Shelling Face Posets of Stratified Spaces of Electrical Networks & of Phylogenetic Trees

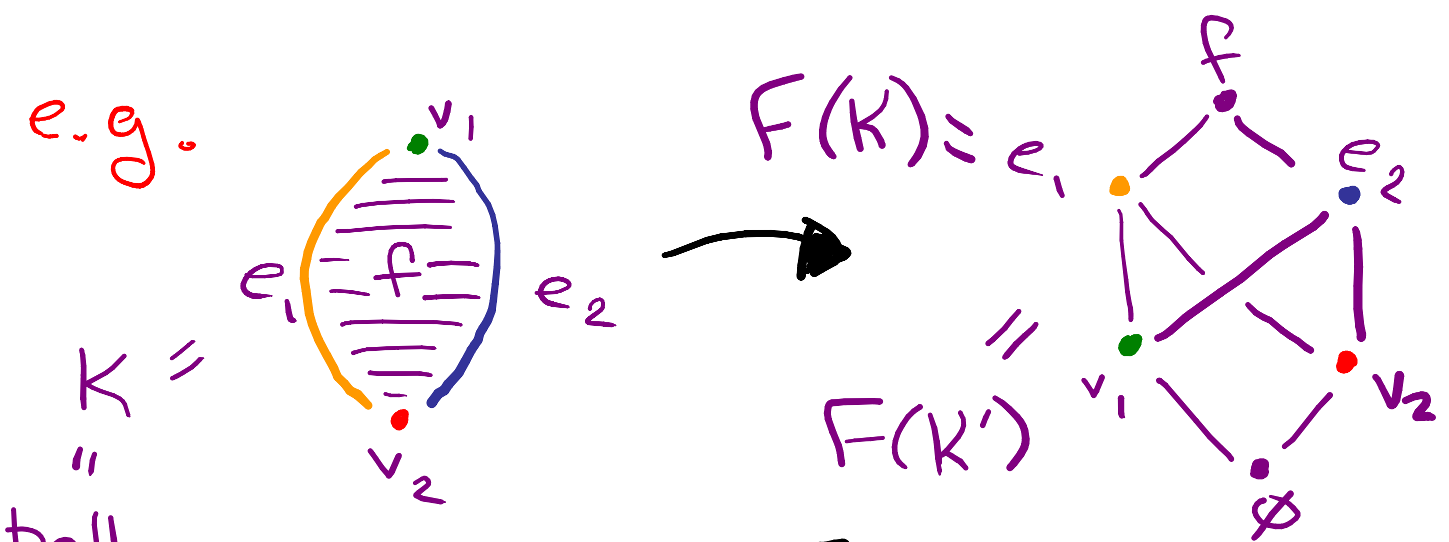
Patricia Hersh
University of Oregon

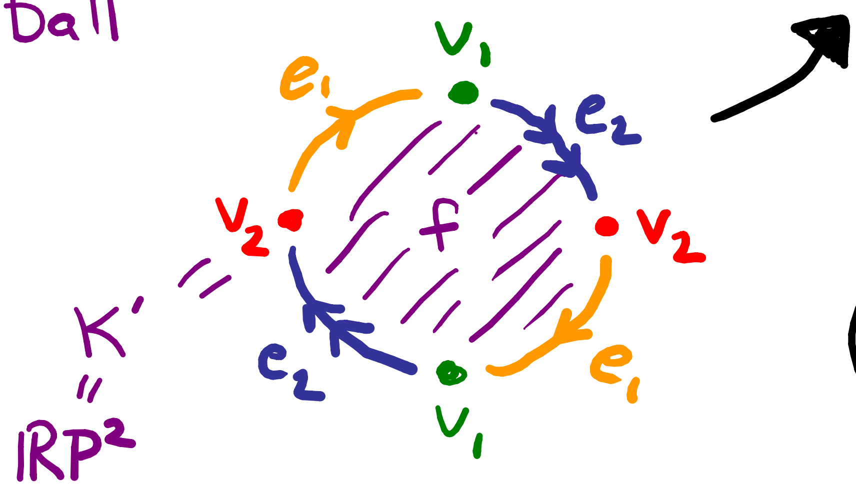
- joint work with Rick Kenyon

(slides at https://pages.uoregon.edu/plhersh/talks.html)

CW Complexes & their Face Posets

e.g.



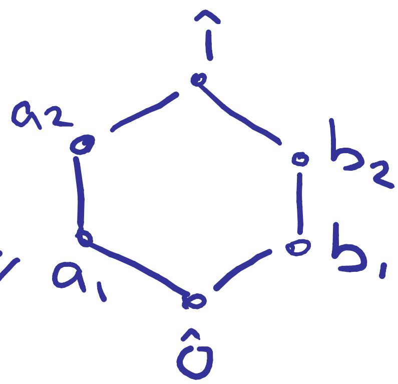


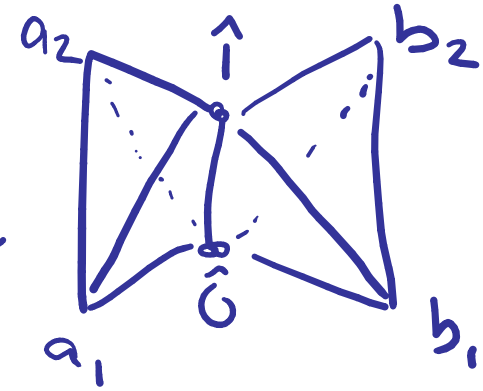
"closure poset"
or "face poset"

$(u \leq v \iff u \leq \bar{v})$

Recall: A CW complex is comprised of open cells each homeomorphic to an open ball. A regular CW complex further has cell closures homeomorphic to closed balls. e.g. simplicial complexes

Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-"chains" $v_0 < \dots < v_i$ in P

e.g. $P =$ 

$\Delta(P) =$ 

Key Property (Hall; popularized by Rota):

$$\mu_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = \begin{array}{l} -1 + \#\text{vertices} \\ \quad - \#\text{edges} \\ \quad + \#\,2\text{-faces} \dots \end{array}$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \dots$$

- $(u,v) = \{ z \in P \mid u < z < v \}$
- $u \lessdot v$ means $u < v$ & $\nexists z$ s.t. $u < z < v$
- saturated chains u to v := $u \lessdot \dots \lessdot v$

Background on Face Posets & CW Posets

- A graded poset with $\hat{0}$ & $\hat{1}$ is **Eulerian** if $\mu(u,v) = (-1)^{rk(v)-rk(u)}$ for all $u \leq v$.

- A graded poset P is a **CW poset** if
  (1) $\hat{0} \in P$
  (2) P has at least one other element
  (3) $\Delta(\hat{0}, u) \cong S^{rk(u)-2}$ for $u \neq \hat{0}$

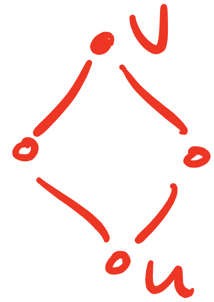
Thm (Björner): P is CW poset $\iff$ there exists regular CW complex with P as poset of closure rel'ns

Cor: CW Poset $\Rightarrow$ Eulerian

Some CW Posets

- all graded, thin, shellable posets (Danaraj-Klee)

"thin" $\iff$



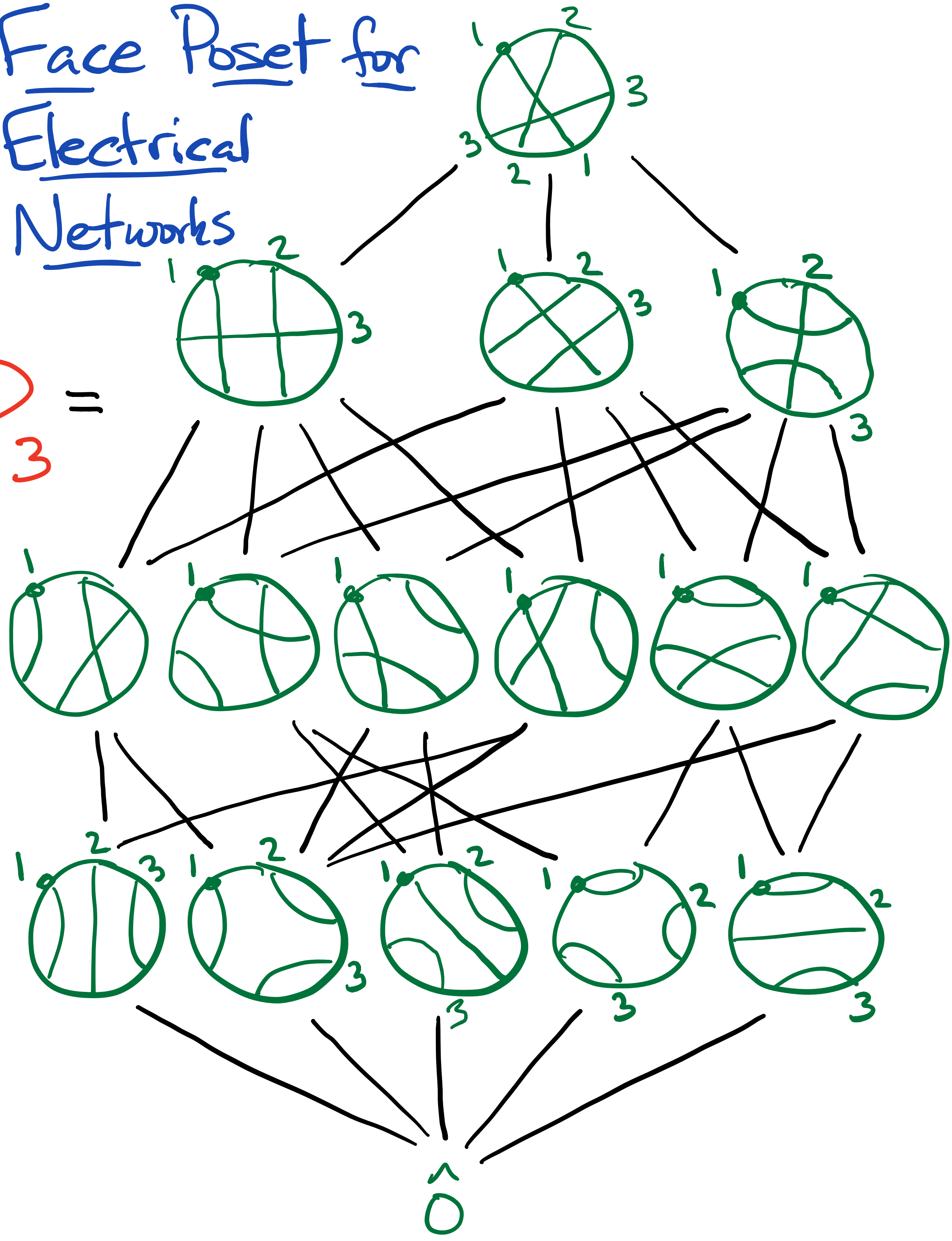
$rk(v) - rk(u) = 2$ then $|[u,v]| = 4$

- Bruhat order (Björner-Wachs; Dyer)

- Face posets of stratified spaces of electrical networks
  - conjectured by Thomas Lam
  - proved by H.-Kenyon (a main topic for today's talk)

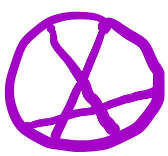
Face Poset for Electrical Networks

$P_3 =$



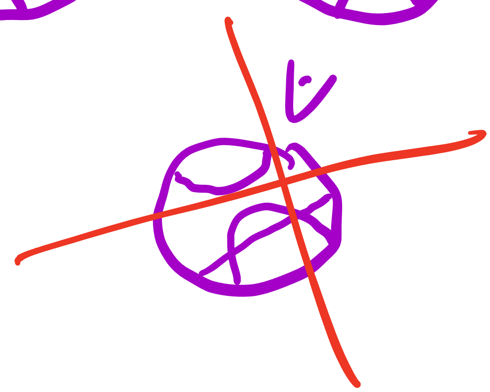
The Uncrossing Poset (Face Poset for Electrical Networks)

- $\hat{1}$:= wire diagram w/ all $\binom{n}{2}$ crossings of n wires



- $u \lessdot v$ if u obtained from v uncrossing pair of wires without introducing double crossing

- $\hat{0}$ adjoined below Catalan many atoms



- P_n := uncrossing poset with n wires

A Conjecture of Thomas Lam

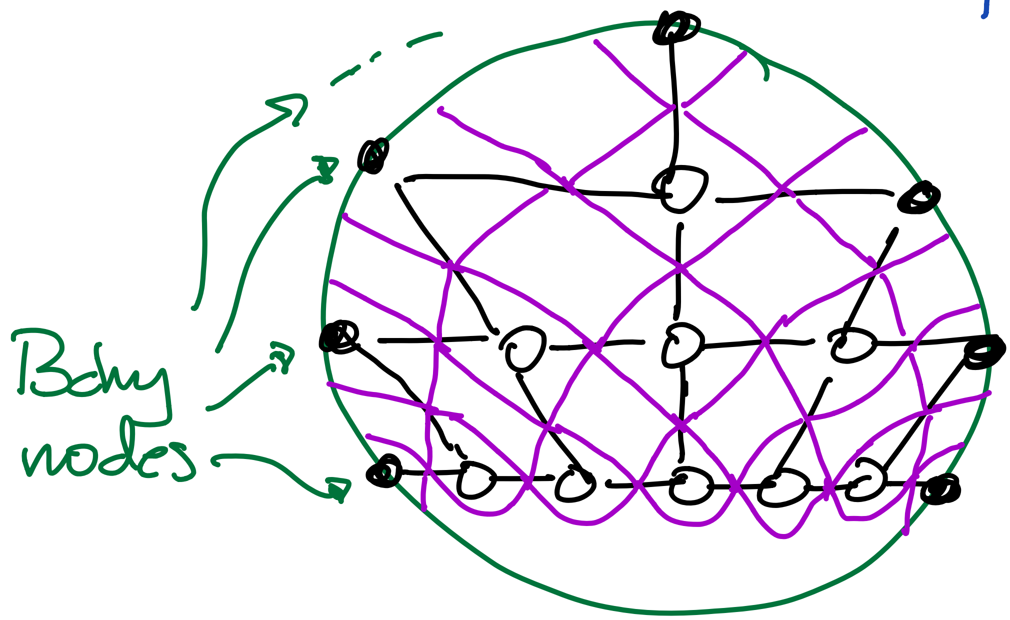
Thm (Lam): The uncrossing poset is Eulerian.

Conjecture (Lam): The uncrossing poset is lexicographically shellable.

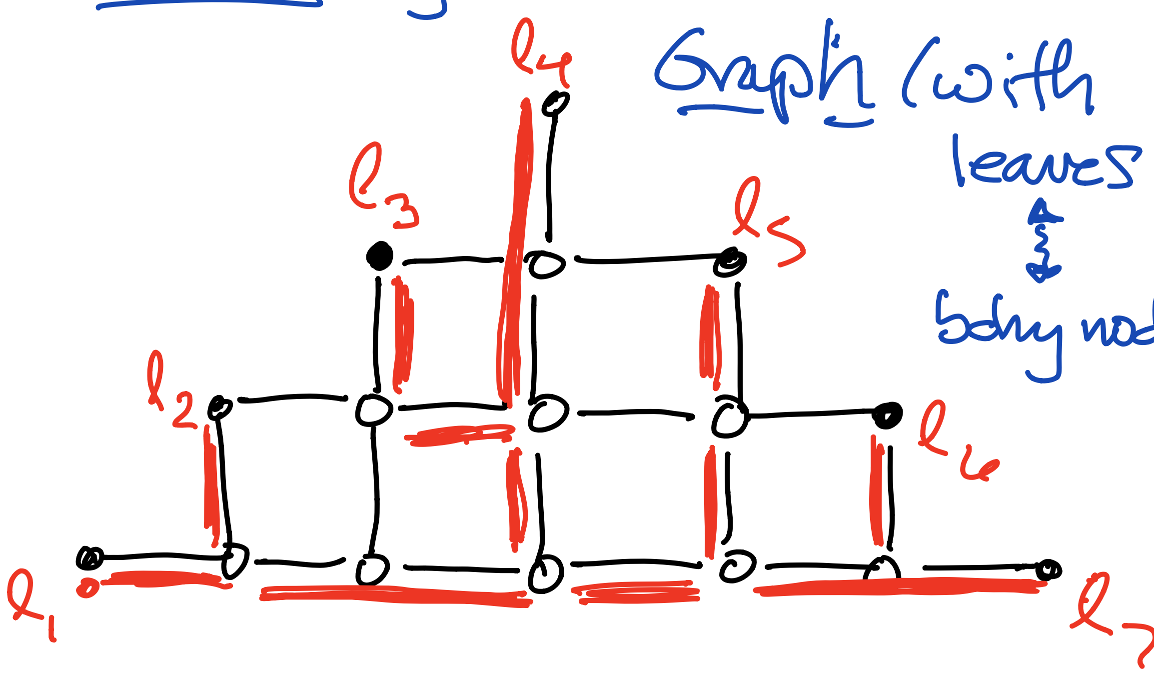
Thm (H.-Kenyon): Uncrossing posets are dual EC-shellable.

Cor: They are CW posets.

Dual Graph for $\hat{1} \in P_7$

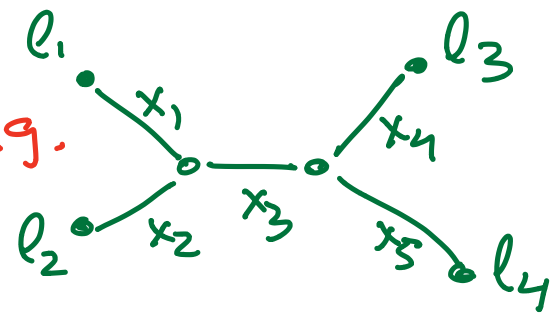


Embedding Tree in Such Dual Graph (with leaves $\leftrightarrow$ bdry nodes)



Edge Product Space, E_n, of Phylogenetic Trees with n leaves

Step 1: Given tree T with (independent) edge probabilities $x_1, x_2, \ldots, x_m$ & leaves $l_1, l_2, \ldots, l_n$,

e.g. 

may calculate vector $m_T(x_1, \ldots, x_m) \in (0,1]^{\binom{n}{2}}$ of probabilities for each pair of leaves that path between them is fully present

e.g. $(\underbrace{x_1x_2}_{p_{12}}, \underbrace{x_1x_3x_4}_{p_{13}}, \underbrace{x_1x_3x_5}_{p_{14}}, \underbrace{x_2x_3x_4}_{p_{23}}, \underbrace{x_2x_3x_5}_{p_{24}}, \underbrace{x_4x_5}_{p_{34}})$

Step 2: Define edge-product space E_n for trees with n leaves as

$$\bigcup_{T \text{ trees}} \mathrm{im}(\underline{m}_T) \subseteq [0,1]^{\binom{n}{2}}$$

w/ leaves $\{l_1, \ldots, l_n\}$ & nonleaves all $\deg \geq 3$

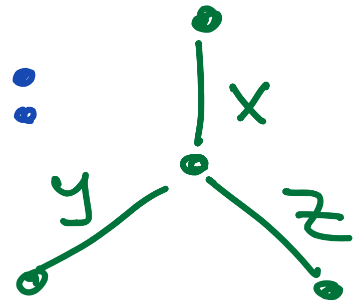
Theorem (Gill-Linusson-Moulton-Steel):
There exists a shelling for each interval in the face poset for E_n, namely the Tuffley poset.

Corollary (Gill-Linusson-Moulton-Steel):
The edge product space of phylogenetic trees is a regular CW complex.

Theorem (Stadnyk): The Tuffley poset in its entirety is not shellable.

Theorem (H.-Kenyon): Each interval in the Tuffley poset is isomorphic to an interval in the uncrossing poset, hence has an explicitly constructed shelling by virtue of the uncrossing poset shelling.

Another Example:



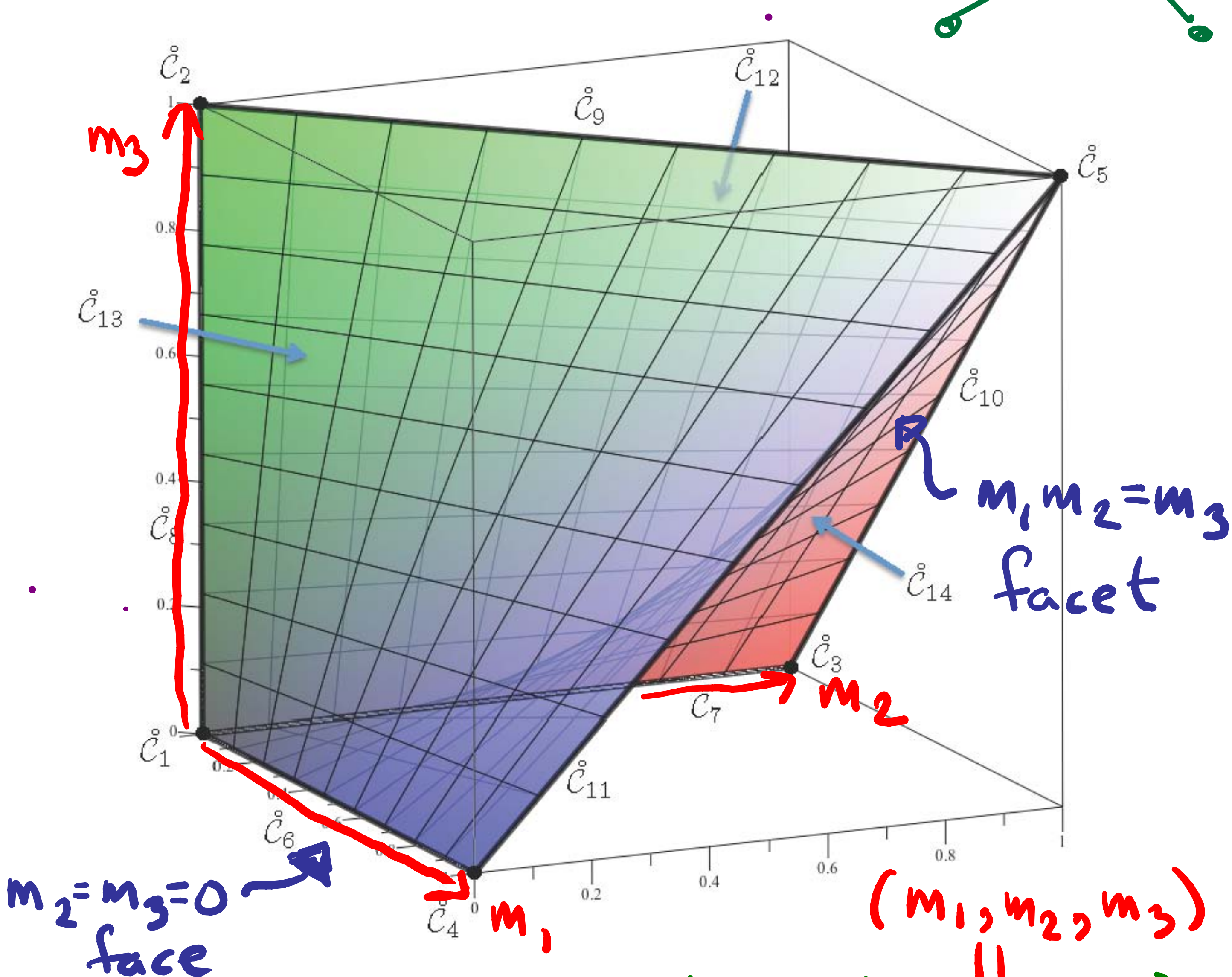
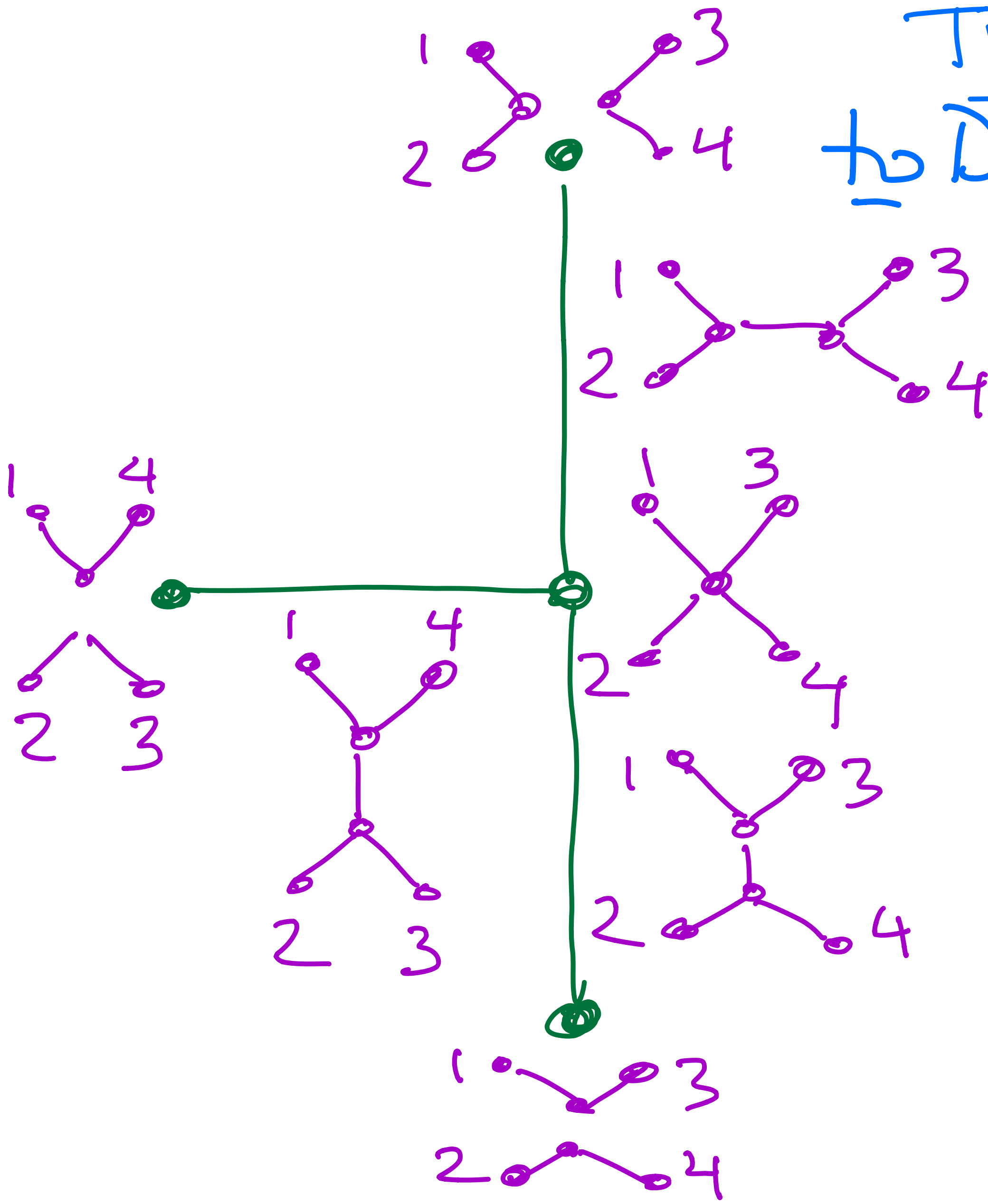


Image of $(x,y,z) \mapsto (xy, xz, yz)$

(bounding surfaces $m_1 m_2 \leq m_3$, $m_1 m_3 \leq m_2$, $m_2 m_3 \leq m_1$ in $[0,1]^3$)

Small Example of Edge Product Space of Phylog. Trees (up to Deconing)



Motivations for Edge Product Space of Phylogenetic Trees

(1) Determining edge lengths from distances between leaves

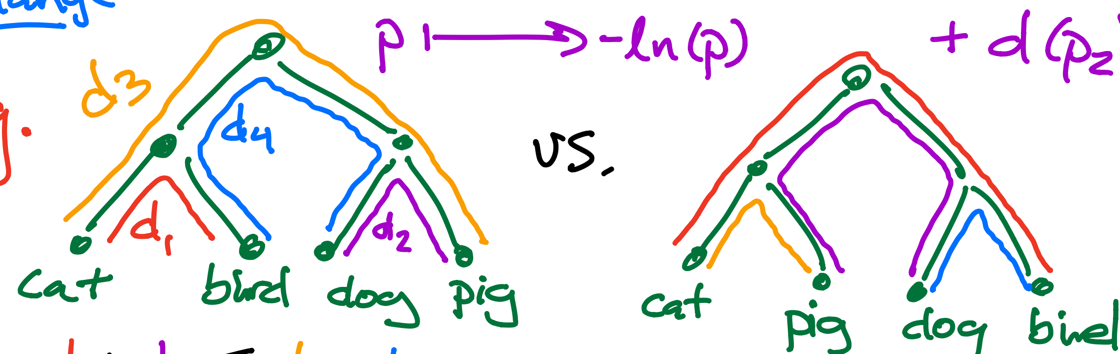
(data measurable from DNA)

(2) Determining tree type (ie. which cell tree lives in)

Convention change: $(0,1] \xrightarrow[\text{homeom}]{d} \mathbb{R}_{\geq 0}$ $\quad p_1 \cdot p_2 \mapsto d(p_1) + d(p_2)$

$p \longmapsto -\ln(p)$

e.g.



$d_1 + d_2 \leq d_3 + d_4$

(1) + (2) $\Rightarrow$ likely phylogenetic history from "distances"

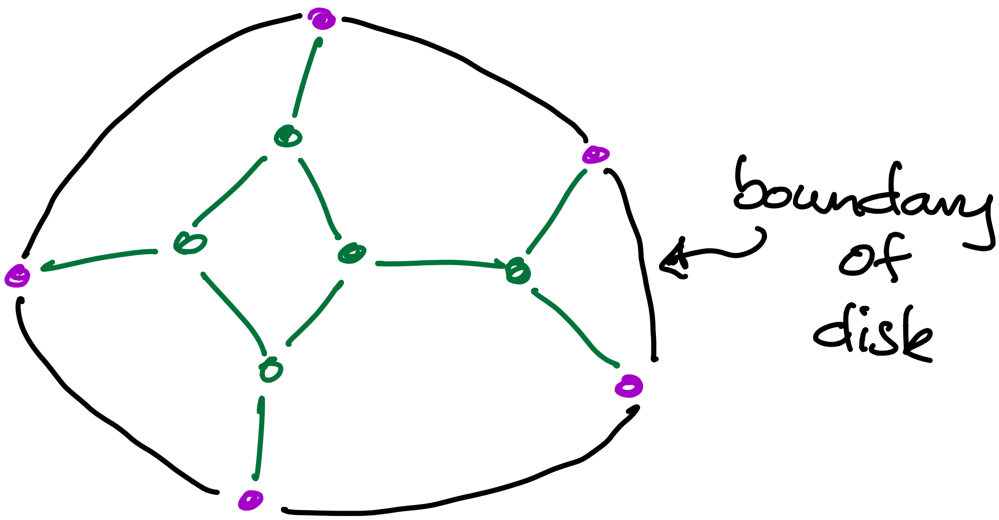
Some Earlier Work on E_n

- Billera-Holmes-Vogtmann: uniqueness of geodesics
- Owen-Provan: efficient algorithm to calculate geodesics
- Moulton-Steel: CW complex
- Ardila-Klivans: homeomorphism (via subdivision) to order complex of partition lattice
- Basu-Gabrielov-Vorobjov: "toric cubes" are regular CW complexes

Note: E_n is compactification of "tree space" in BHV, OP, AK

Space of "Response Matrices" of a Planar Electrical Network

e.g.



I = interior nodes

N = boundary nodes

$$\underbrace{\Delta}_{\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}} \underbrace{\begin{pmatrix} v_N \\ v_I \end{pmatrix}}_{\text{vector of voltages}} = \underbrace{\begin{pmatrix} c_N \\ 0 \end{pmatrix}}_{\text{vector of currents}}$$

$$\underbrace{\left(A - BC^{-1}B^T\right)}_{\text{"response matrix" of the network}} v_N = c_N$$

A Goal: Given a graph G, study the space of response matrices as image of

$$f: \left\{ \begin{matrix} \text{conductance} \\ \text{vectors} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{response} \\ \text{matrices} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \text{conductance} \\ \text{vectors} \end{matrix} \right\} = \left(\mathbb{R}_{\geq 0} \cup \{\infty\}\right)^{|E|}$$

Note:

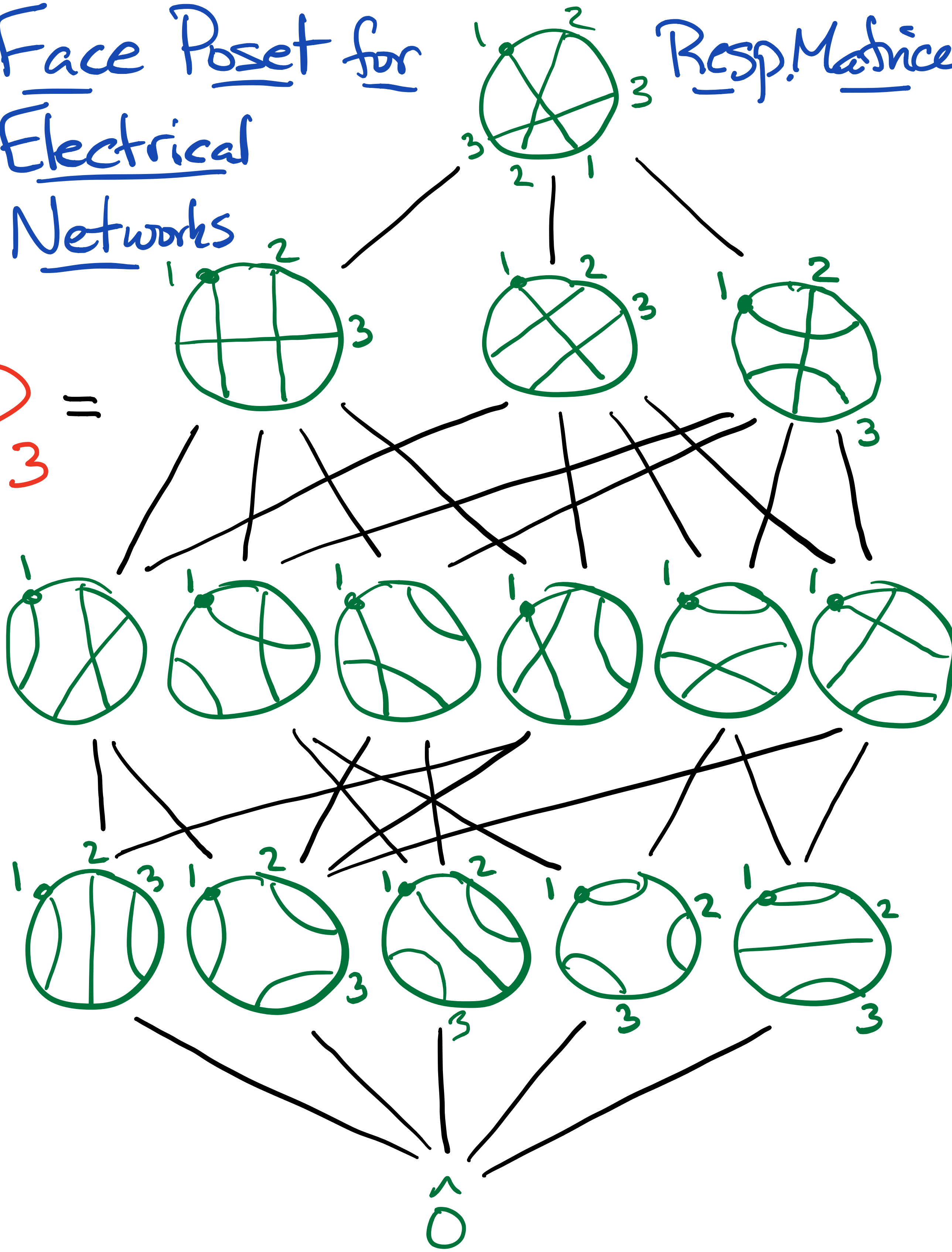
contracting an edge $\rightsquigarrow$ sending conductance to ∞ (i.e. resistance to 0)

deleting an edge $\rightsquigarrow$ sending conductance to 0 (resistance to ∞)

Secondary Goal: Study fibers of f

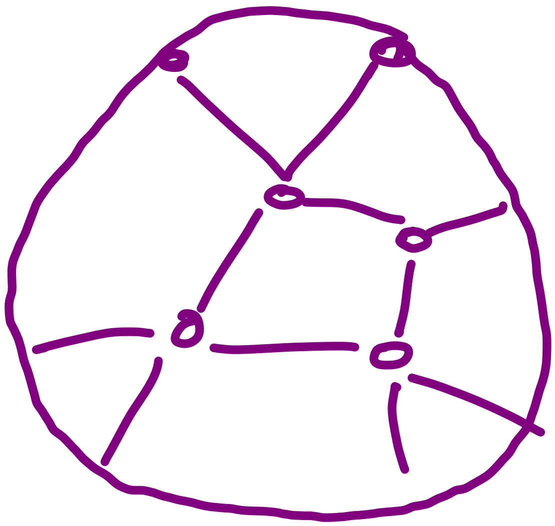
Face Poset for Resp. Matrices for Electrical Networks

$P_3 =$

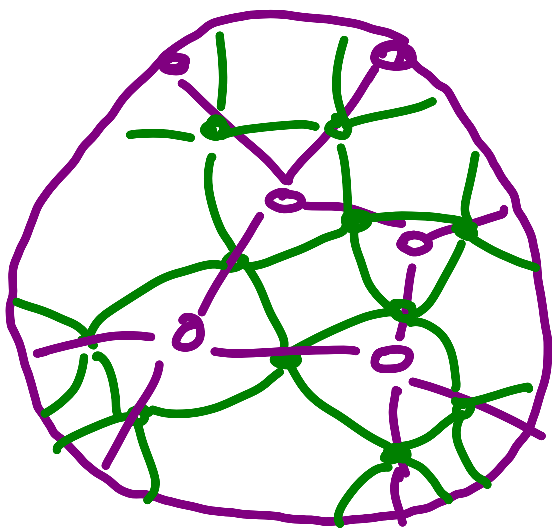


Correspondence: From Graphs to Uncrossing Wire Diagrams "Medial Graphs"

$\Gamma =$

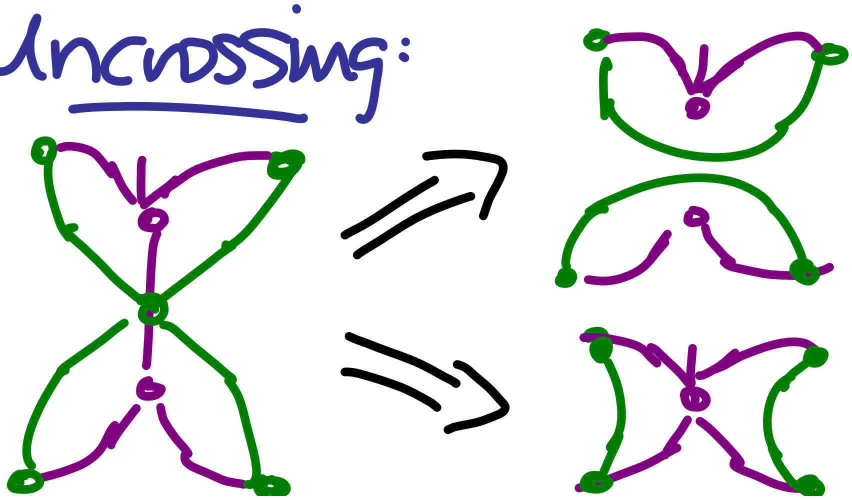


$G(\Gamma) =$



= wire diagram

Uncrossing:



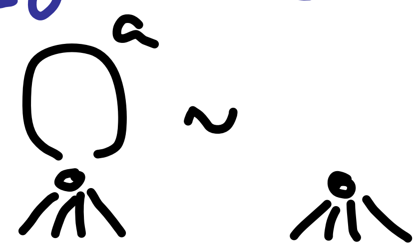
deletion

contraction

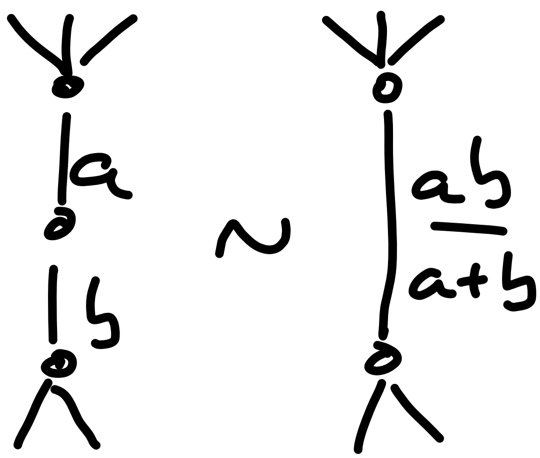
Turning to Fibers via "Electrical Equivalence"

(1)

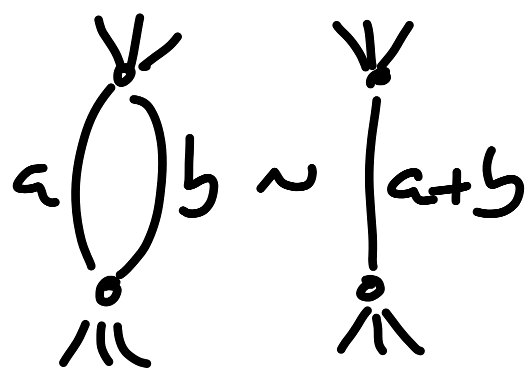
(2)



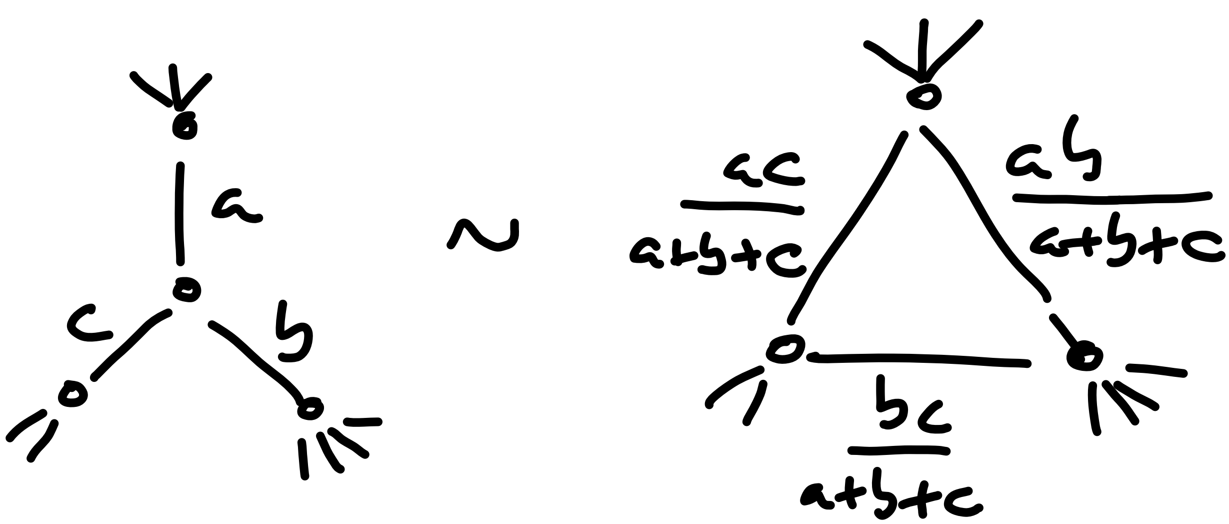
(3)



(4)



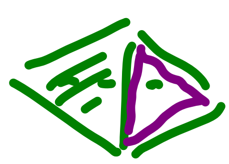
(5) "Y-Δ moves"



Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \dots, F_k$, a **shelling**, on facets s.t. $\bar{F}_j \cap (\bigcup_{i<j} \bar{F}_i)$ is pure, codimension one subcomplex of $\bar{F}_j$ for each $j>1$ (hence is $\partial \bar{F}_j$ or has a cone point).

OR

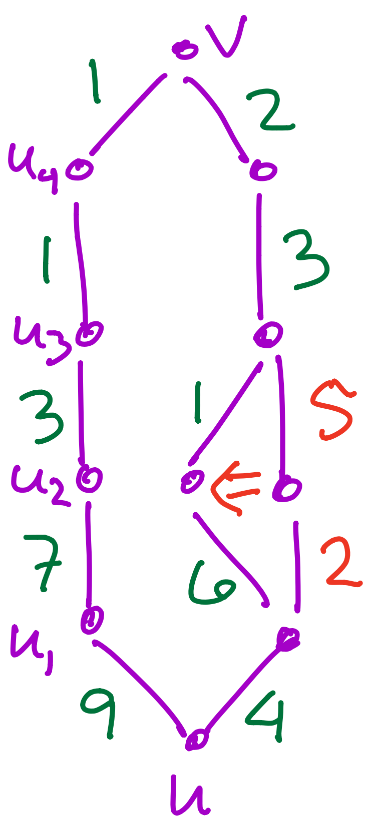


- Each facet attachment preserves homotopy type or closes off a new sphere

Lexicographic Shellability for $\Delta(P)$

(Björner, Björner-Wachs)

A poset is dual EL-shellable if it admits edge labeling λ (called dual EL-labeling) of cover relations $x \lessdot y$ s.t. for each $u < v$ in P:

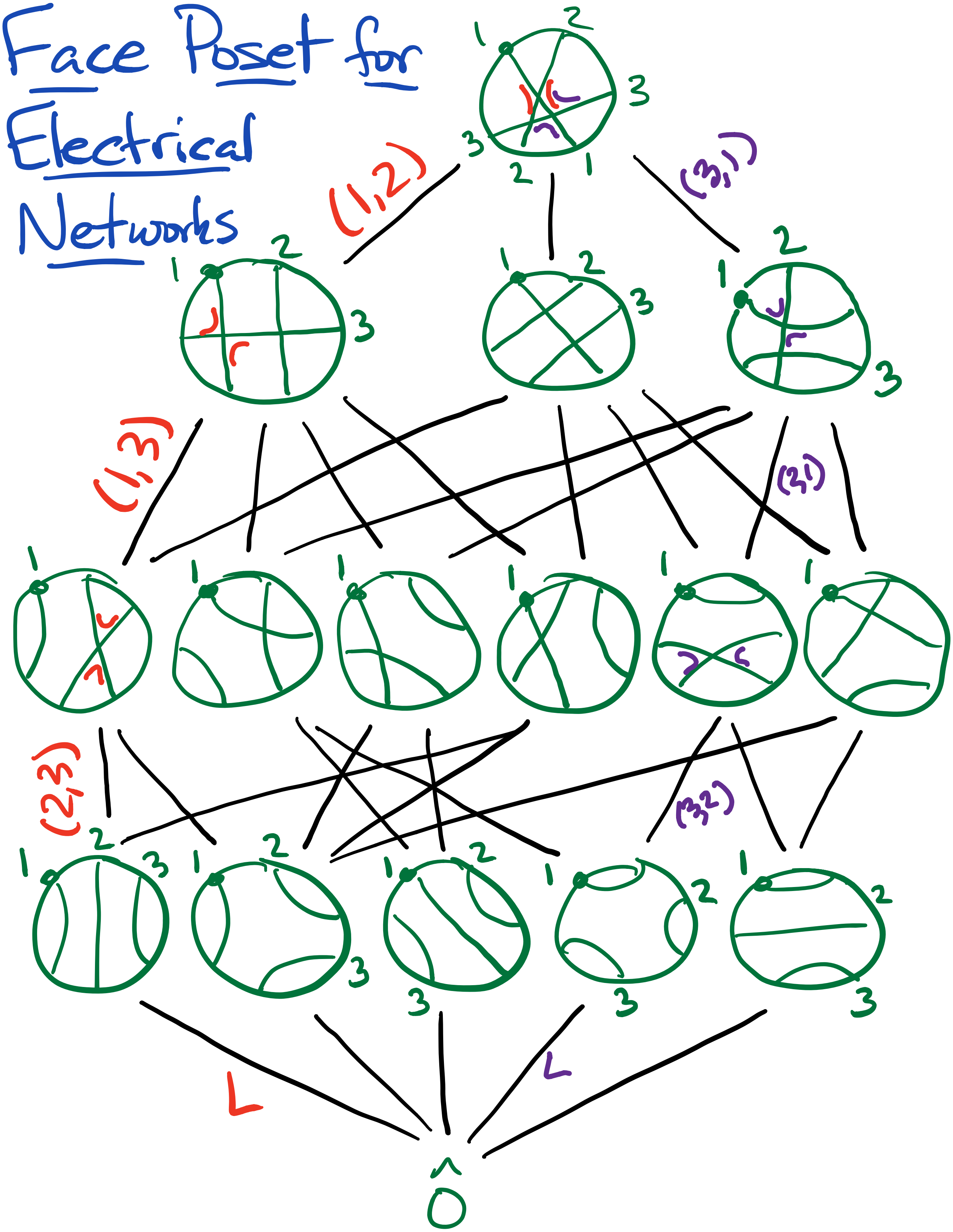


(1) $\exists!$ saturated chain $u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_k \lessdot v$ s.t. $\lambda(u_k, v) \le \lambda(u_{k-1}, u_k) \le \cdots \le \lambda(u, u_1)$

<u>and</u>

(2) $(\lambda(u_k, v), \ldots, \lambda(u, u_1))$ is smaller in dictionary order than all other label sequences from v to u

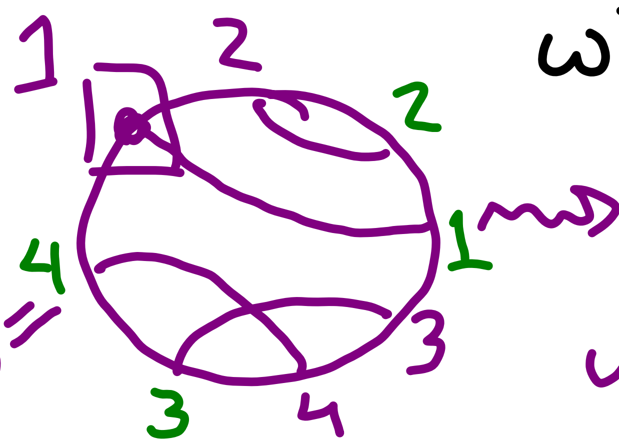
Face Poset for Electrical Networks



Edge Labeling

Step 1: Define **word** of wire diagram D, denoted $w(D)$, as sequence of $2n$ wire endpts encountered clockwise starting with "root" 1.

e.g. $D =$



$12213434 = w(D)$

Step 2: Label $D \lessdot D'$ for $i<j$ as

- (i,j) if $ijij$ in $w(D')$ becomes $ijji$ in $w(D)$
- (j,i) if $ijij$ becomes $iijj$

Label Ordering

- labels $\{(i,j) \mid 1 \le i < j \le n\} \cup \{L\}$
  $\cup \{(j,i) \mid 1 \le i < j \le n\}$

- $(i,j) < L < (r,s)$ for all $i<j$
  and all $r>s$

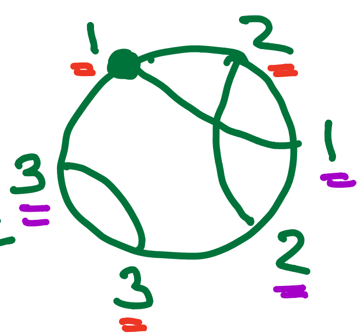
- $(1,2) < (1,3) < (1,4) < \_\_\_ < (1,n) < (2,3)$
  $< (2,4) < \_\_\_ < (2,n) < (3,4) < \_\_ < (3,n)$
  $< \_\_\_ < (n-1,n)$

- $(n,n-1) < (n,n-2) < (n-1,n-2) < (n,n-3)$
  $< (n-1,n-3) < (n-2,n-3) < \_\_\_$
  $< (2,1)$

Rk: finite type A reflection order $\{(i,j)\}$ then L, then reversal for $\{(j,i)\}$

"Start Sets" and Connection to Type A Bruhat Order

The start set of D, denoted $S(D)$, is the subset of $\{1,2,\ldots,2n\}$ of positions in $w(D)$ where 1st copies of letters occur.

e.g. $D =$ 

$w(D) = \underline{1}\,\underline{2}\,1\,2\,\underline{3}\,3$

$S(D) = \{1,2,5\}$

$\pi(D) = 123 \in S_3$

Prop'n: If $D' < D$ and $S(D') = S(D)$, then $[D', D] \cong \underbrace{[\pi(D'), \pi(D)]}_{\text{subwords of } w(D') \,\&\, w(D)}{}_{\text{Bruhat}}$

- uncrossing order labeling coincides here w/ Dyer's Bruhat order labeling

Concluding Remarks

1. Galashin-Karp-Lam recently announced proof that stratified space of response matrices is regular CW complex.
2. Analogy w/ my past work tot. nonneg. part unip. radical of Borel?
3. Work in progress: graphical analogue of subword complexes to describe fibers.

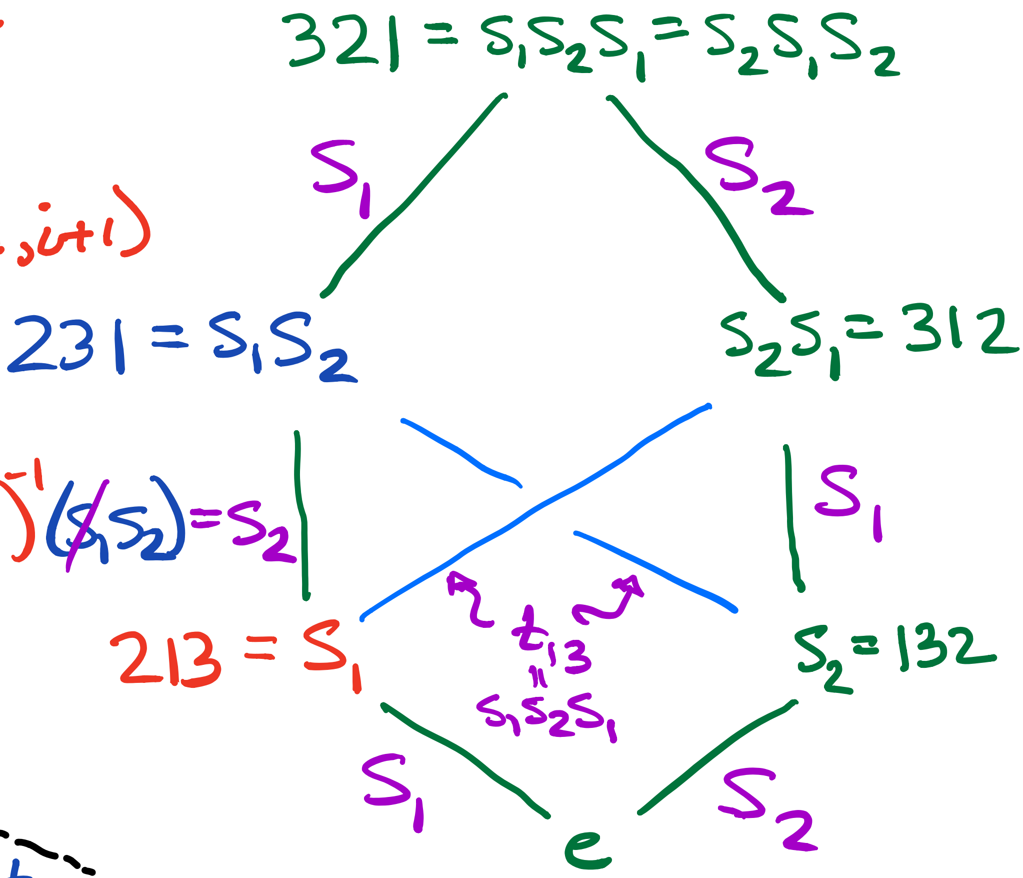
Thanks for your attention!

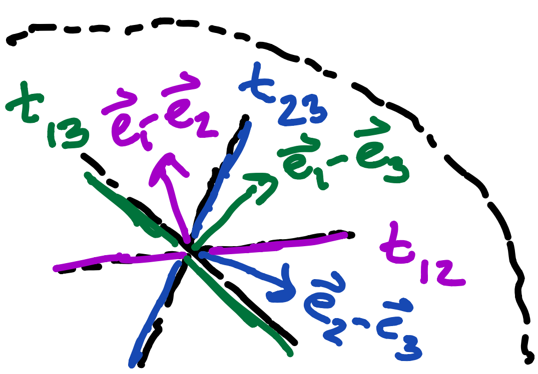
Bruhat Order of a (Finite) Reflection Group / Coxeter Group & Dyer's EL-Labeling

e.g.

$s_i := (i, i+1)$

$(\cancel{s_1})^{-1}(\cancel{s_1}s_2) = s_2$





EL-labeling: $u \lessdot v = ut$

$\lambda(u,v) := u^{-1}v = t$

Dyer's EL-labeling (cont)

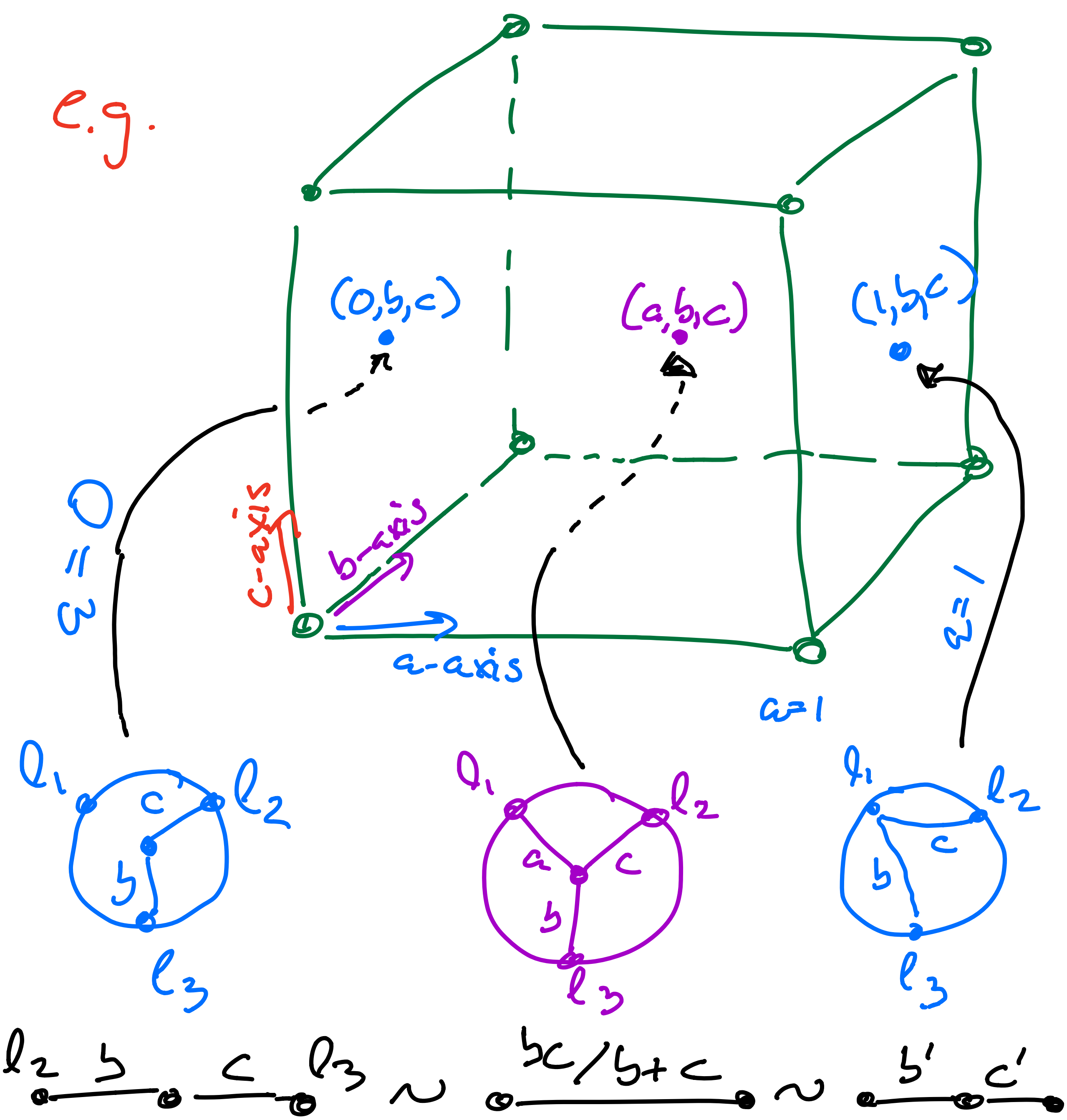
- Use any "reflection order" to totally order edge labels
- Dyer proved these exist & induce EL-labelings

Defn: A total order on positive roots (& assoc'd reflections) is reflection order if $\alpha < c_1\alpha + c_2\beta < \beta$ or $\beta < c_1\alpha + c_2\beta < \alpha$ for each such triple of positive roots w/ $c_1, c_2 > 0$

e.g $(1,2) < (1,3) < (2,3)$ or $(2,3) < (1,3) < (1,2)$ in type A

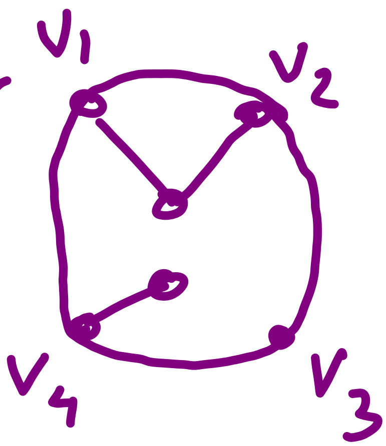
Stratification into "Cells" for Space of Response Matrices

e.g.



Minors of Response Matrix via "Groves"

$\Pi(\underset{\text{graph}}{\underset{\sim}{G}})$:= set partition of bdry nodes into connected components

e.g. $\Pi\Bigg($ 

$\Bigg) = 12|3|4$

Thm (Special Case of Next Result):

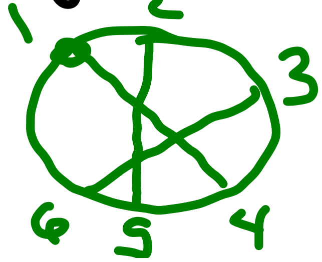
$$L_{ij}(G) = \frac{\displaystyle\sum_{\substack{G' \le G \\ \Pi(G') = ij|\text{singletons}}} wt(G')}{\displaystyle\sum_{\substack{G' \le G \\ \Pi(G') = \text{all singletons}}} wt(G')}$$

$wt(G')$ = Product of edge weights (i.e. conductances)

Lam's Dual Embedding of Uncrossing Order into $\tilde{S}_{2n}$ Bruhat Order

- D_e = fully crossed diagram $\rightsquigarrow$ $1D \in \tilde{S}_{2n}$
  $\shortparallel$
  g_2 s.t.
  $i \to i+n$

- $D \lessdot D'$ where D obtained from D' by i,j wire endpoint swap $\rightsquigarrow$ $g_D = (i,j)\, g_{D'}\, (i,j)$

e.g. $D' =$ 

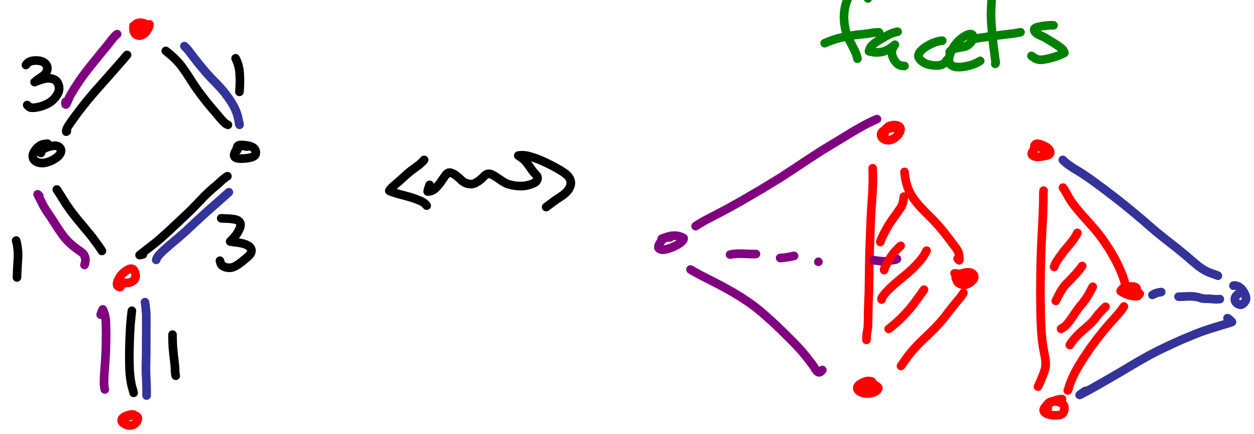
$D =$ 

$$g_D = (1,2)\, g_{D'}\, (1,2) = (4,5)\, g_{D'}\, (4,5)$$

Thm (Björner): EL-labeling $\Rightarrow$ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in labeling" $\leftrightsquigarrow$ codim. one overlap of facets



- "descending" chains $\leftrightsquigarrow$ facets attaching along entire bdry $\leftrightsquigarrow$ spheres
- $\mu_P(u,v) = \pm \#$ descending chains u to v (for P graded)