Cascade Lectures in Combinatorics

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Matroid Type Complexes with Orthogonality Relations

(joint work with Kevin Piterman)

Volkmar Welker

(Universität Marburg)

Recall: MATROID

A matroid over a finite set M is given by its set $\mathcal{I}(M)$ of independent sets satisfying

- $\emptyset \in \mathcal{I}(M)$
- $J \subseteq I \in \mathcal{I}(M) \Rightarrow J \in \mathcal{I}(M)$
- $I, J \in \mathcal{I}(M)$, $\# J > \# I \Longrightarrow$ exists $e \in J \setminus I$ with $I \cup \{e\} \in \mathcal{I}(M)$

Ex: $\mathbb{F}_q$ finite field with q elements

$M = \mathbb{P}\mathbb{F}_q^r = \{\text{lines in } \mathbb{F}_q^r\}$

$$I(\mathbb{P}\mathbb{F}_q^r) = \left\{ \{\mathbb{F}_q v_1, \dots, \mathbb{F}_q v_s\} \;\middle|\; v_1, \dots, v_s \text{ linearly independent} \right\}$$

Recall: SIMPLICIAL COMPLEX

Δ simplicial complex over (finite) ground set Ω if

- $\Delta \subseteq 2^{\Omega}$
- $A \subseteq B \in \Delta \Rightarrow A \in \Delta$

Ex: M matroid $\Rightarrow \mathcal{I}(M)$ simplicial complex

Δ simplicial complex

- $A \in \Delta$ is called a face

  $\dim A = \# A - 1$ dimension of A

- $\dim \Delta = \max_{A \in \Delta} \dim A$

  dimension of Δ

Ex:

$$\dim I(\mathbb{P}\mathbb{F}_q^r) = r-1$$

Δ simplicial complex

- face $A \in \Delta$ is maximal face if $A \subseteq B \in \Delta \Rightarrow A = B$

Ex. maximal faces of $\mathcal{I}(\mathbb{F}_q^r)$ are

$\{\mathbb{F}_q v_1, \ldots, \mathbb{F}_q v_r\}$ for

$v_1, \ldots, v_r$ base of $\mathbb{F}_q^r$

Recall: INVARIANTS OF SIMPLICIAL COMPLEXES

- $\tilde{\chi}(\Delta) = \sum_{i \geq 0} (-1)^{i-1} \cdot \# \{ A \in \Delta \mid \# A = i \}$

reduced Euler characteristic

- $\Omega = \{ w_1 < \cdots < w_n \}$

$\tilde{C}_i(\Delta, \mathbb{Z}) = \bigoplus_{\substack{A \in \Delta \\ \# A = i+1}} \mathbb{Z} e_A \,, \; i \geq -1 \,,$

ith chain group

$$\partial_i e_A = \sum_{l=0}^{i} (-1)^l e_{A \setminus \{\omega_{j_l}\}} \qquad A = \{\omega_{j_0} < \cdots < \omega_{j_i}\}$$

i-th differential

$$\cdots\cdots \xrightarrow{\partial_1} C_0(\Delta, \mathbb{Z}) \xrightarrow{\partial_0} C_{-1}(\Delta, \mathbb{Z}) \xrightarrow{\partial_{-1}} 0$$

simplicial chain complex

Fact!: $\partial_i \circ \partial_{i+1} = 0 \ (\Leftrightarrow \ker \partial_i \subseteq \operatorname{Im} \partial_{i+1})$

$$\tilde{H}_i(\Delta, \mathbb{Z}) = \operatorname{Im}\partial_{i+1} \big/ \ker \partial_i$$

i-th reduced homology group with coefficients in $\mathbb{Z}$

Recall: COHEN-MACAULAY SIMPLICIAL COMPLEX

Δ CM over $\mathbb{Z}, \mathbb{R}$ if for all $A \in \Delta$ and

$$\operatorname{link}_\Delta A = \left\{ B \in \Delta \;\middle|\; \begin{array}{l} A \cap B = \emptyset \\ A \cup B \in \Delta \end{array} \right\}.$$

(the Link of A in Δ)

we have $\tilde{H}_i(\operatorname{link}_\Delta A, {}^{\mathbb{Z}}_{\mathbb{R}}) = 0$ for $i < \dim(\operatorname{link}_\Delta(A))$

Simple fact:

Δ CM over $\mathbb{Q}$, $\mathbb{Z}$

$$\Rightarrow \widetilde{H}_{\dim \operatorname{link}}(\operatorname{link}_\Delta A, \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}^{|\widetilde{\chi}(\operatorname{link}_\Delta A)|} \\ \mathbb{Z}^{|\widetilde{\chi}(\operatorname{link}_\Delta A)|} \end{cases}$$

$A = \emptyset \Rightarrow \operatorname{link}_\Delta \emptyset = \Delta$

$$\Rightarrow \widetilde{H}_{\dim \Delta}(\Delta, \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}^{|\widetilde{\chi}(\Delta)|} \\ \mathbb{Z}^{|\widetilde{\chi}(\Delta)|} \end{cases}$$

SIMPLICIAL COMPLEXES ASSOCIATED TO MATROIDS

M matroid

1. $\mathcal{I}(M)$ independence complex

well known

THM: $\mathcal{I}(M)$ shellable ($\Rightarrow$ homotopy CM

$\Rightarrow$ CM over $\mathbb{Z}$

$\Rightarrow$ CM over k)

Ex: $M = \mathbb{P}\mathbb{F}_q^r$

$$\#\{A \in I(M) \mid \#A = i\} = \frac{1}{i!}\,\frac{q^r-1}{q-1}\cdots\frac{q^r-q^{i-1}}{q-1}$$

$$\tilde{\chi}\left(I(\mathbb{P}\mathbb{F}_q^r)\right) = \sum_{i\geq 0}(-1)^{i-1}$$

$$= \begin{cases} 0 & r=1 \\ -\frac{q(q-1)}{2} & r=2 \\ \frac{q(q+1)(q^2+3q+3)(q-1)^2}{6} & \\ -\frac{q(q^2+q+1)(q^7+ \quad \cdots\cdots \quad -4q^3-12q-12)(q-1)^2}{24} & \end{cases}$$

1 $\mathcal{F}(M)$ Lattice of flats

$F \subseteq M$ flat of M if for any $x \in M \setminus F$

$$\max\left\{ \# I \;\middle|\; \begin{array}{l} I \in \mathcal{I}(M) \\ I \subseteq F \end{array} \right\}$$

$$\shortparallel$$

$$\max\left\{ \# I \;\middle|\; \begin{array}{l} I \in \mathcal{I}(M) \\ I \subseteq F \cup \{x\} \end{array} \right\}$$

Ex: $M = \mathbb{P}\mathbb{F}_q^r$

$F \subseteq M$ flat $\Leftrightarrow \exists\, U \leq \mathbb{F}_q^r : F = \{\text{lines in } U\}$

$\mathcal{F}(M) = \{F \subseteq M \mid F \text{ a flat}\}$

lattice of flats

- partially ordered by inclusion
- lattice

$\emptyset, M \in \mathcal{F}(M)$ unique maximal and minimal element

P poset with unique minimal element $\hat{0}$

maximal " $\hat{1}$

$\Delta(P) = \{\hat{0} < p_0 < \dots < p_i < \hat{1}\}$ order complex

! often write P for $\Delta(P)$!

well known

THM: $\Delta(\mathcal{F}(M))$ is shellable ($\Rightarrow$ homotopy CM

$\Rightarrow$ CM over $\mathbb{Z}$

$\Rightarrow$ CM over k)

Ex: $M = \mathbb{P}\mathbb{F}_q^r$

$\mathcal{F}(M)$ lattice of subspaces of $\mathbb{F}_q^r$

$$\widetilde{H}_{r-2}(\mathcal{F}(M), \mathbb{Z}) = \mathbb{Z}^{q^{\binom{r}{2}}}$$

$$\Rightarrow \widetilde{\chi}(\mathcal{F}(M)) = (-1)^r \cdot q^{\binom{r}{2}} = \mu(\mathcal{F}(M))$$

Additional structure:

$$\langle\, , \rangle : \mathbb{F}_q^r \times \mathbb{F}_q^r \to \mathbb{F}_q^r$$

non degenerate form

particularly interested in the unitary case

Why? end of talk

$$\langle\, , \rangle : \mathbb{F}_{q^2}^r \times \mathbb{F}_{q^2}^r \to \mathbb{F}_{q^2}^r$$

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \right\rangle = x_1 y_1^q + \cdots + x_r y_r^q$$

2 Lattice of flats $\mathcal{F}(\mathcal{R}(\mathbb{F}_q^r))$

= Lattice of subspaces of $\mathbb{F}_q^r$

$\rightsquigarrow$ poset $\mathcal{F}_{\langle,\rangle}(\mathbb{P}\mathbb{F}_{q^2}^r)$ of non-degenerate subspaces of $\mathbb{F}_{q^2}^r$

$\rightsquigarrow$ no "nice" Möbius number/Euler char

! Other option

building theory

$\mathcal{F}(\mathbb{P}(\mathbb{F}_q^r)) \rightsquigarrow$ Poset of isotropic subspaces

shellable, CM, $\tilde{\chi} - (-q)^{\binom{\cdot}{2}}$

Ennola phenomenon

Thm: (Devillier, Gramlich, Mühlherr)

$$2^{r-2} < \frac{q^2}{q+1}$$

$$\begin{pmatrix} SU_r \\ SU_i \; Sq_i \end{pmatrix}$$

- $\mathcal{F}_{\langle,\rangle}(\mathbb{F}_{q^2}^{r}) \simeq \bigvee$ spheres

$\Downarrow$

- homotopically CM

$\Downarrow$

- CM over $\mathbb{Z}$

$\Downarrow$

- CM over k

Structure of links

! Also for infinite fields !

$$2^{r-2} < \frac{q^2}{q+1}$$

$$\begin{array}{ccc} \parallel & & \parallel \\ 4 & > & \frac{9}{4} \end{array}$$

$r=4$, $q=3$ $\quad \overline{\mathbb{F}_{3^2}}^{\times}$

Indeed:

$$\widetilde{H}_1\left(\Delta \mathcal{F}_{<,>}(\mathbb{P}\mathbb{F}_{q^2}^2), \mathbb{Z}\right)$$

$$= \mathbb{Z}/3\mathbb{Z}$$

No example: $\mathcal{F}_{<,>}(\mathbb{P}(\mathbb{F}_{q^2}^r))$ not CM over $\mathbb{R}$

1

Independence complex $I(\mathbb{P}(\mathbb{F}_{q^2}^{n}))$

$\rightsquigarrow$ complex of (partial) frames

$I_{\langle,\rangle}(\mathbb{P}\mathbb{F}_{q^2}^{n})$

$$I_{\langle,\rangle}(\mathbb{P}\mathbb{F}_{q^2}^{n}) = \left\{ \{\mathbb{F}_{q^2}v_1, \dots, \mathbb{F}_{q^2}v_i\} \;\middle|\; \begin{array}{l} v_l \perp v_k \quad 1 \leq l < k \leq i \\ v_k \text{ not isotropic} \\ 1 \leq k \leq i \end{array} \right\}$$

complex of partial frames

Fact: $r \geq 2$: $\mathbb{F}_q^2 v_1, \dots, \mathbb{F}_q^2 v_{r-1} \in \mathcal{I}_{\langle,\rangle}(\mathbb{H}_{q^2}^r)$

$\Rightarrow \exists!\, \mathbb{F}_q^2 v_r$ such that

$\mathbb{F}_q^2 v_1, \dots, \mathbb{F}_q^2 v_r$ is a full frame

$\shortparallel$

maximal simplex of

$\mathcal{I}_{\langle,\rangle}(\mathbb{H}_{q^2}^r)$

ordered by inclusion

$$\hat{\mathcal{I}}_{\langle,\rangle}(\mathbb{H}_{q^2}^r) = \left\{ A \subseteq \mathcal{I}_{\langle,\rangle}(\mathbb{H}_{q^2}^r) \;\middle|\; \# A \neq r-1 \right\}$$

Easy: $r \geq 2$

$$\tilde{H}_i\left(\hat{\mathcal{I}}_{\langle,\rangle}(\mathbb{H}_{q^2}^r), \mathbb{Z}\right) = \tilde{H}_i\left(\mathcal{I}_{\langle,\rangle}(\mathbb{H}_{q^2}^r), \mathbb{Z}\right)$$

Indeed:

$$I_{\langle 1\rangle}(\mathbb{P}^r_{\mathbb{F}_q}) \simeq \hat{I}_{\langle 1\rangle}(\mathbb{P}^r_{\mathbb{F}_q})$$

never CM $/\mathbb{Z}$ or $\mathbb{k}$ for $r \geq 2$

$\chi(\cdot) \neq 0$

potentially CM

better $q \neq 2$

Main tool: orthogonality graph

$G = (V, E)$

non degenerate 1-dim subspace

pairs of orthogonal subspaces

Fact: G "=" 1-skeleton

$$I_{\langle,\rangle}(\mathbb{P}\mathbb{F}_{q^2}^r)$$
$$\updownarrow$$
$$\widehat{I}_{\langle,\rangle}(\mathbb{P}\mathbb{F}_{q^2}^r) \quad r \geq 4$$

$I_{\langle,\rangle}(\mathbb{P}\mathbb{F}_{q^2}^r)$ clique complex

$\hat{I}_{<,>}(\mathbb{F}_{q^2}^r)$	THM: $I_{<,>}(\mathbb{F}_{q^2}^r)$
?CM?	$r \geq 5$: simply connected except $r=6, q=2$
~~CM~~ ?CM?	$r=4, q=2,3$: not simply connected
H_1 torsion or 0	$r=4, q \geq 4$: π_1 has finite abelianization
CM	$r=3, q \geq 3$: $\simeq \bigvee S^1$
~~CM~~	$r=3, q=2$: $\simeq$ points
——	$r=2$: points $\simeq$

$\pi_1 = C_2 \times C_2$

Proof: Uses results from work of Aschbacher / Segev applied to orthogonality graph

$\pi_1\left(\begin{matrix}\text{Clique complex}\\ \text{of graph}\end{matrix}\right) = 0 \Leftarrow$ every n-gon is „trivial" in clique complex for $n \leq 2\,\mathrm{diam}(\text{graph}) + 1$

Show: orthogonality graph has small diameter

THM: $r < q+1$

$$\Longrightarrow \widehat{I}_{<,>}(\mathbb{P}\mathbb{F}_q^r) \text{ CM over } \mathbb{R}$$

Main tools:

- links in $I_{<,>}(\mathbb{P}\mathbb{F}_{q^2}^r)$ and $\widehat{I}_{<,>}(\mathbb{P}\mathbb{F}_{q^2}^r)$ are homotopy equivalent for faces of codim ≥ 2 in $I_{<,>}(\mathbb{P}\mathbb{F}_q^r)$
- $\mathrm{Link}_{I_{<,>}(\mathbb{P}\mathbb{F}_{q^2}^r)}(A) = I_{<,>}(\mathbb{P}\mathbb{F}_q^{r-\#A})$
- 1-skeleta of links are orthogonality graphs

Garland Method:

$$\tilde{H}_k(\Delta, \mathbb{R}) = 0 \text{ if}$$

$$\lambda \geq \frac{k}{k+1}$$

Compute λ for the orthogonality graph

smallest eigenvalue > 0 for the normalized Laplacian of the 1-skeleton of the link of any $(k-1)$-face

(all links must be connected)

Motivation:

- G finite group
- p prime
- $A_p(G)$ poset of non trivial elementary abelian p-subgroups of G

Conjecture: (Quillen 1970s)

$\tilde{H}_i(A_p(G),\mathbb{Z}) = 0 \iff \exists$ non-trivial normal p-subgroup

"$\Leftarrow$" ✓

($\exists$ elementary abelian)

" $\Longrightarrow$ " early 90s

Aschbacher/Smith $p>5$ and

G does not "involve" a unitary group

2020– Piterman/Smith $p=3,5$

same unitary restriction

Known: $$\tilde{H}_{r-2}\left(\mathcal{I}_{<,>}(\mathbb{F}_{q^2}^{r}), \mathbb{Z}\right) \neq 0$$

+ same for a complex similar to $\mathcal{I}_{<,>}(\mathbb{F}_{q^2}^{r})$

$\Longrightarrow$ can remove the unitary restriction

THANK

YOU !