Webs and canonical bases in degree two

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@ Cascade lectures in combinatorics

April 2 2022

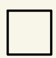
GL_n rep theory

Representation of GL_n $\begin{cases} \text{homom} & GL_n \to GL_m \\ \text{action} & GL_n \circlearrowright \mathbb{C}^m \end{cases}$

E.g.

$GL_n \circlearrowright \mathbb{C}^n$

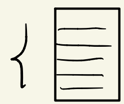
defining rep



$GL_n \longrightarrow GL(\mathbb{C})$

$g \longmapsto \det g$

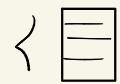
determinant rep

n 

$GL_n \circlearrowright \Lambda^k \mathbb{C}^n$

$k \in [n]$

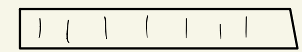
fundamental reps

k 

$GL_n \circlearrowright \mathrm{Sym}^k \mathbb{C}^n$

$k \in \mathbb{N}$

symmetric powers



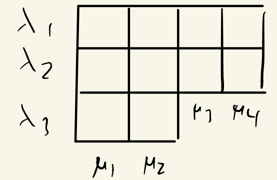
Fact Irreducible polynomial reps of GL_n $\longleftrightarrow$ partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ $\longleftrightarrow$ Young diagrams

Notation: $\mathbb{S}_\lambda \mathbb{C}^n$ for irrep labeled by λ "Schur functor"

$\dim \mathbb{S}_\lambda \mathbb{C}^n = \# SSYT(\lambda, [n])$ semistandard Young tableaux

$\mathrm{character}(\mathbb{S}_\lambda \mathbb{C}^n) = s_\lambda(x_1, \dots, x_n)$ Schur polynomial

Explicit constructions of $\mathbb{S}_\lambda \mathbb{C}^n$



- image of certain map $\bigotimes_i \Lambda^{\mu_i} \mathbb{C}^n \longrightarrow \bigotimes_j \mathrm{Sym}^{\lambda_j} \mathbb{C}^n$
- $\Gamma(GL_n/B, \mathcal{L}_\lambda)$ global sections of line bundle on flag variety Borel Weil
- $\mathbb{C}[GL_n/N] \simeq \bigoplus_\lambda \mathbb{S}_\lambda \mathbb{C}^n$ as GL_n-reps.

  $\rightsquigarrow$ explicit construction of $\mathbb{S}_\lambda$ in terms of matrix minors

- other constructions via Springer Fibers / affine Grassmannian Kamnitzer "lectures on geom. constructions..."

Problem Construct + compare bases for $\mathbb{S}_\lambda \mathbb{C}^n$ w/ "good properties"

basis elts indexed by SSYT's

Explicit construction of $\mathbb{S}_\lambda \mathbb{C}^n$ when $\lambda = d^k$ is $k \times d$ rectangle

$k \times n$ matrix of variables x_{ij} $i \in [k]$ $j \in [n]$

Plücker coordinate $\Delta_I = \det(x_{ij})_{i \in [k], j \in I}$

Then $\mathbb{S}_{d^k} \mathbb{C}^n \underset{\text{as } GL_n \text{ reps}}{\simeq} \mathrm{span}_{\mathbb{C}} (\underbrace{\Delta_{I_1} \cdots \Delta_{I_d}}_{GL_n \text{ acts on } k \times n \text{ matrices, hence on } \Delta_I\text{'s.}} : I_1, \dots, I_d \in \binom{[n]}{k})$

<u>Eg</u> $\mathbb{S}_{\square\!\square} \mathbb{C}^n$ is spanned by $\Delta_{ij} = x_{1i} x_{2j} - x_{1j} x_{2i}$ $\quad i,j \in [n]$

$\mathbb{S}_{\boxplus} \mathbb{C}^n$ . . . . $\Delta_{ij} \Delta_{kl}$'s $\}$ these satisfy linear relations

e.g. $\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} = 0.$

"Plücker relations"

"Classical" bases

1) standard monomial basis — $\Delta_{I_1} \cdots \Delta_{I_d}$ with $\begin{array}{|c|c|c|} I_1 & \cdots & I_d \end{array}$ a semistandard tableau.

2) noncrossing monomial basis — . . . a noncrossing tableau.
Peterson Pylyavskyy Speyer

3) Gelfand-Tsetlin basis — defined using $GL_1 \subset GL_2 \subset \cdots \subset GL_n$

"Modern" bases

(Gelfand, Zelevinsky, De Concini, Kazhdan, Retakh, Berenstein, ...)

4) (dual) canonical	recursive def'n | simple modules over KLR algebras / quantum affine sl_k	Lusztig Kashiwara	these overlap a lot but don't all coincide!
5) semicanonical	modules over preprojective algebra of type A	Lusztig	
6) geometric Satake	irreducible components in affine Grassmannian slices	Mirković Vilonen	
7) Θ-basis	scattering diagrams, positive tropical points in cluster varieties	Gross Hacking Keel Kontsevich	

What makes a basis good?

- positive structure constants

- multiplicative properties

- cyclic symmetry properties w/respect to $i \longmapsto i+1 \bmod n$

- restriction to positroid varieties

- each basis elt is a subtraction-free rational expression in Δ_I's

- nice action of Chevalley generators $e_i \in \mathfrak{b} \subset \mathfrak{g}$

- homogeneous / "weight basis" labeled by SSYT's

products of basis elts are sums of basis elts

if $xy \in$ basis then x, y quasicommute in a quantum deformation

$\Longrightarrow$ cyclic sieving phenomenon for SSYT$((d^k), [n])$

Reiner Stanton White
Rhoades

nonvanishing basis elts are a basis for the positroid coordinate ring

Lam

$\Longrightarrow$ basis elts are positive on the positive Grassmannian

e_i's $\underbrace{\text{permute basis elts}}_{\text{crystal axioms}}$ (or kill them) up to lower order terms

≥ 0 structure constants for e_i - action

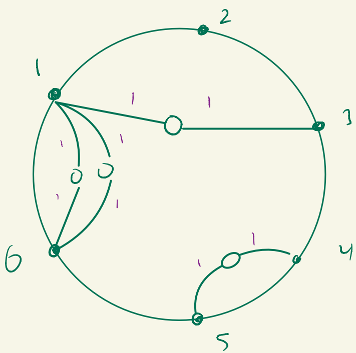
"biperfect"

A pictorial approach to bases

(from knot theory...)

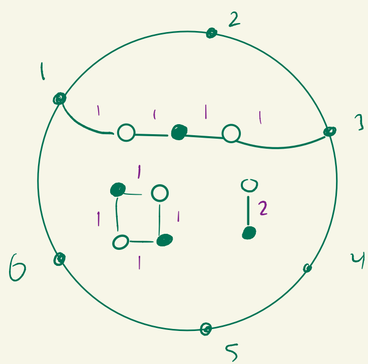
SL_k web diagram $\left\{ \begin{array}{l} \text{planar, bipartite graph in disk with } n \text{ boundary vertices} \\ \mathbb{N}\text{-weighted edges, } \sum_{e \sim v} wt(e) = k \quad \forall v \\ \partial \text{ vertices are black, } \partial \text{ edges have weight one} \end{array} \right.$

$$d = \frac{\# \partial \text{ edges}}{k}$$



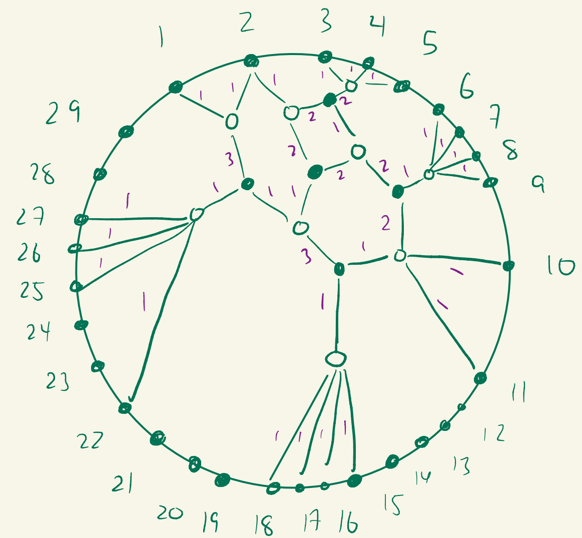
$k = 2$

$d = 4$



$k = 2$

$d = 1$

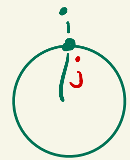


$k = 5, \quad d = 4$

Web polynomials

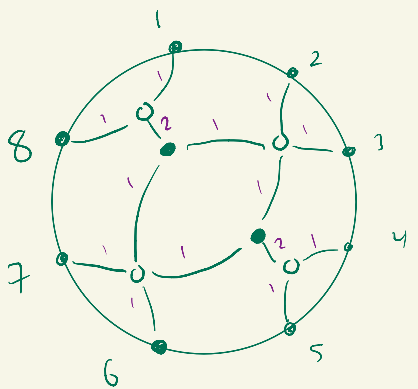
Web $W \longmapsto$ polynomial $[W]$

$$[W] := \sum_{\substack{\ell : E(W) \to 2^{[k]} \\ \text{proper weighted coloring}}} \underbrace{\mathrm{sgn}(\ell)}_{\text{sign of the boundary labels}} \ \underbrace{\mathrm{monom}(\ell)}_{\text{monomial in } x_{ij}\text{'s}}$$

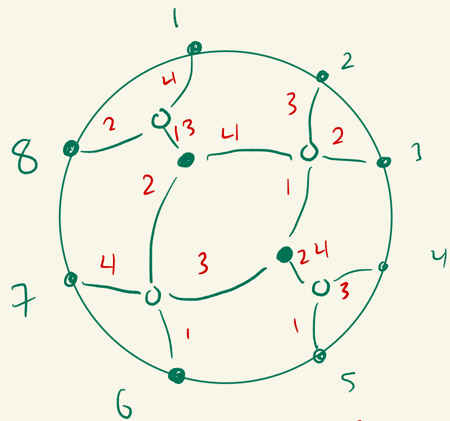


"proper": disjoint around interior vertices

"weighted": $\#\ell(e) = wt_W(e)$



SL_4 web W



proper coloring ℓ

$\mathrm{monom}(\ell) =$

$x_{41}x_{32}x_{23}x_{34}x_{15}x_{16}x_{47}x_{28}$

$\mathrm{sign}(\ell) = (-1)^{\#\text{ of inversions}}$

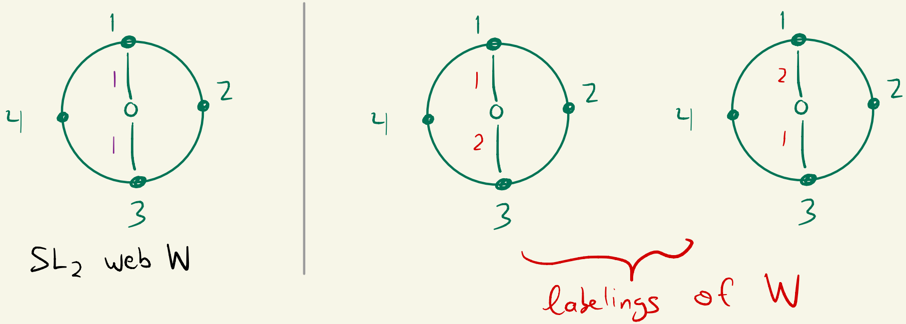
(inversions of $4,3,2,3,1,1,4,2$)

Properties of web polynomials

- $[W] \in \mathbb{S}_{d^k} \mathbb{C}^n$. These polys <u>span</u> and all linear relations between them are known (SL_k skein relations).

  Kuperberg
  Cautis-Kamnitzer-Morrison
  F. - Lam - Le

- Each Plücker coordinate is the web poly of a "claw graph on ∂ vertices I".



$$[W] = x_{11}x_{23} - x_{21}x_{13} = \Delta_{13}$$

- $[W \cup W'] = [W][W']$: can work algebraically with web polynomials using skein relns

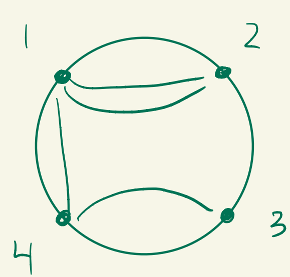
- Often study webs which have degree one at each ∂ vtx. "standard webs"

Web bases for small k

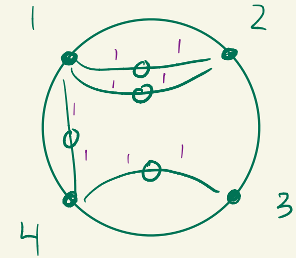
$k=2$: noncrossing matching basis

equinumerous with $SSYT(2\times d, [n])$

coincides w/ canonical basis



$\Delta_{12}^2 \Delta_{14} \Delta_{34}$



SL_2 web

Kuperberg

$k=3$: webs whose

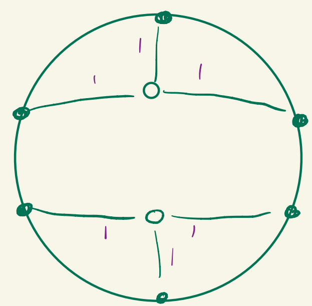
interior cycles have ≥ 6 sides

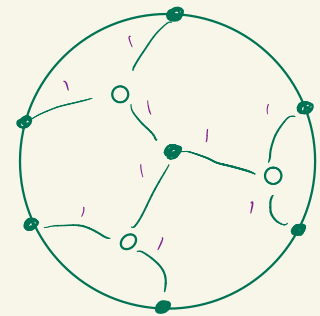
boundary cycles have ≥ 4 sides

equinumerous with $SSYT(3\times d, [n])$

known to disagree w/ canon basis

but there's lots of overlap





$k>3$: open problem (e.g.) to find a basis which is invariant under the cyclic symmetry of $\mathbb{S}_{d^k}\mathbb{C}^n$.

Web bases for small d

$d=1$: Canonical basis is the Plücker coordinates and its a web basis

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$d=2$: Transition matrix between web basis + standard monomial basis is known.

Lam $\left\{ \text{Canonical basis elts} \longleftrightarrow SL_2 \text{ web immants} \right.$ } defined using double dimers on plabic graphs

$\exists$ basis elts which are not cluster monomials once $k \geq 4$

$\exists$ recent work trying to understand cluster variables in this degree

$\exists$? a connection w/ enumerative geometry (DT-PT correspondence) Jenne-Webb-Young

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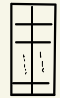
Thm (F.) The canonical basis of $S_{2k}\mathbb{C}^n$ consists of web polynomials.

$d=2$ $(\forall k,n)$

Cor $\exists$ cyclically invariant basis of web polynomials when $d=2$.

Each SL_2 web immanant (for $Gr(k,n)$) is an SL_k web polynomial.

Construction: SYT() $\hookrightarrow$ SL_k web on $2k$ boundary vertices

$\underbrace{}_{k \times 2}$

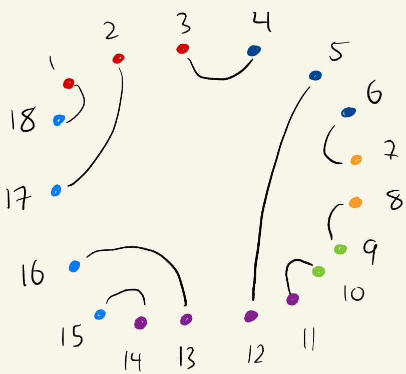
Catalan object

Straight forward to extend to SSYT's

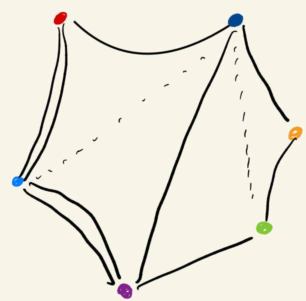
SYT

1	4
2	7
3	9
5	11
6	12
8	15
10	16
13	17
14	18

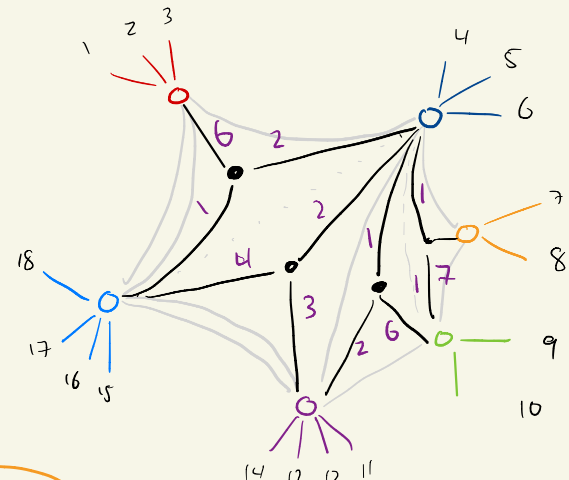
noncrossing matching



multi triangulation of polygon



SL_k web



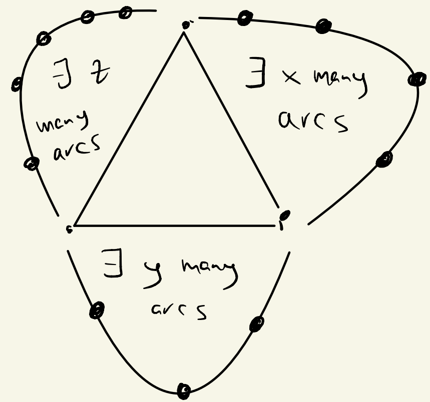
1^{st} column = left end points

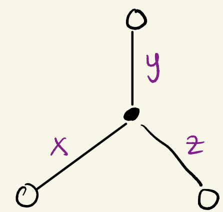
merge vertices + choose dashed diagonals

see next page

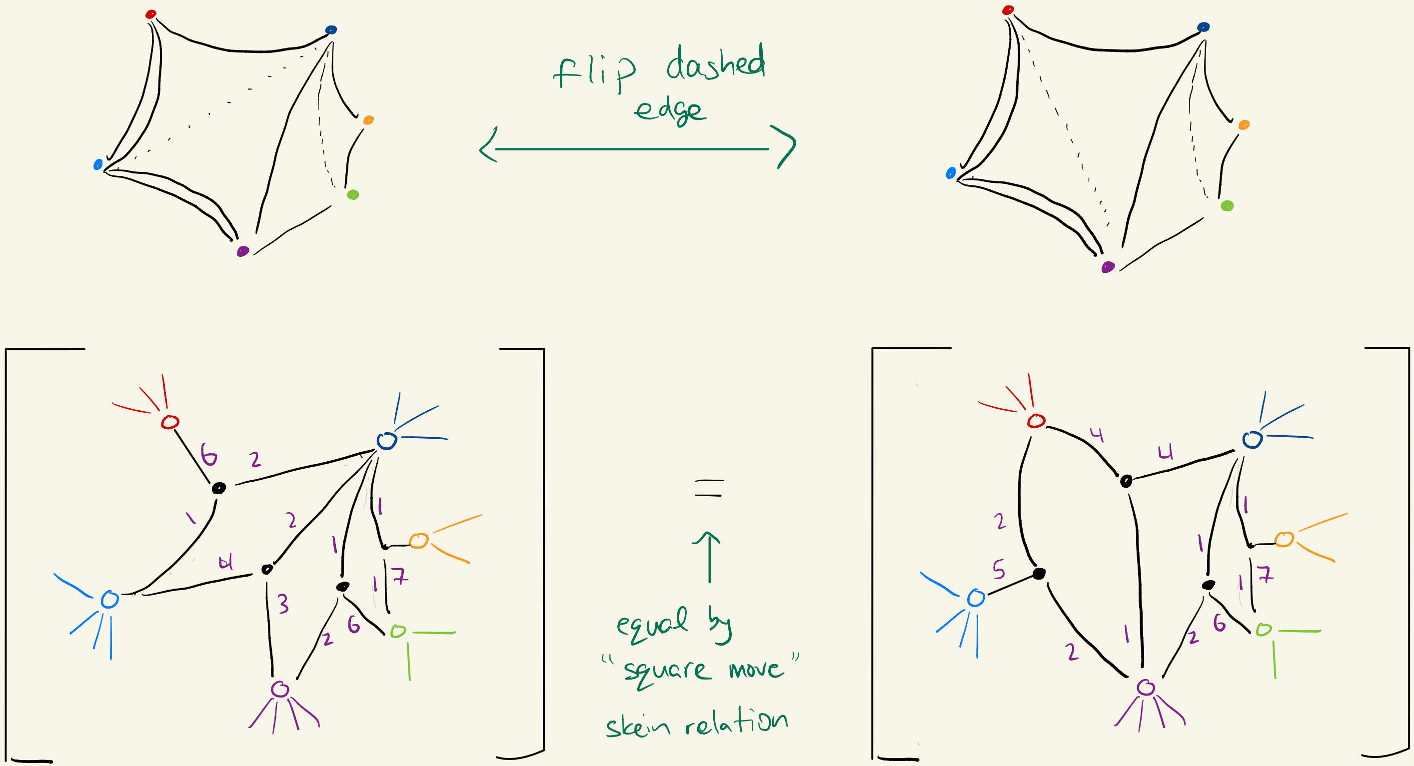
Rule:

multi triangulation T	SL_k web W
boundary vertex	white vtx
triangle	interior black vtx



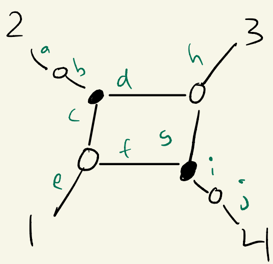


<u>Rmk</u> different dashed edges yield different webs but <u>same</u> web polynomial



Plabic graphs + Plücker coords

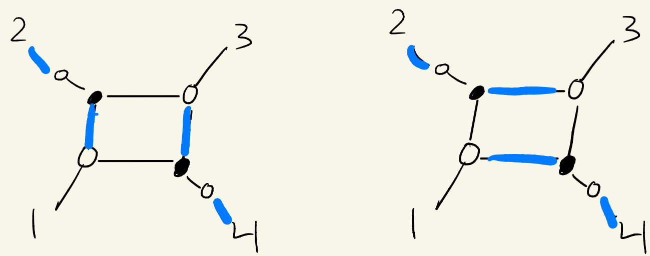
Gen fcn of dimer covers of plabic graph which use d vertices I " = " Plücker coordinate Δ_I



plabic graph

$n=4$

$k=2$



acgj   adfj

dimer covers with boundary $\{2,4\}$

$$\Delta_{24} = aj\,(cg+df)$$

d-fold dimer cover $\longleftrightarrow$ element of $\underbrace{S_{d^k}\,\mathbb{C}^n}_{\text{space of } SL_k \text{ webs}}$

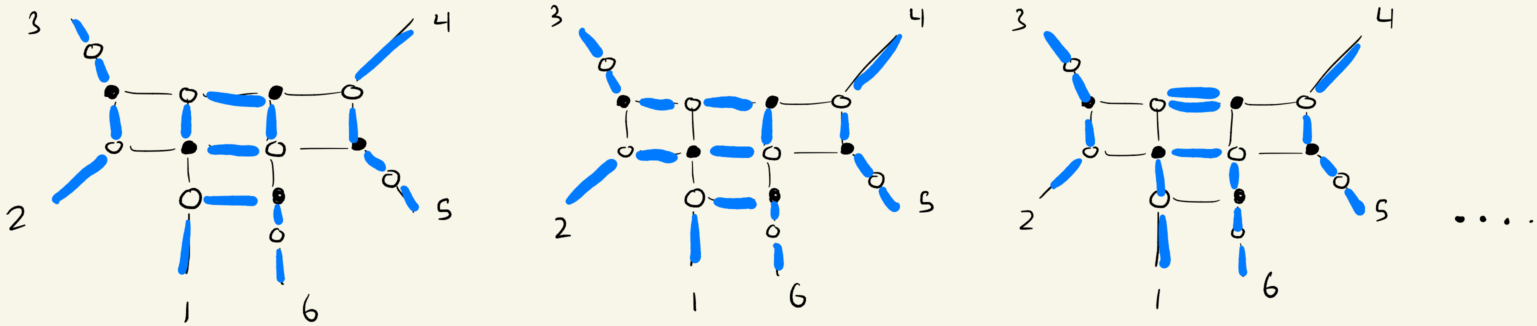
"SL_d-web subgraph"

F.-Lam-Le

SL_2 web immanant: Gen fcn for double dimer covers w/ fixed connectivity

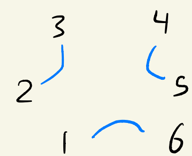
Rhoades–Skandera
Lam

eg

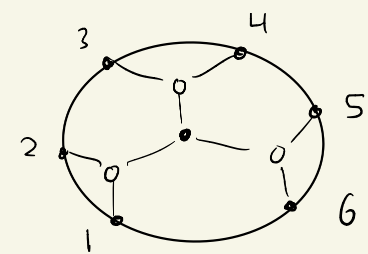


$n = 6$
$k = 3$
$d = 2$

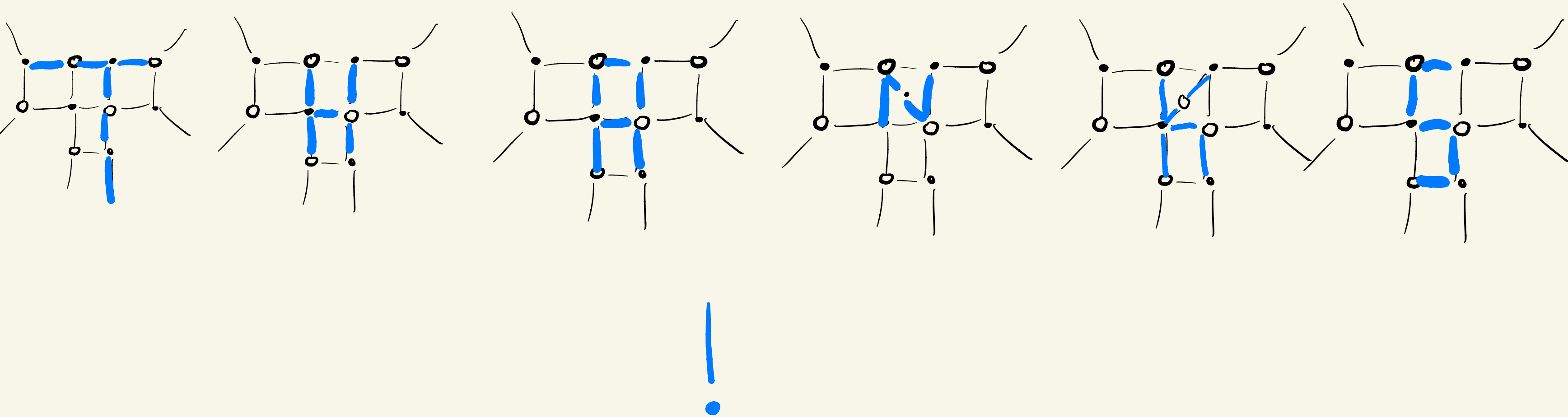
are double dimers with connectivity



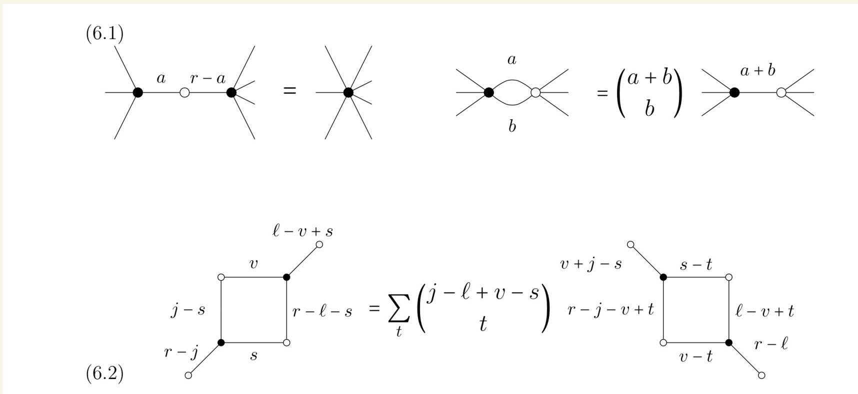
Our thm $\Rightarrow$: this gen fcn is the web polynomial



$\in \mathbb{S}_{(2,2,2)}\mathbb{C}^6$


THANKS!

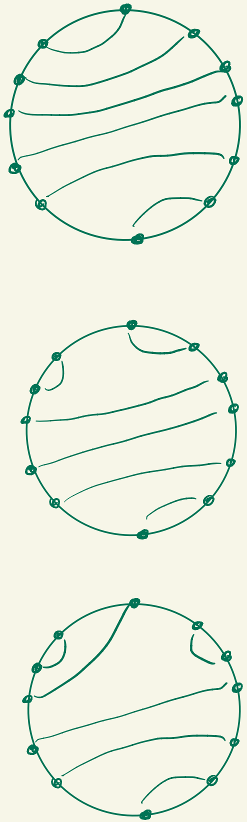
SL_r Skein relations



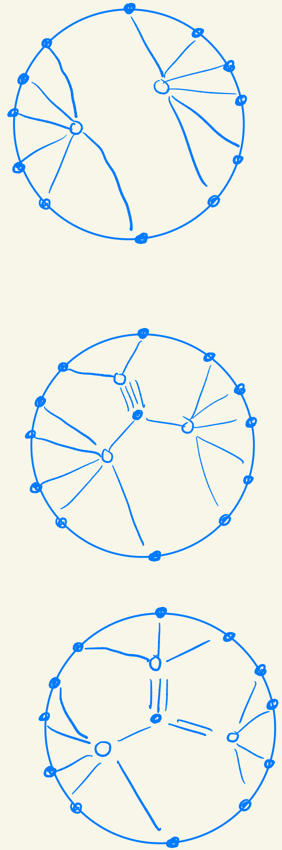
$(a, b, \dots \in [r])$

Tableau-to-web examples

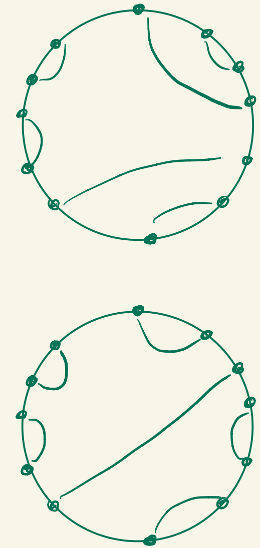
SL_2



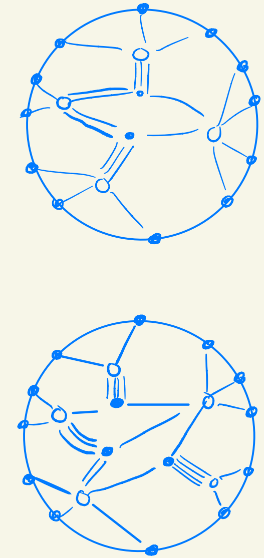
SL_6



SL_2

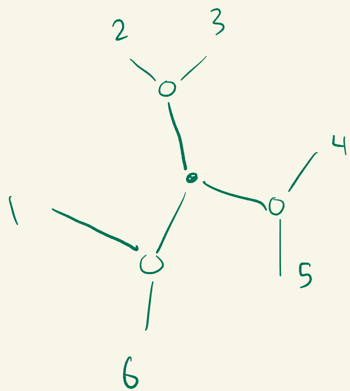


SL_6



Web polys vs Plü coords

web polynomial



can be expressed as

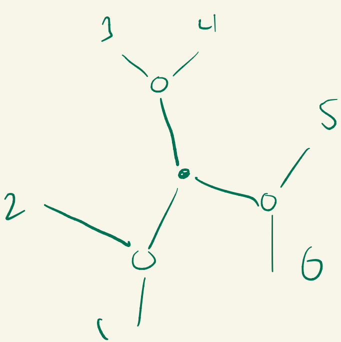
$$\Delta_{145}\Delta_{236} - \Delta_{123}\Delta_{456}$$

$$= \Delta_{136}\Delta_{245} - \Delta_{345}\Delta_{126}$$

$$= \Delta_{235}\Delta_{146} - \Delta_{156}\Delta_{234}$$

$$= \Delta_{135}\Delta_{246} - \Delta_{125}\Delta_{346} - \Delta_{124}\Delta_{356} - 2\Delta_{123}\Delta_{456}$$

standard monomial expression.



$$= \Delta_{124}\Delta_{356} - \Delta_{123}\Delta_{456}$$

standard monomial expression