(i_1, \dots, i_d)}(k_1, t_2, \dots, t_d) = p$$

has a solution.

This has $x_{i_1}(-k_1)p \in U(\delta(i_2, \dots, i_d))$

Lemma: If i_1 is redundant in $(i_1, \dots, i_d)$, then there exists $t_1^{\max} > 0$ s.t. $f_{(i_1, \dots, i_d)}(k_1, t_2, \dots, t_d) = p$ has solution $\Leftrightarrow k_1 \in [0, t_1^{\max}]$.

Moreover, $k_1 \in [0, t_1^{\max})$

implies $x_{i_1}(-k_1) \in U(\delta(i_2, \dots, i_d))$.

Lemma: Given i_j nonredundant in $(i_j, \dots, i_d) \neq$ given $k_1, \dots, k_{j-1} \geq 0$ s.t.

$$x_{i_{j-1}}(-k_{j-1}) \dots x_{i_j}(-k_j) p \in U(\delta(i_j, \dots, i_d))$$

then there is a unique value

$$k_j = f_j(k_1, \dots, k_{j-1}) \in \mathbb{R}_{\geq 0} \text{ for } t_j \text{ s.t.}$$

$$f_{(i_1, \dots, i_d)}(k_1, k_2, \dots, k_{j-1}, k_j, t_{j+1}, \dots, t_d) = p$$

has solution with $t_{j+1}, \dots, t_d \in \mathbb{R}_{\geq 0}$.

e.g. $f_{(1,2,1,2)}^{-1}(x_1(5) \underbrace{x_2(7) x_1(3)})$

has $t_1 \in [0, 5]$ and

$$\bullet f_2(k_1) = \frac{21}{8-k_1} = "k_2"$$

$$\bullet f_3(k_1, k_2) = 8 - k_1 \quad \bullet f_4(k_1, k_2, k_3) = 7 - k_2$$

$$\begin{pmatrix} 1 & 8 & 35 \\ & 1 & 7 \\ & & 1 \end{pmatrix}$$

Lemma: Given i_ℓ redundant in $(i_1, \dots, i_d)$ $\nexists$ any $k_1, \dots, k_{\ell-1} \geq 0$ s.t.

$$x_{i_{\ell-1}}(-k_{\ell-1}) - x_{i_1}(-k_1) p \in U(\delta(i_\ell - i_d))$$

then t_ℓ takes exactly the values in $[0, K]$ for some $K > 0$.

" $t_\ell^{\max}(k_1, \dots, k_{\ell-1})$ "

Example: $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & \frac{1}{14} \\ & & 1 \end{pmatrix}$ then

$f_{(1,2,1,2)}^{-1}(M)$

$\{(t_1, t_2, t_3, t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$

achieves every $t_1 \in [0, \frac{e}{14}]$

" $t_1^{\max}$ "

Useful Characterization of which
Parameters are Uniquely Determined
by Parameters to Left

Lemma: $S^c = \{j_1, \dots, j_s\} \subseteq \{1, \dots, d\}$ indexes
 rightmost subword of $(i_1, \dots, i_d)$ which
 is reduced word for $\delta(i_1, \dots, i_d)$

$\Longleftrightarrow S^c = \{j \in [d] \mid i_j \text{ is nonredundant in } (i_j, \dots, i_d)\}$

e.g. $(1, 2, 1, 2, 4, 1, 5, 2, 4, 5)$ $d=10$


$S^c = \{4, 6, 7, 8, 9, 10\}$ since $(2, 1, 5, 2, 4, 5)$
 is rightmost reduced word for $\delta(1, 2, 1, \dots, 4, 5)$

$S = \{1, 2, 3, 5\}$ Notation: $S = \{j'_1, \dots, j'_{d-s}\}$

Domain for Homeom. from $[0,1]^S$ to Union of States

Notation: $S^c = \{j_1, \dots, j_{\ell(\omega)}\}$ s.t.
 $(i_{j_1}, \dots, i_{j_{\ell(\omega)}})$ is rightmost reduced
 word for $\omega = s(i_1, \dots, i_d)$ in $(1, \dots, i_d)$.

$$D^{<\max} := \left\{ (t_1, \dots, t_d) \mid t_j < t_j^{\max} (t_1, \dots, t_{j-1}) \right. \\ \left. \text{for all } j \in S \right\}$$

Thm (DHM): $D^{<\max} = \bigcup_{\sigma \in \text{Stats } \overline{\sigma}} \sigma$
 s.t. $\forall v \in \overline{\sigma}$
 $[0,1]^{d-\ell(\omega)}$ for v of support S^c

Cor: Each statum $\cong (0,1)^S$ for some $S \geq 0$

Generalization of Lusztig Result

Lemma: Given $S(i_1, \dots, i_d) = w \neq 1$ $D_{k_{j_1}, \dots, k_{j_s}} =$

$$\left\{ (t_1, \dots, t_d) \in \mathbb{R}_{>0}^d \mid \begin{array}{l} t_{j'_e} > 0 \text{ for } j'_e \in S^c \\ t_{j_r} = k_{j_r} \text{ for } j_r \in S \end{array} \right\}$$

for any fixed $k_{j_1}, \dots, k_{j_s} \geq 0$, then

$f_{(i_1, \dots, i_d)}|_{D_{k_{j_1}, \dots, k_{j_s}}}$ is a homeomorphism to its image within $U(w)$.

e.g. $\in D_{3,7} \cong \mathbb{R}_{>0}^4$

$\in U(s_4 s_1 s_2 s_1)$

$(t_2, t_4, t_5, t_6) \mapsto x_1(3)x_4(t_2)x_2(7)x_1(t_4)x_2(t_5)x_1(t_6)$

Consequence: Within $D^{<max}$,

redundant parameter values determine values for nonredundant parameters

First Example, Revisited

$$x_1(t_1)x_1(t_2)x_1(t_3) = x_1(5)$$

$$f_{(1,1,1)}''(t_1, t_2, t_3)$$

Constraints for $D^{\max}$ part of Fiber

- $0 \leq t_1 < t_1^{\max} = 5$
- $0 \leq t_2 < t_2^{\max}(k_1) = 5 - k_1$
- $t_3 = 5 - k_1 - k_2 = f_3(k_1, k_2)$

$$[0,1)^2 \cong \sum_{\substack{\sim \\ [0,1)}} (k_1, k_2, k_3) \left| \begin{array}{l} 0 \leq k_1 < 5 \\ 0 \leq k_2 < 5 - k_1 \\ k_3 = 5 - k_1 - k_2 \end{array} \right. \right\}$$

this uses continuity of $t_2^{\max} \doteq f_3$

Continuity Lemmas

Lemma: Given i_ℓ non-redundant in $(i_1, \dots, i_d)$ and $k_1, \dots, k_{\ell-1} \geq 0$ s.t.

(1) $k_i < t_i^{\max}(k_1, \dots, k_{i-1}) \forall i \in S \neq \emptyset$

(2) $\exists (k_1, \dots, k_{\ell-1}, t_\ell, \dots, t_d) \in f_{(i_1, \dots, i_d)}^{-1}(p) \cap \mathbb{R}_{\geq 0}$

then f_ℓ is continuous function, letting $f_\ell(k_1, \dots, k_{\ell-1})$ be the unique value for t_ℓ in $f_{(i_1, \dots, i_d)}^{-1}(p) \cap \mathbb{R}_{\geq 0}^d$ s.t. $t_i = k_i$ for $i=1, 2, \dots, \ell-1$.

e.g. $f_{(1,2,1,2)}^{-1}(x_1(5)x_2(7)x_1(3))$

• $t_1^{\max} = 5$

• $f_3(k_1, k_2) = 8 - k_1$

• $f_2(k_1) = \frac{21}{8 - k_1}$

• $f_4(k_1, k_2, k_3) = 7 - k_2$
 $= 7 - \frac{21}{8 - k_1}$

Idea: $x_{i_1}(k_1) \dots x_{i_{e-1}}(k_{e-1}) x_{i_e}(t_e) \dots x_{i_d}(t_d) = p$



$$x_{i_e}(t_e) \dots x_{i_d}(t_d) = x_{i_{e-1}}(-k_{e-1}) \dots x_{i_1}(-k_1) p$$

$$\text{so } \{(t_e, \dots, t_d) | (k_1, \dots, k_{e-1}, t_e, \dots, t_d) \in f_{(i_e, \dots, i_d)}^{-1}(p)\}$$

||

$$f_{(i_e, \dots, i_d)}^{-1}(\underbrace{x_{i_{e-1}}(-k_{e-1}) \dots x_{i_1}(-k_1) p}_{\text{continuous function of } k_1, \dots, k_{e-1} \text{ for fixed } p})$$

so f_e is coordinate fn composed with

$f_{(i_e, Q)}^{-1}$ where (i_e, Q) is reduced subword of $(i_e, \dots, i_d)$

Lemma: If i_e is redundant in $(i_1, \dots, i_d)$, then $t_e^{\max}$ is continuous fn of $k_1, \dots, k_{e-1}$

Proof Idea:

$$t_e^{\max}(k_1, \dots, k_{e-1}) = f_e(k_1, \dots, k_{e-1})$$

for word

$(i_1, \dots, i_d)$

$f_{(i_1, \dots, i_d)}^+(p)$

for subword $(i_1, \dots, i_e, Q)$

$f_{(i_1, \dots, i_e, Q)}^+(p)$ where

(i_e, Q) is reduced word

so f_e cont. $\Rightarrow t_e^{\max}$ continuous

Example: $f_{(1,1,1,2)}(t_1, t_2, t_3, t_4) = x_1(7)x_2(5)$

$$t_2^{\max}(k_1) = 7 - k_1 \quad (\text{for } f_{(1,1,1,2)})$$

$$\parallel \quad f_{(1,1,-,2)}(t_1, t_2, -, t_4) = x_1(7)x_2(5)$$

$f_2(k_1)$ (for $f_{(1,1,2)}$) $(1,1,2)$ nonreduced
 $(1,-,2)$ reduced $Q=(2)$

Remark: We focus on $D^{\max}$ where

$t_\ell < k_\ell^{\max}$ for all $\ell \in S$ because

$t_\ell = k_\ell^{\max}$ part much less well behaved.

Summary: $(0,1)^S \xrightarrow{\cong} \bigcup_{v \leq \bar{\sigma} \leq \bar{\tau}} \sigma$ which

restricts to $(0,1)^S \xrightarrow{\cong} \hat{\tau}$, showing each stratum is open ball.

An Open Qn: Given word Q , a subword $(i_{j_1} \dots i_{j_\ell})$ that is reduced word for $\delta(Q)$ and constants $\{k_j \geq 0 \mid j \notin \{j_1, \dots, j_\ell\}\}$,

is $(t_{j_1}, \dots, t_{j_\ell}) \mapsto f_Q(u_1, \dots, u_d)$

for $u_j = \begin{cases} t_j & \text{if } j \in \{j_1, \dots, j_\ell\} \\ k_j & \text{otherwise} \end{cases}$ homeom to image?

e.g. $(t_1, t_6) \mapsto x_1(t_1)x_2(3)x_1(2)x_2(7)x_1(\pi)x_2(t_6)$

Equations w/ Unique Solution in $\mathbb{R}_{>0}$ by Lusztig Injectivity Result

$$f_{(1,2)}: (t_1, t_6) \mapsto \begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_6 \\ & 1 \end{pmatrix} = \underbrace{M}_{\text{constant matrix in } U(S_1, S_2)}$$

e.g. $t_1 = M_{12}; \quad t_1 t_6 = M_{13}$
 $t_6 = M_{23}$

Modified Equations for

$$(t_1, t_6) \mapsto \begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t_6 \\ & 1 \end{pmatrix}$$

e.g.

$$\begin{array}{cccccc} \underbrace{x_1(t_1)}_{s_1} & \underbrace{x_2(3)}_{s_2} & \underbrace{x_1(2)}_{s_1} & \underbrace{x_2(7)}_{s_2} & \underbrace{x_1(\pi)}_{s_1} & \underbrace{x_2(t_6)}_{s_2} \\ \text{---} & & \text{---} & & \text{---} & \end{array}$$

$$t_1 + 2 + \pi = \underbrace{M_{12}}_{\text{"S}_1\text{" minor}} \quad 3 + 7 + t_6 = \underbrace{M_{23}}_{\text{"S}_2\text{" minor}}$$

$$\underbrace{t_1 \cdot 3} + \underbrace{t_1 \cdot 7} + \underbrace{t_1 t_6} + 2 \cdot 7 + 2 \cdot t_6 + \pi t_6 = \underbrace{M_{13}}_{\text{"S}_1 \text{S}_2\text{" minor}}$$

Conjecture (Davis-H.-Miller):

$f_{(i, -id)}^{-1}(p) \cap \mathbb{R}_{\geq 0}^d$ is regular

CW complex homeomorphic

to interior dual block

complex of subword complex

$\Delta((i, -id), w)$ where $p \in U(w)$.

Thanks!