Colored vertex models and k-tilings of the Aztec diamond

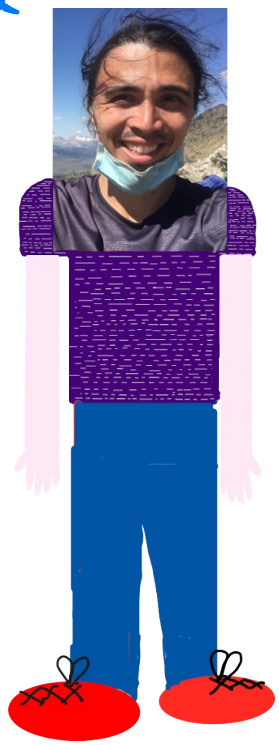
Sylvie Corteel (CNRS, IRIF)
UC Berkeley (on leave)

Calico - April 2022

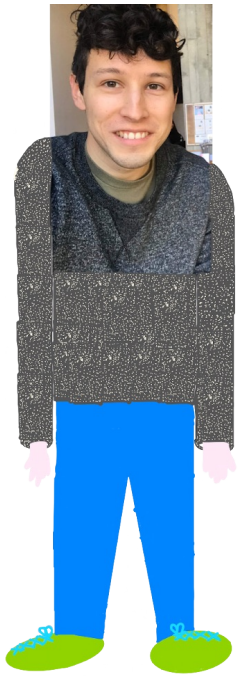
C., Gitlin, Keating & Meza
IMRN (2021)

Gitlin, Keating
Arxiv 2110.10273

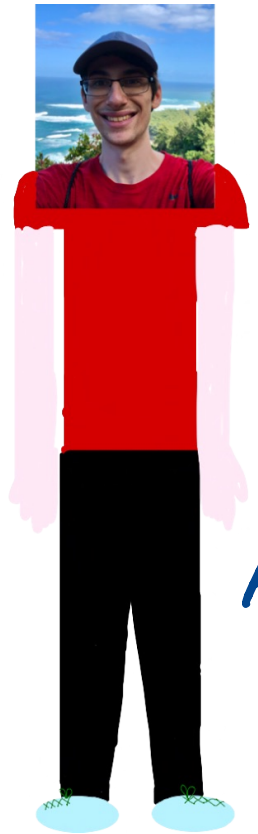
C., Gitlin & Keating
Arxiv 2202.06020



D. Keating
→ Wisconsin



J. Meza
→ Goldman Sachs

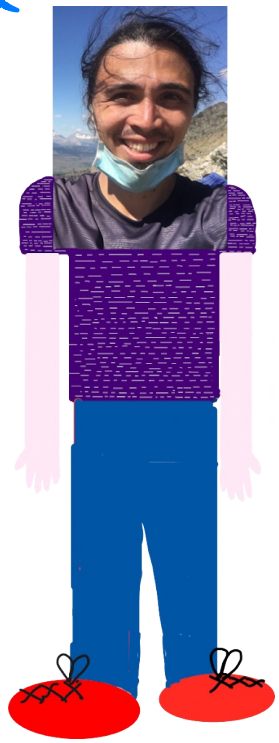


A. Gitlin
Berkeley

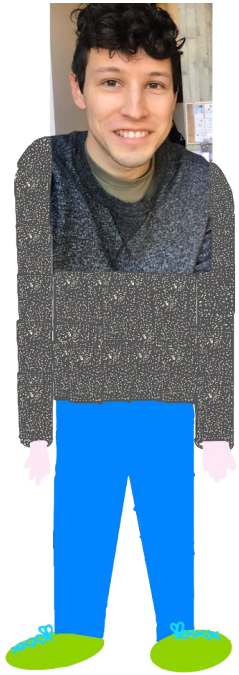
C., Gitlin, Keating & Meza
IMRN (2021)

Gitlin, Keating
Arxiv 2110.10273

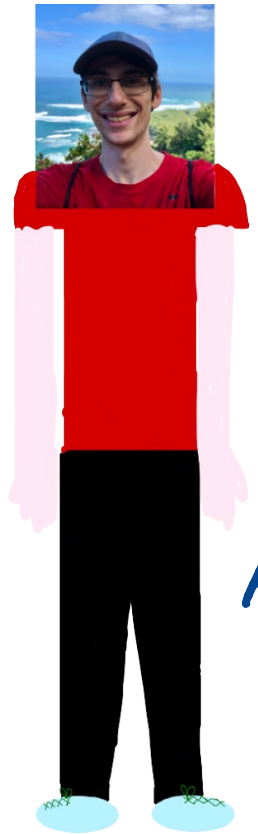
C., Gitlin & Keating
Arxiv 2202.06020



D. Keating
→ Wisconsin



J. Meza
→ Goldman Sachs



A. Gitlin
Berkeley

Many other papers

"Solvable lattice model seminar"

Tuesday 2pm PST

Five vertex models

(Folklore)

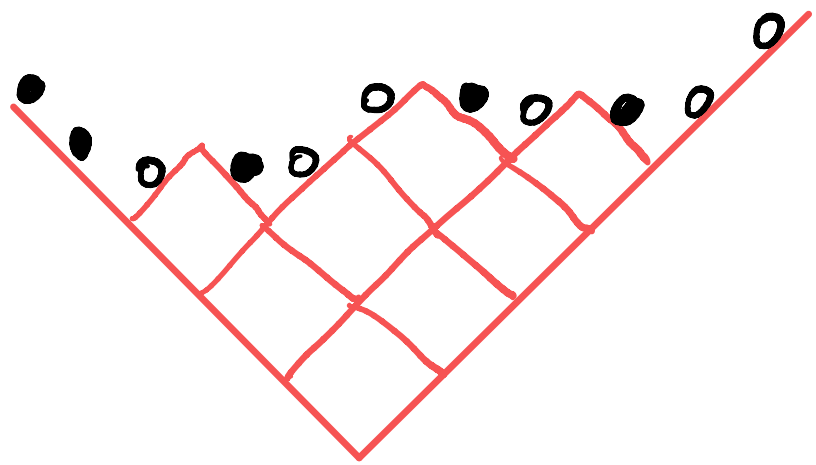
Partition of an integer n

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$$

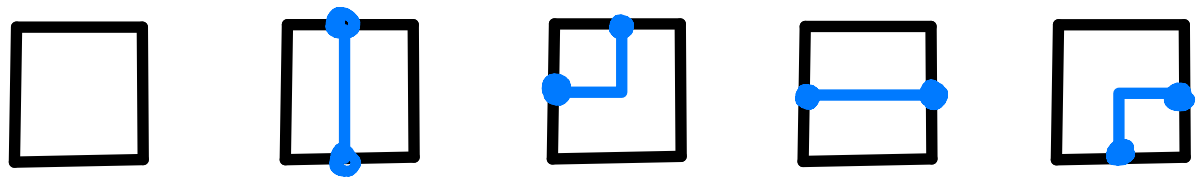
$$\lambda_1 + \dots + \lambda_k = n$$

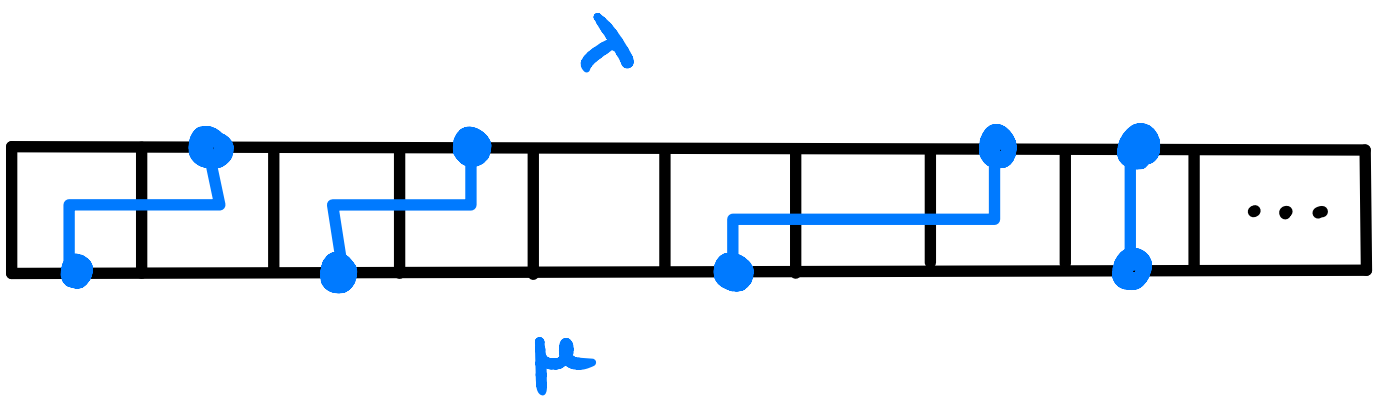
Maya diagram of a partition



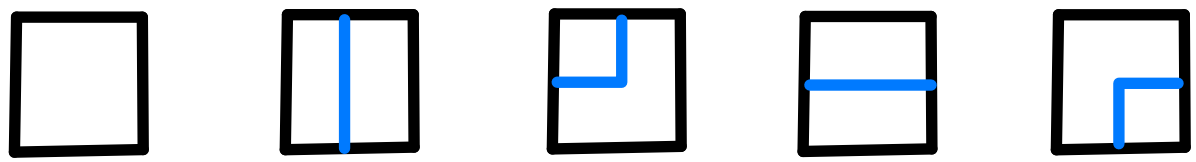
$\lambda = (4,3,1,0,0)$

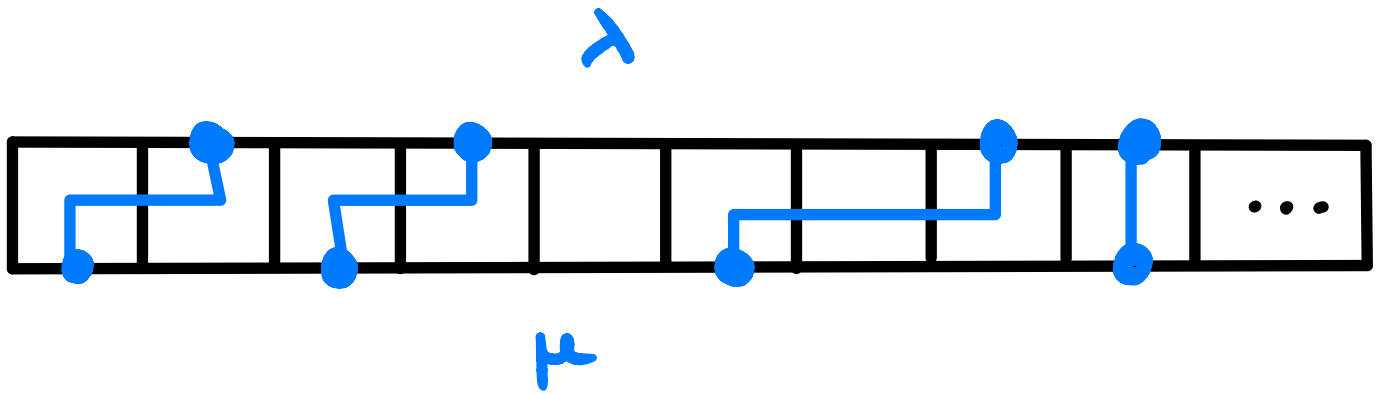
Five vertex model





Five vertex model

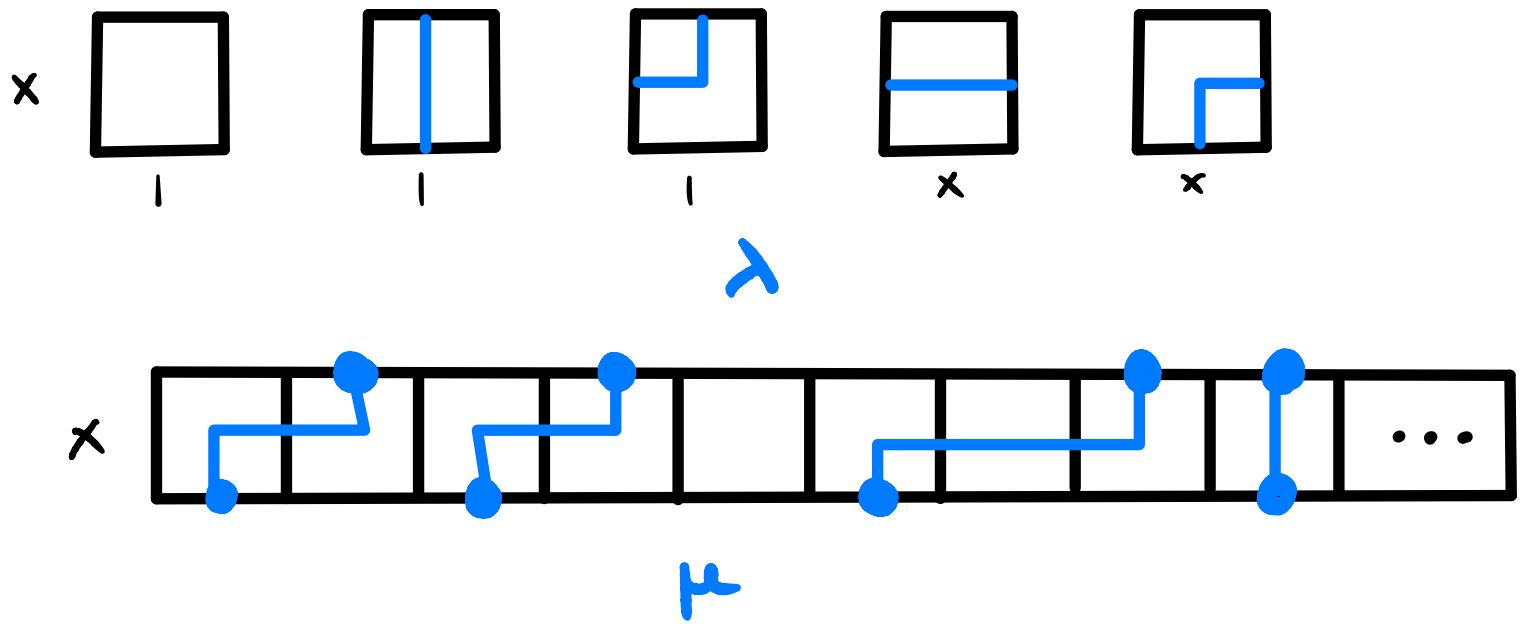




Interlacing $\lambda \succ\!\prec \mu$

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \cdots$$

Five vertex model



Interlacing $\lambda \succ\!\!\prec \mu$

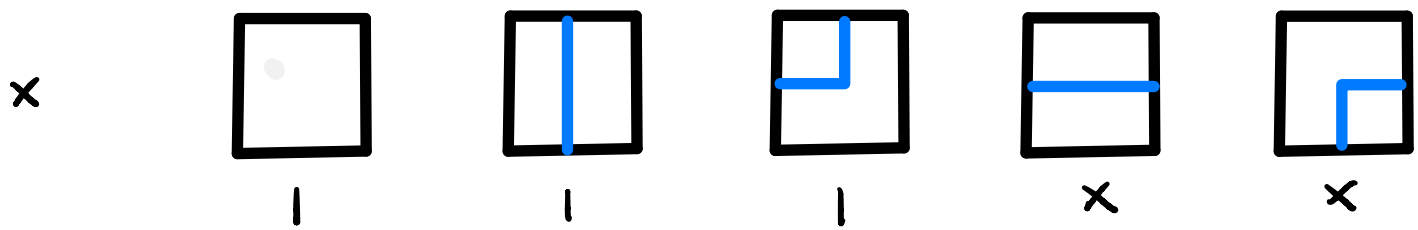
$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \cdots$$

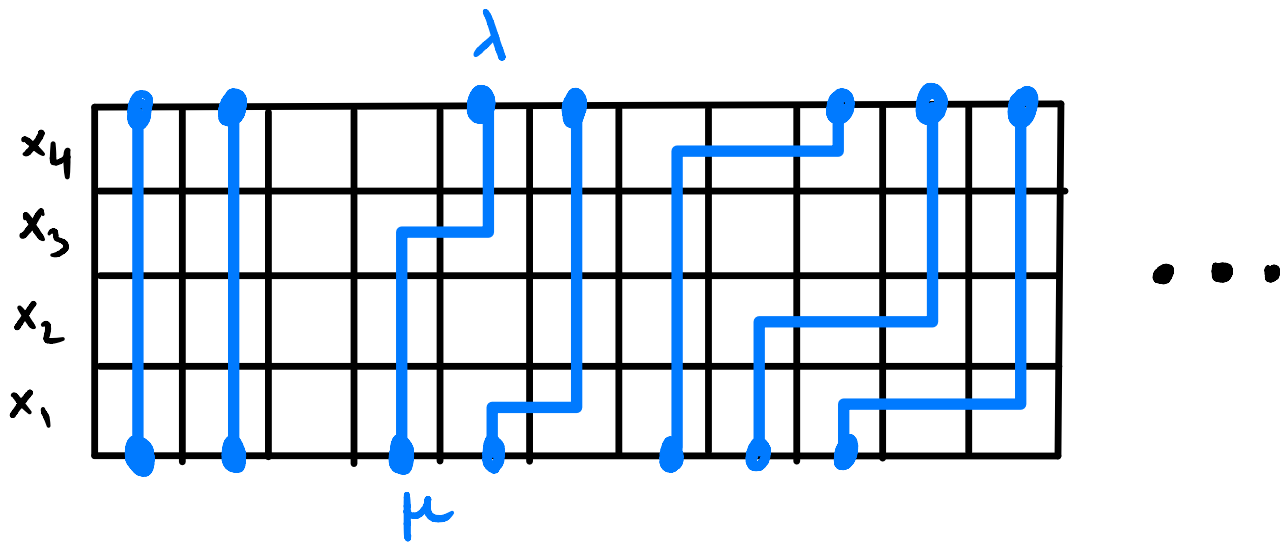
ex $\lambda = (5,5,2,1)$ $\mu = (5,3,1,0)$

$$5 \geq 5 \geq 5 \geq 3 \geq 2 \geq 1 \geq 1 \geq 0$$

Weight x^4

Weight of a configuration

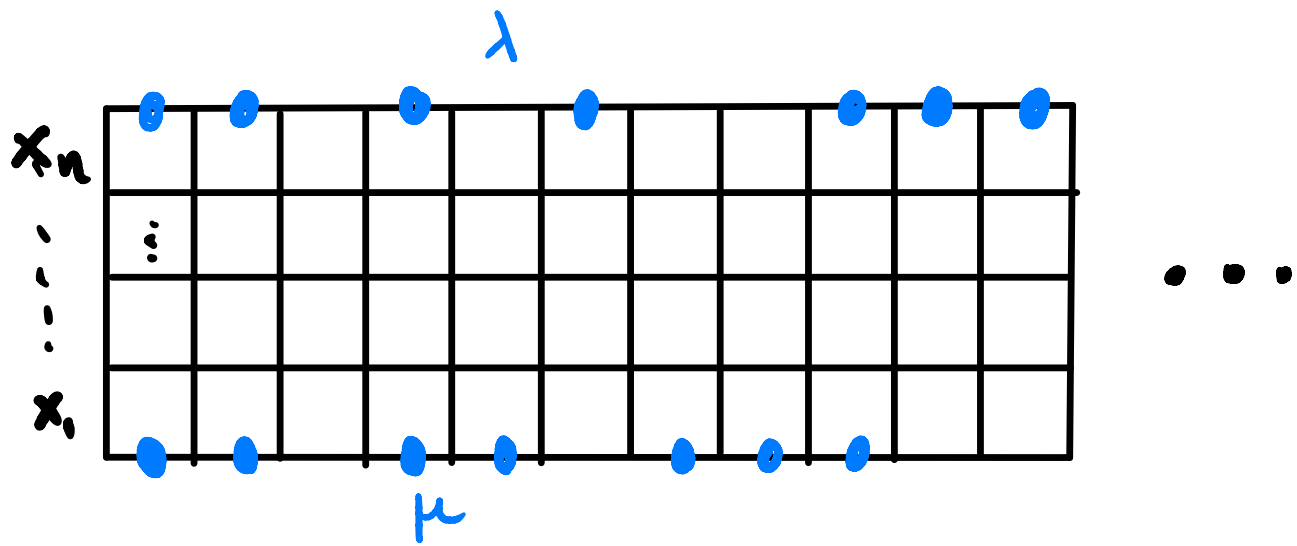




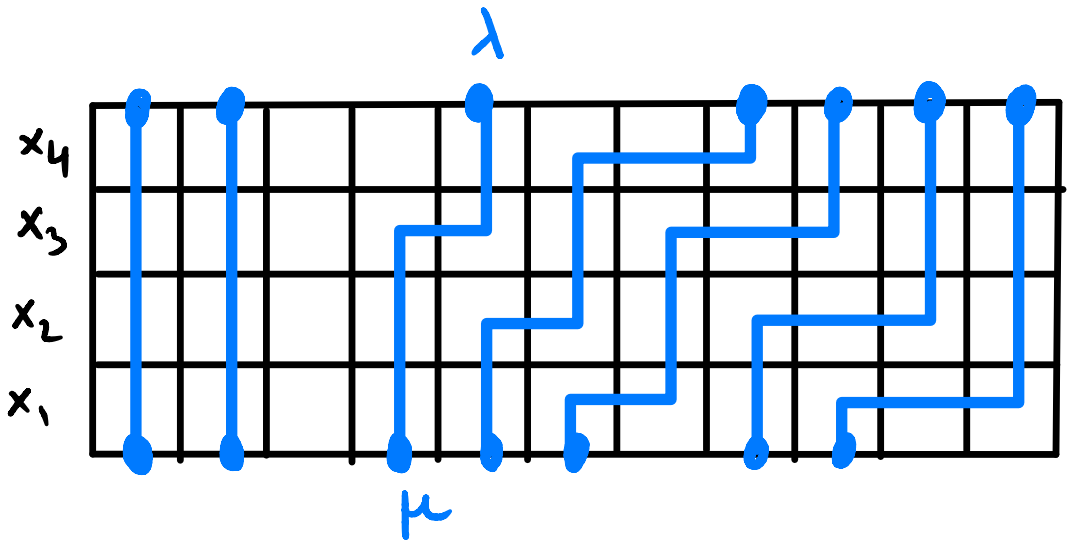
$$wt(c) = \prod_{\text{vertex}} \text{weight of the vertex}$$

<u>ex</u> $x_1^3 \, x_2^2 \, x_3 \, x_1$

Partition function

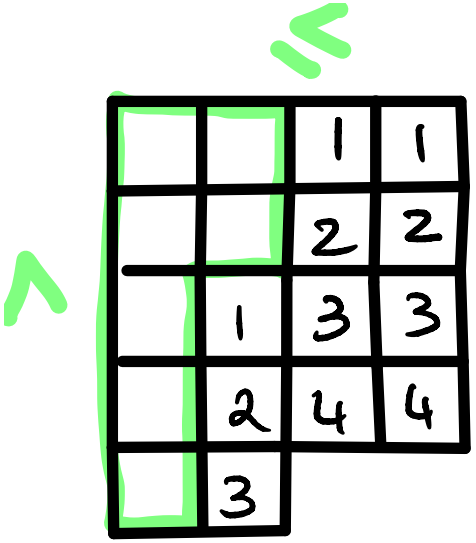


$$Z_{\lambda/\mu}(x_1, x_2, \ldots, x_n) = \sum_{\substack{c \text{ configurations} \\ \text{from } \mu \text{ to } \lambda}} wt(c)$$



$\mu = (2,2,1,1,1)$
$\lambda = (4,4,4,4,2)$

...



Prop

$$Z_{\lambda/\mu}(x_1, x_2, \dots, x_n) = s_{\lambda/\mu}(x_1, \dots, x_n)$$

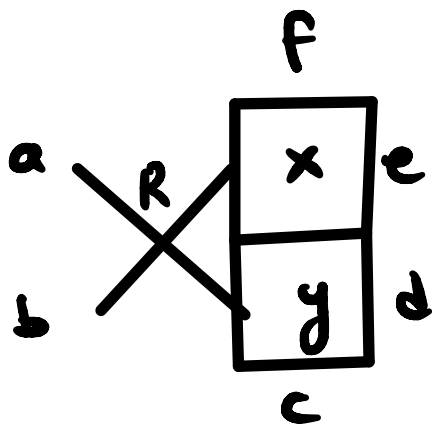
Skew - Schur polynomials

Prop For any permutation $\pi \in S_n$

$$s_{\lambda/\mu}(x_1, \dots, x_n) = s_{\lambda/\mu}(x_{\pi_1}, \dots, x_{\pi_n})$$

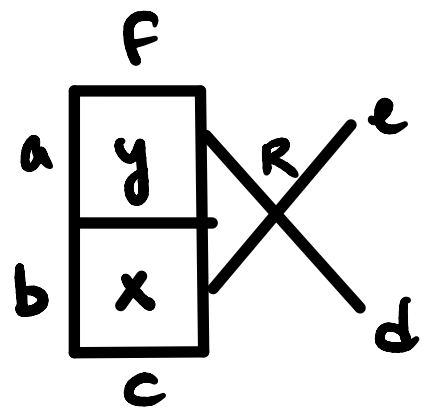
Proof Yang Baxter equation

$$\sum_{\text{conf}}$$



$$=$$

$$\sum_{\text{conf}}$$

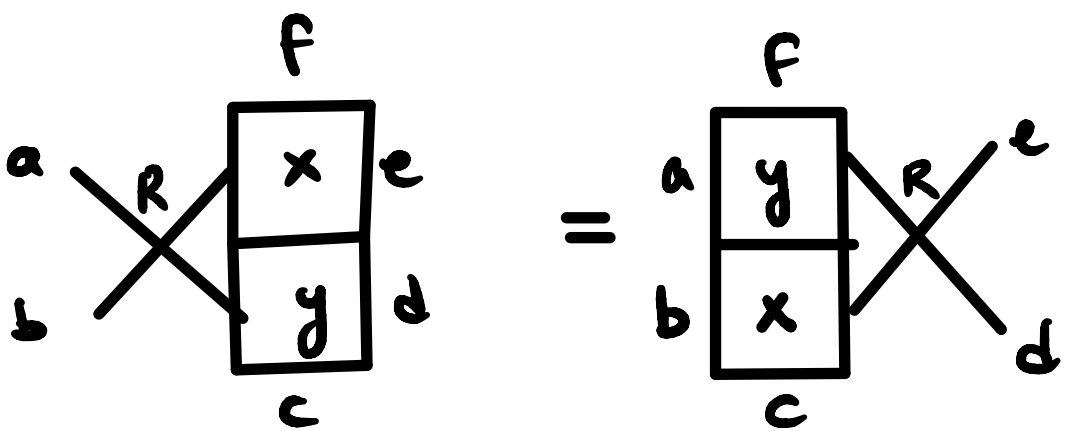


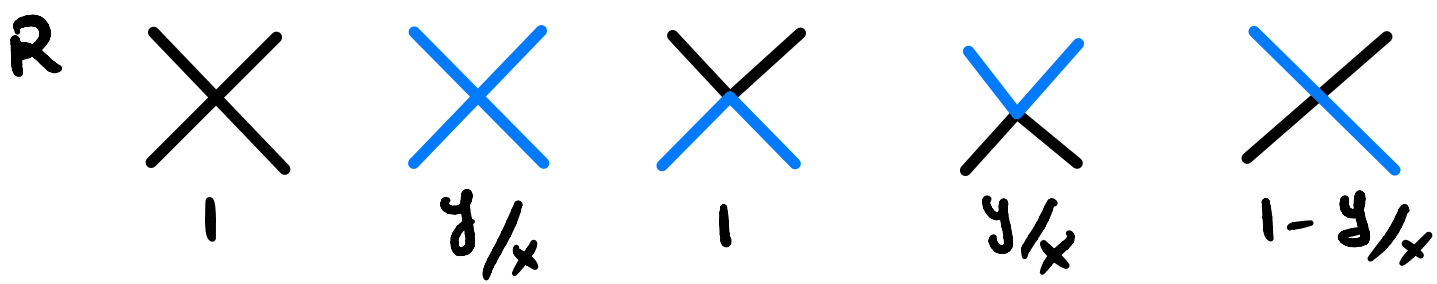
$\forall\ a, b, c, d, e, f \in \{\bullet, \circ\}$

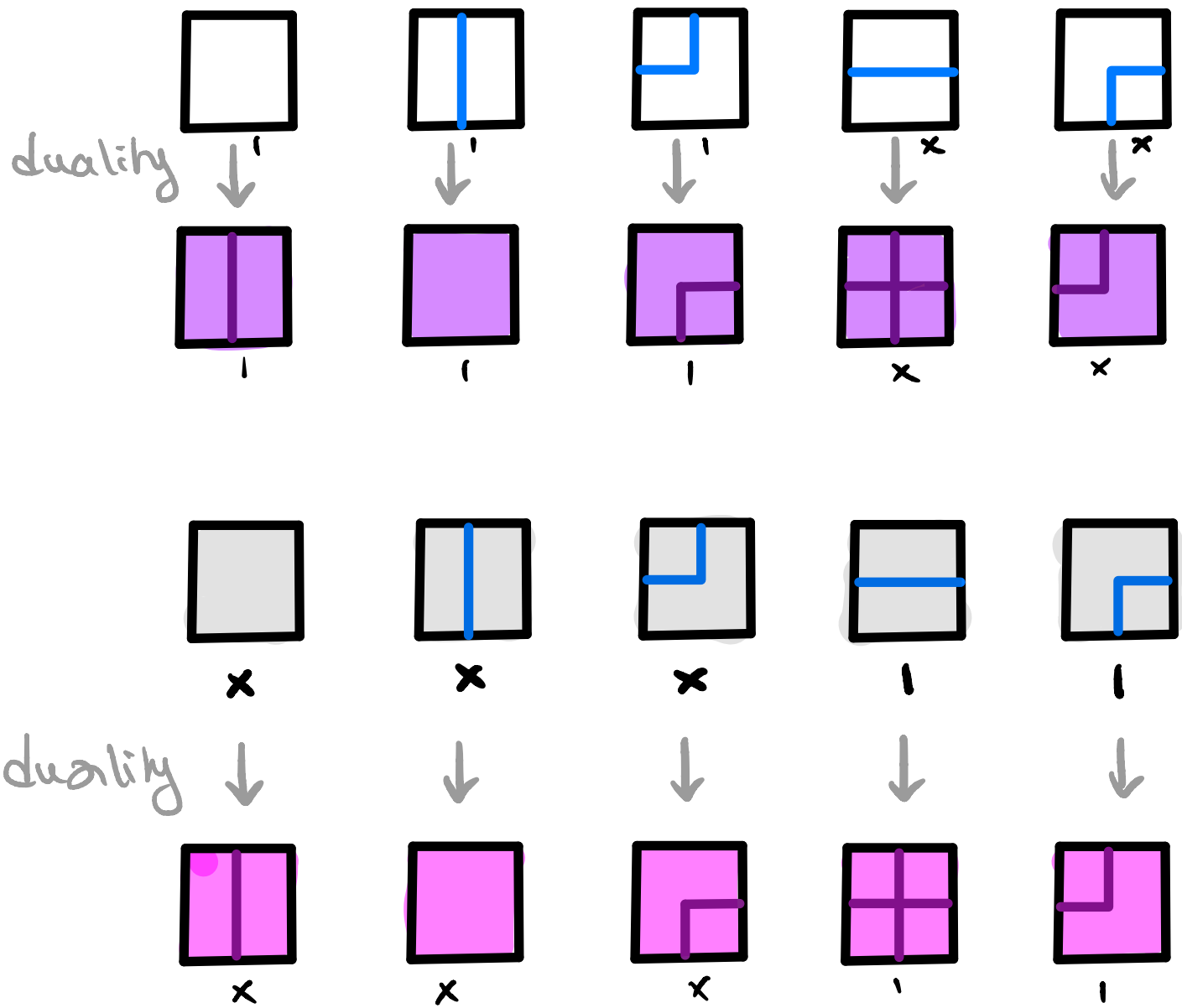
Prop For any permutation $\pi \in S_n$

$$s_{\lambda/\mu}(x_1, \dots, x_n) = s_{\lambda/\mu}(x_{\pi_1}, \dots, x_{\pi_n})$$

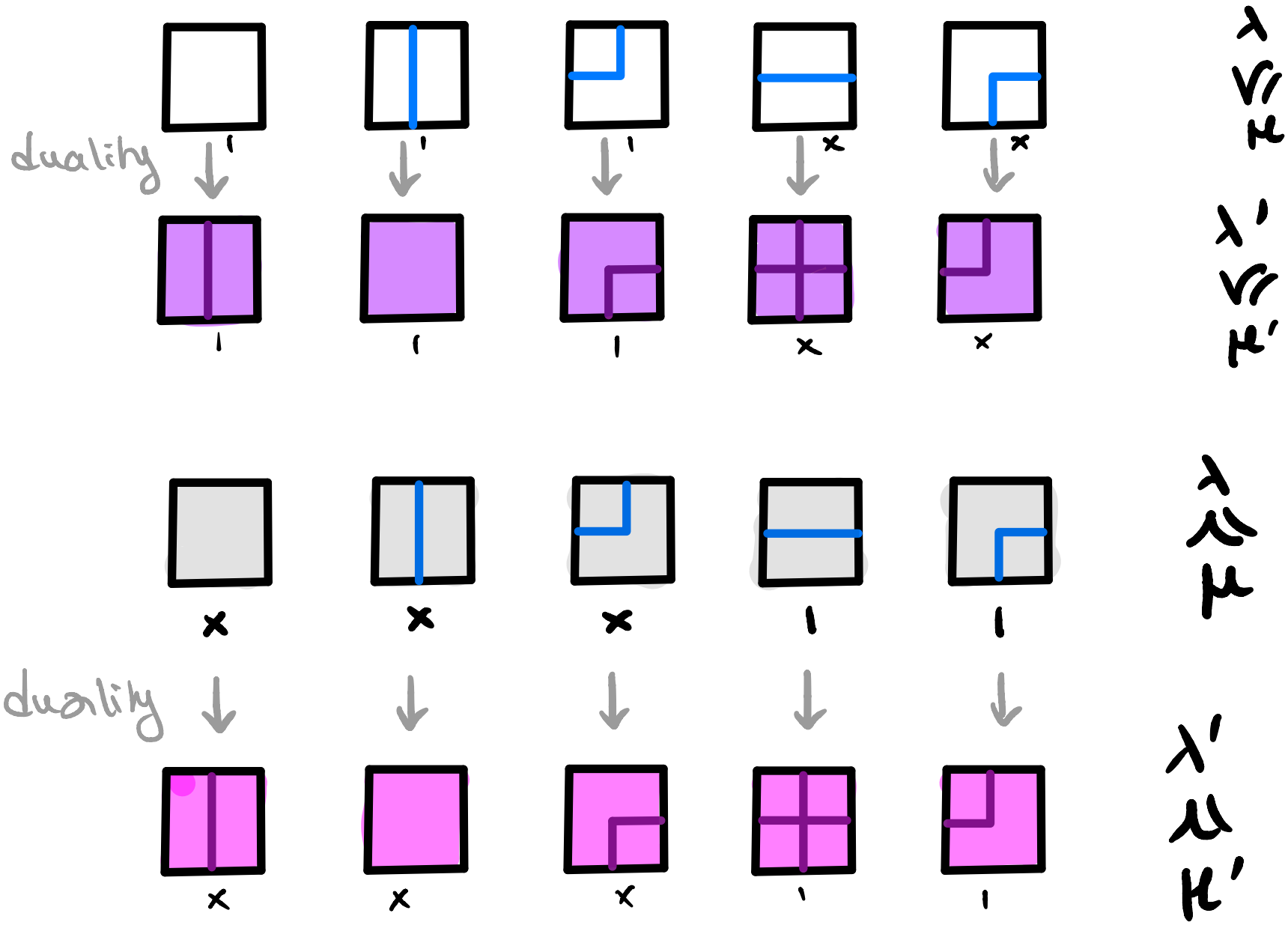
Proof Yang Baxter equation



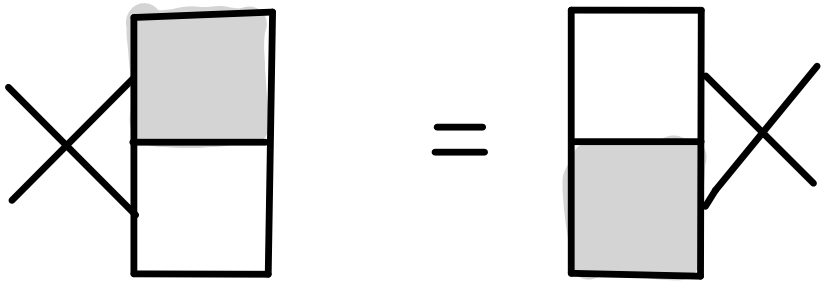



duality
1
1
1
x
x
1
1
1
x
x
x
x
x
1
1
duality
x
x
x
1
1

$\lambda \succ \mu$

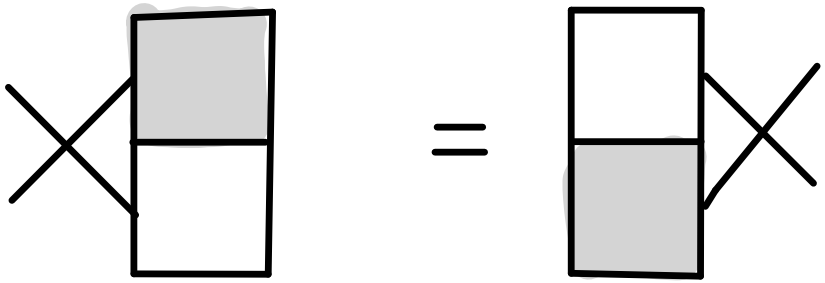


Pyramid partition (Young) Rail Yard Graphs (BBCCR)



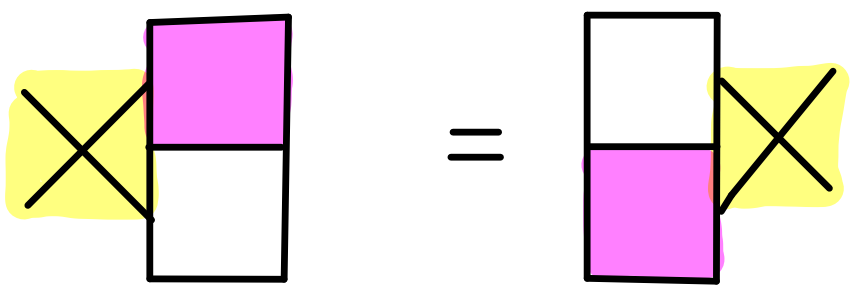
<u>Cauchy identity.</u>

$$\sum_{\lambda} s_{\lambda/\mu}(x)\, s_{\lambda/\nu}(y) = \frac{1}{1-xy} \sum_{\lambda} s_{\mu/\lambda}(y)\, s_{\nu/\lambda}(x)$$



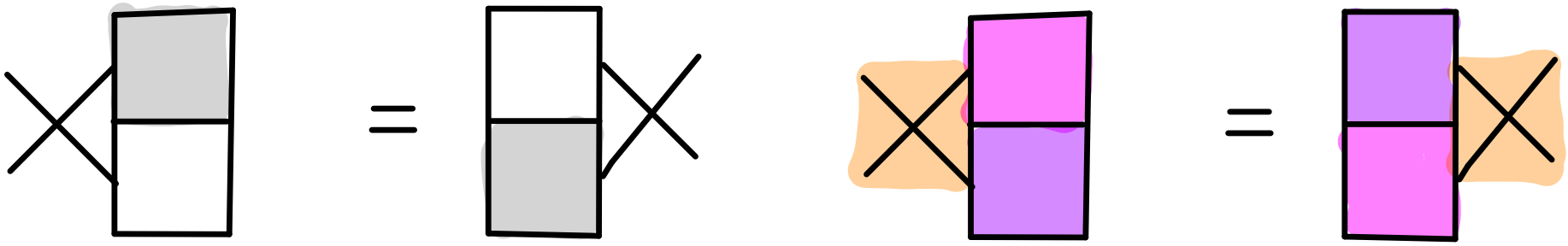
Cauchy identity.

$$\sum_{\lambda} s_{\lambda/\mu}(x)\, s_{\lambda/\nu}(y) = \frac{1}{1-xy} \sum_{\lambda} s_{\mu/\lambda}(y)\, s_{\nu/\lambda}(x)$$



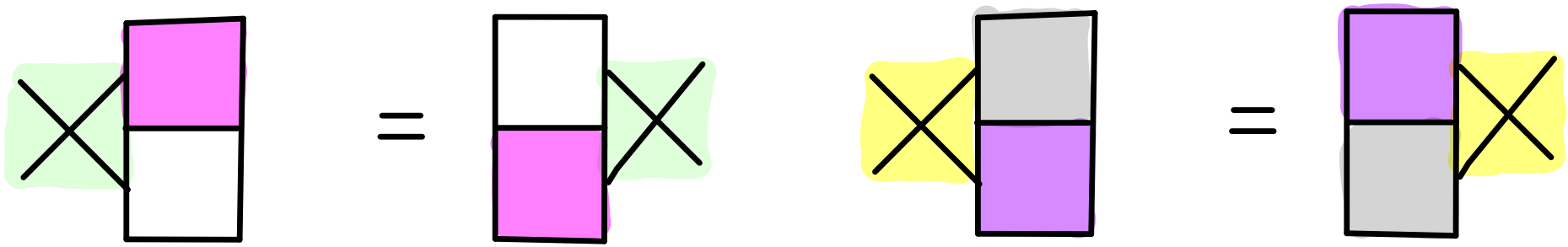
Dual Cauchy identity

$$\sum_{\lambda} s_{\lambda/\mu}(x)\, s_{(\lambda/\nu)'}(y) = (1+xy) \sum_{\lambda} s_{(\mu/\lambda)'}(y)\, s_{\nu/\lambda}(x)$$



Cauchy identity.

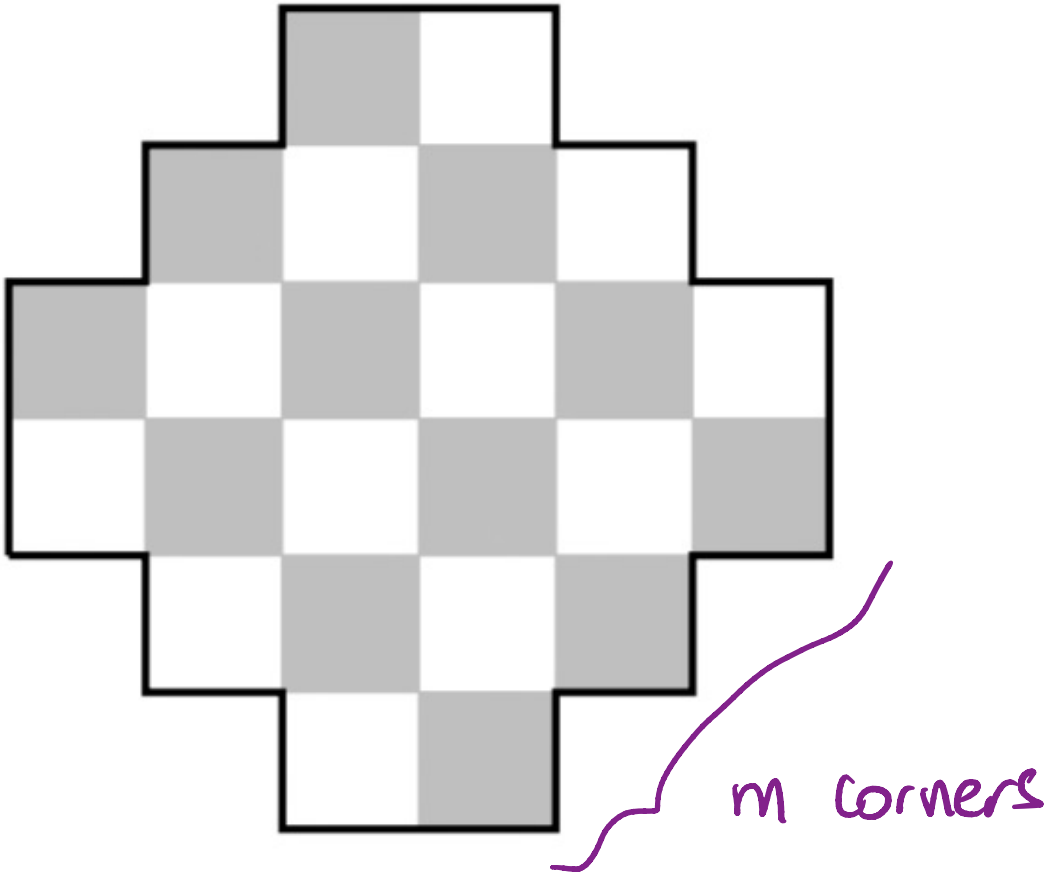
$$\sum_{\lambda} s_{\lambda/\mu}(x)\, s_{\lambda/\nu}(y) = \frac{1}{1-xy} \sum_{\lambda} s_{\mu/\lambda}(y)\, s_{\nu/\lambda}(x)$$



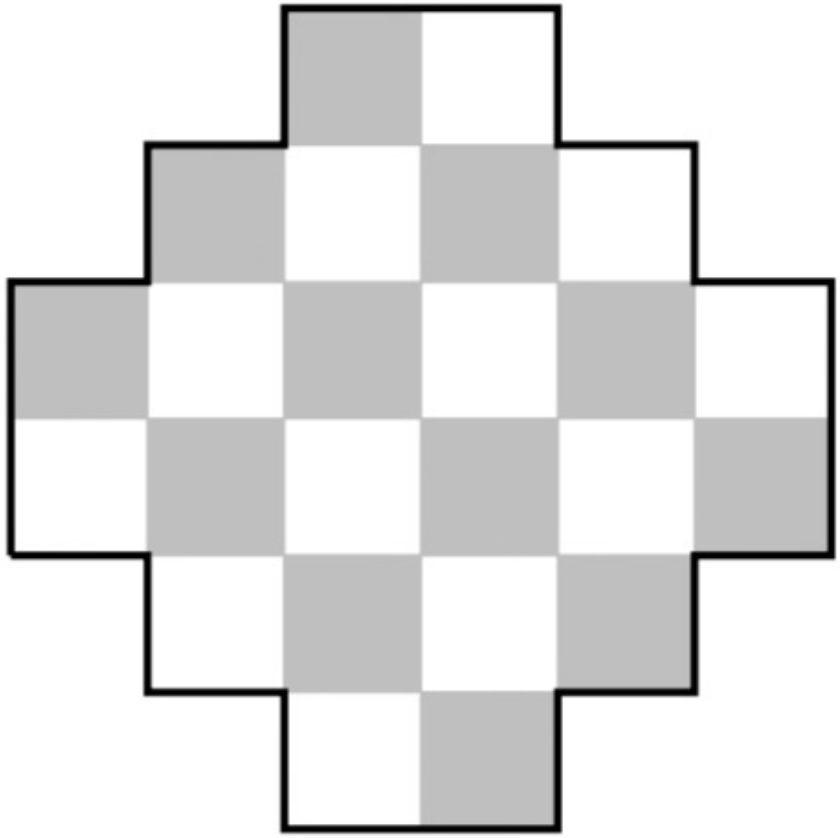
Dual Cauchy identity

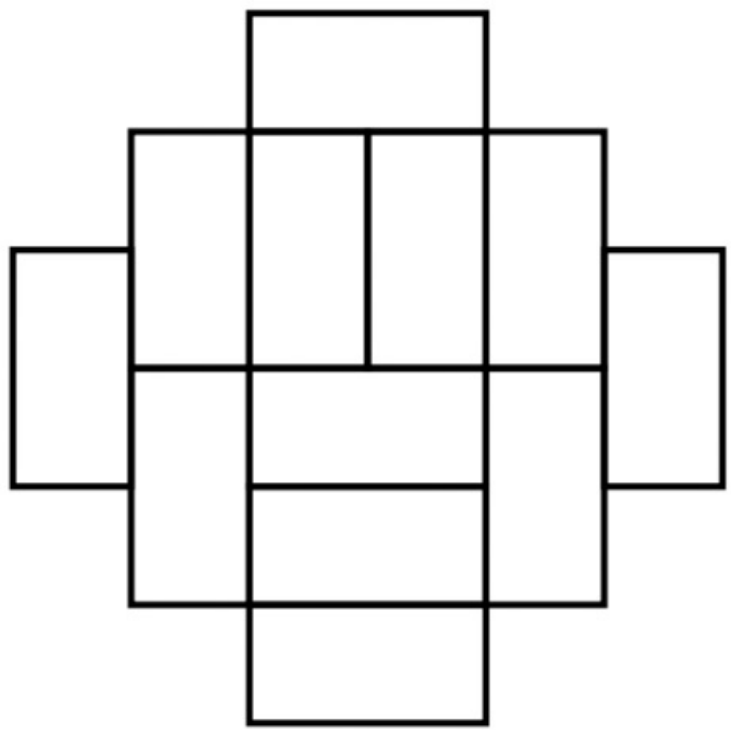
$$\sum_{\lambda} s_{\lambda/\mu}(x)\, s_{(\lambda/\nu)'}(y) = (1+xy) \sum_{\lambda} s_{(\mu/\lambda)'}(y)\, s_{\nu/\lambda}(x)$$

Domino tilings of the Aztec diamond (Propp et al 90s)



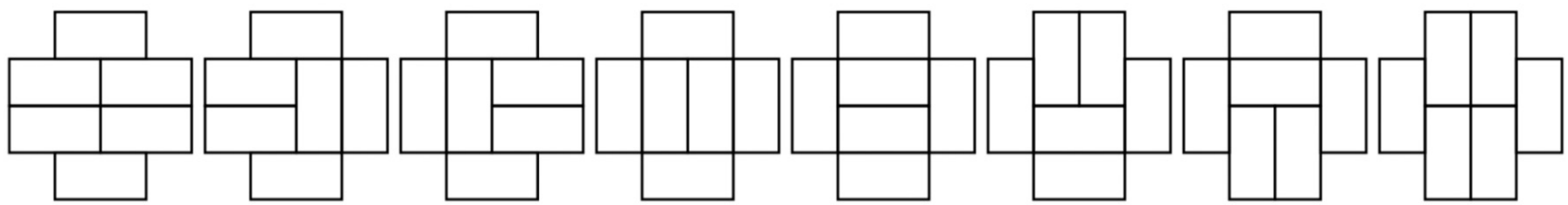
Domino tilings of the Aztec diamond (Propp et al 90s)





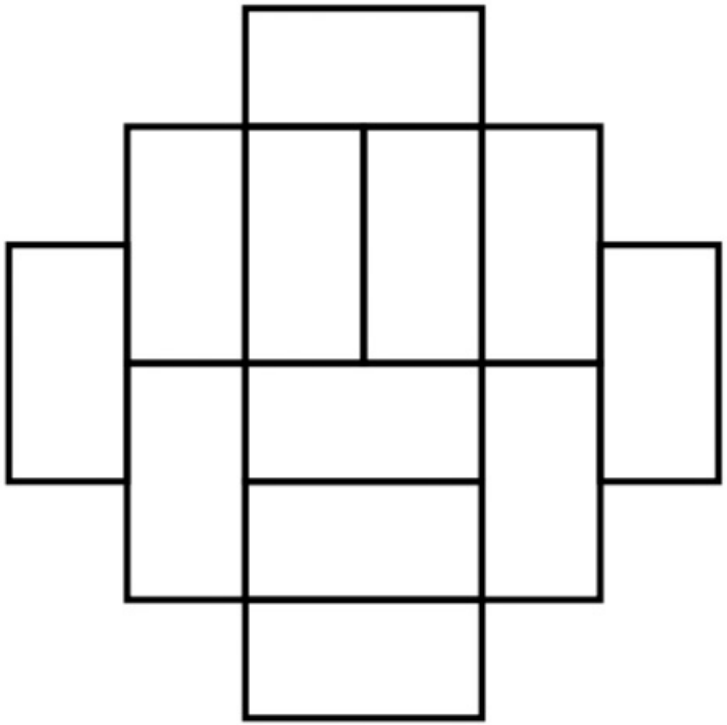
Domino tilings of the Aztec diamond

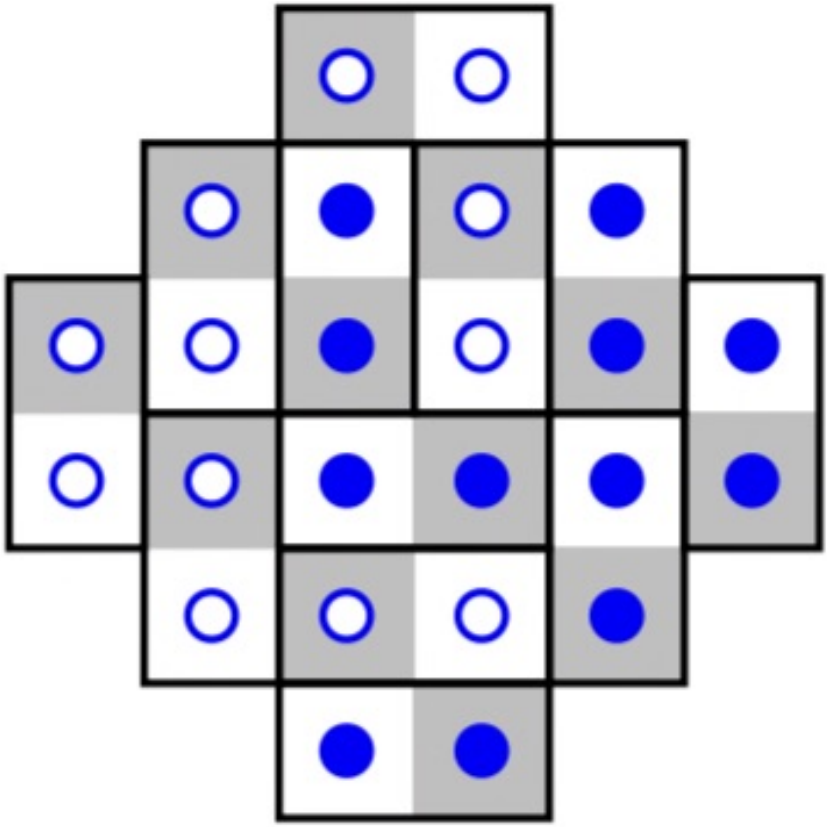
$m = 2$



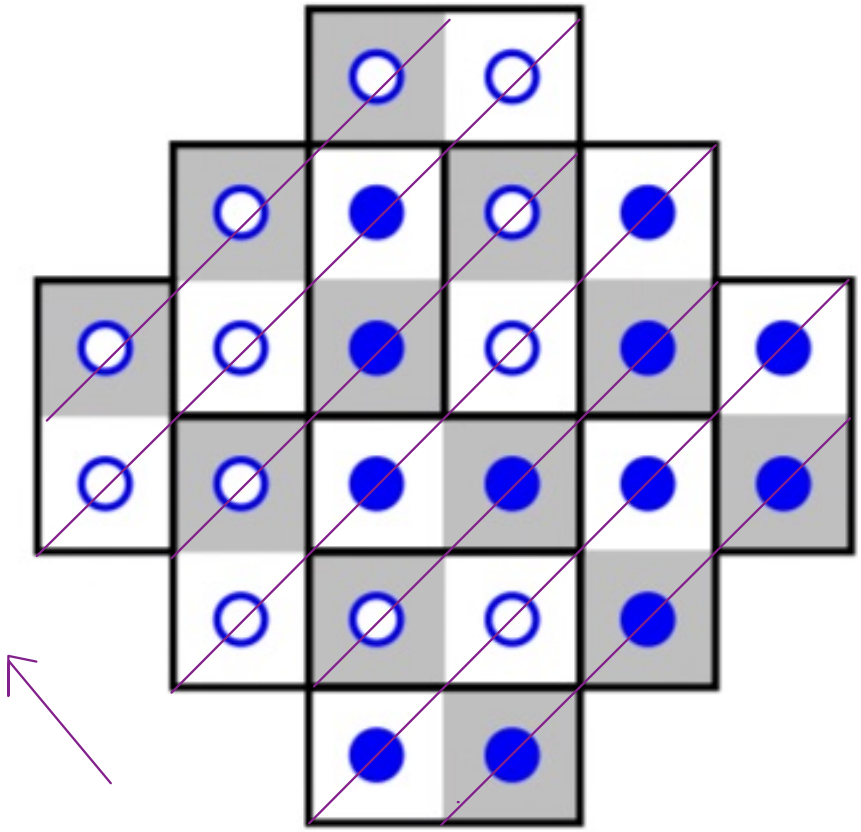
Theorem The number of domino Tilings of the Aztec diamond with m corners is $2^{\binom{m+1}{2}}$

Domino tilings of the Aztec diamond

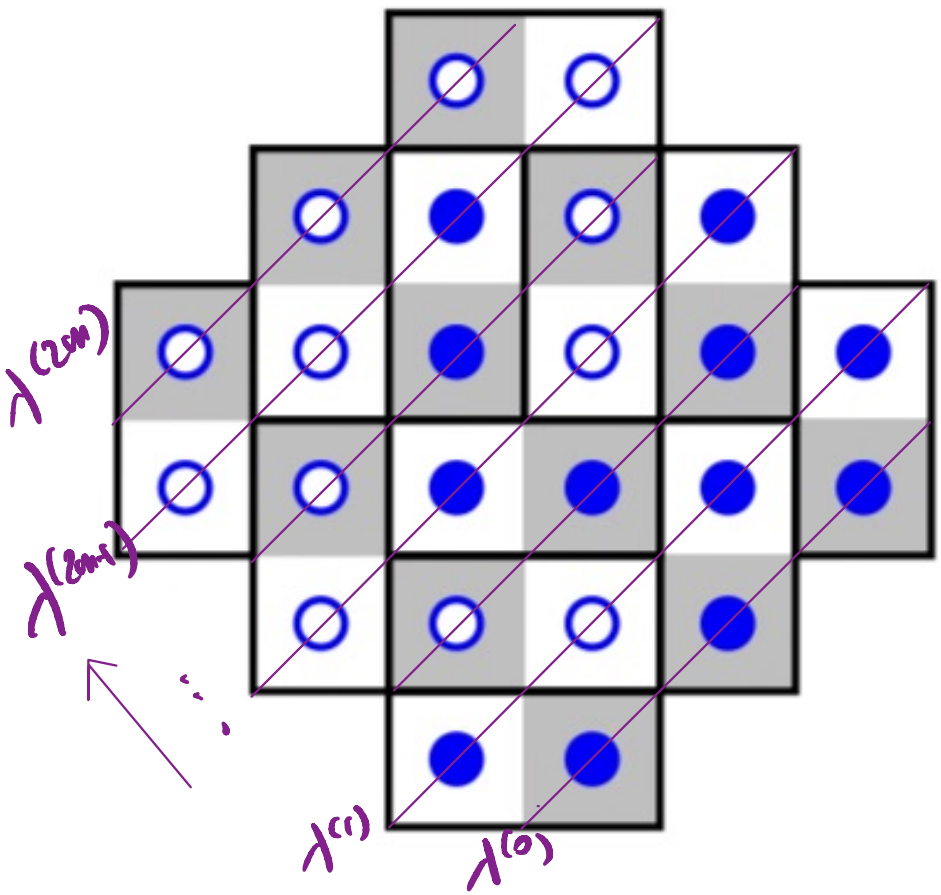




Domino tilings of the Aztec diamond



Domino tilings of the Aztec diamond

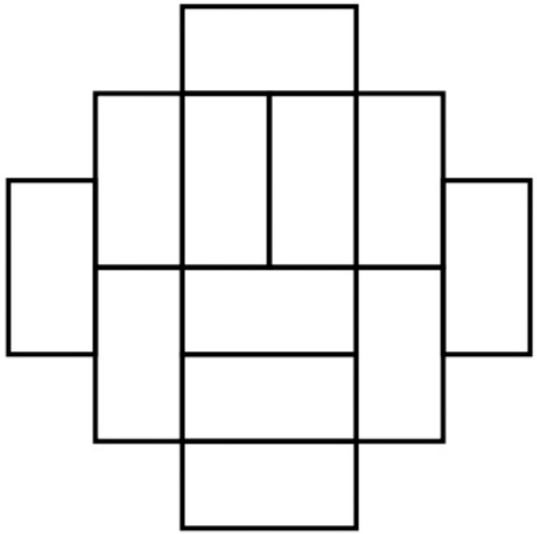


$$\emptyset = \lambda^{(0)} \prec' \lambda^{(1)} \succ \lambda^{(2)} \cdots \prec' \lambda^{(2m-1)} \succ \lambda^{(2m)} = \emptyset$$

$$Z_n(x,y) = \sum_{\Lambda} \prod_i x_i^{|\lambda^{(2i-1)}/\lambda^{(2i-2)}|} \, y_i^{|\lambda^{(2i-1)}/\lambda^{(2i)}|}$$

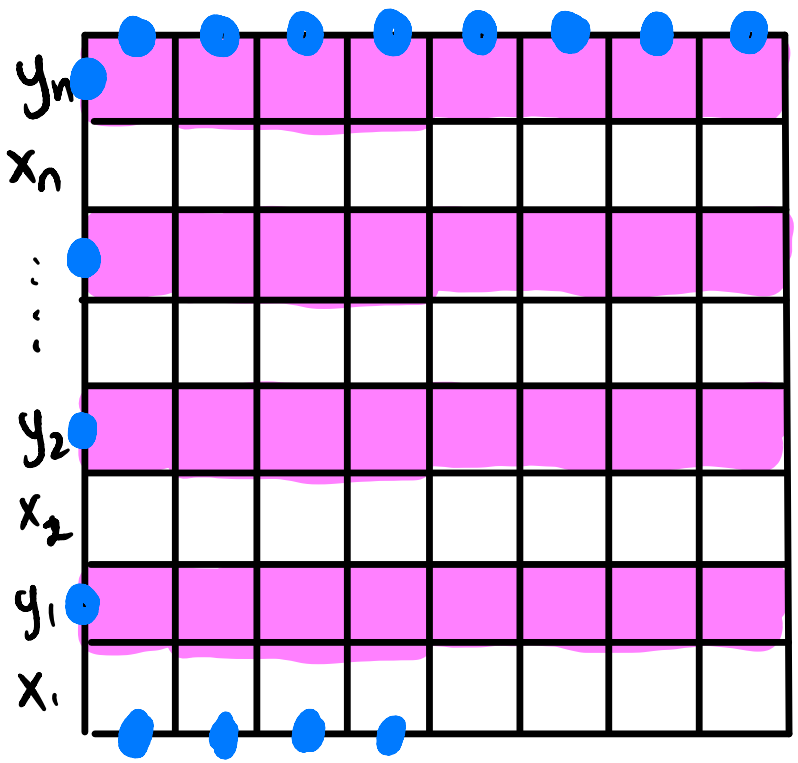
$$= \prod_{1 \le i \le j \le m} (1 + x_i y_j)$$

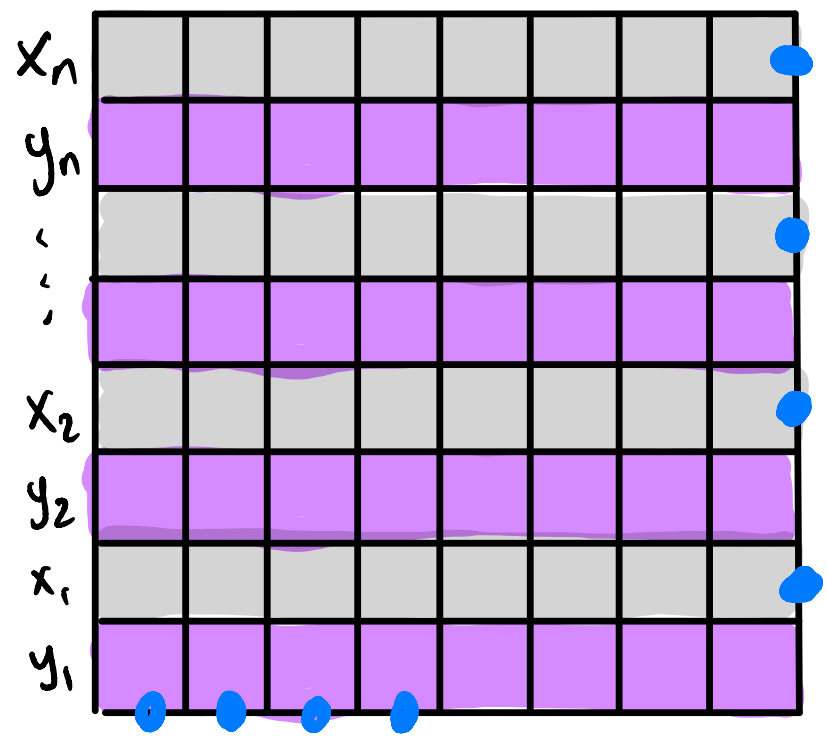
Aztec diamond $\longrightarrow$ Vertex models

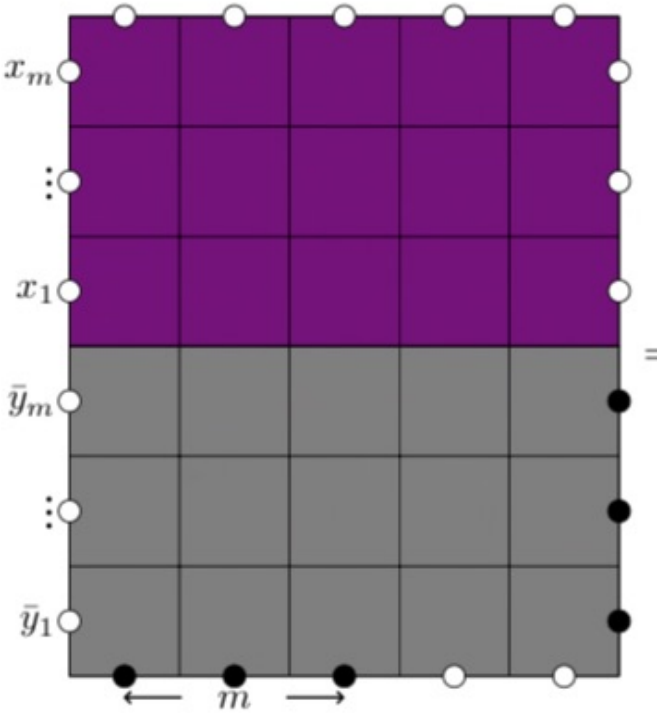


$$\prod_{i \leq j} (1 + x_i y_j)$$

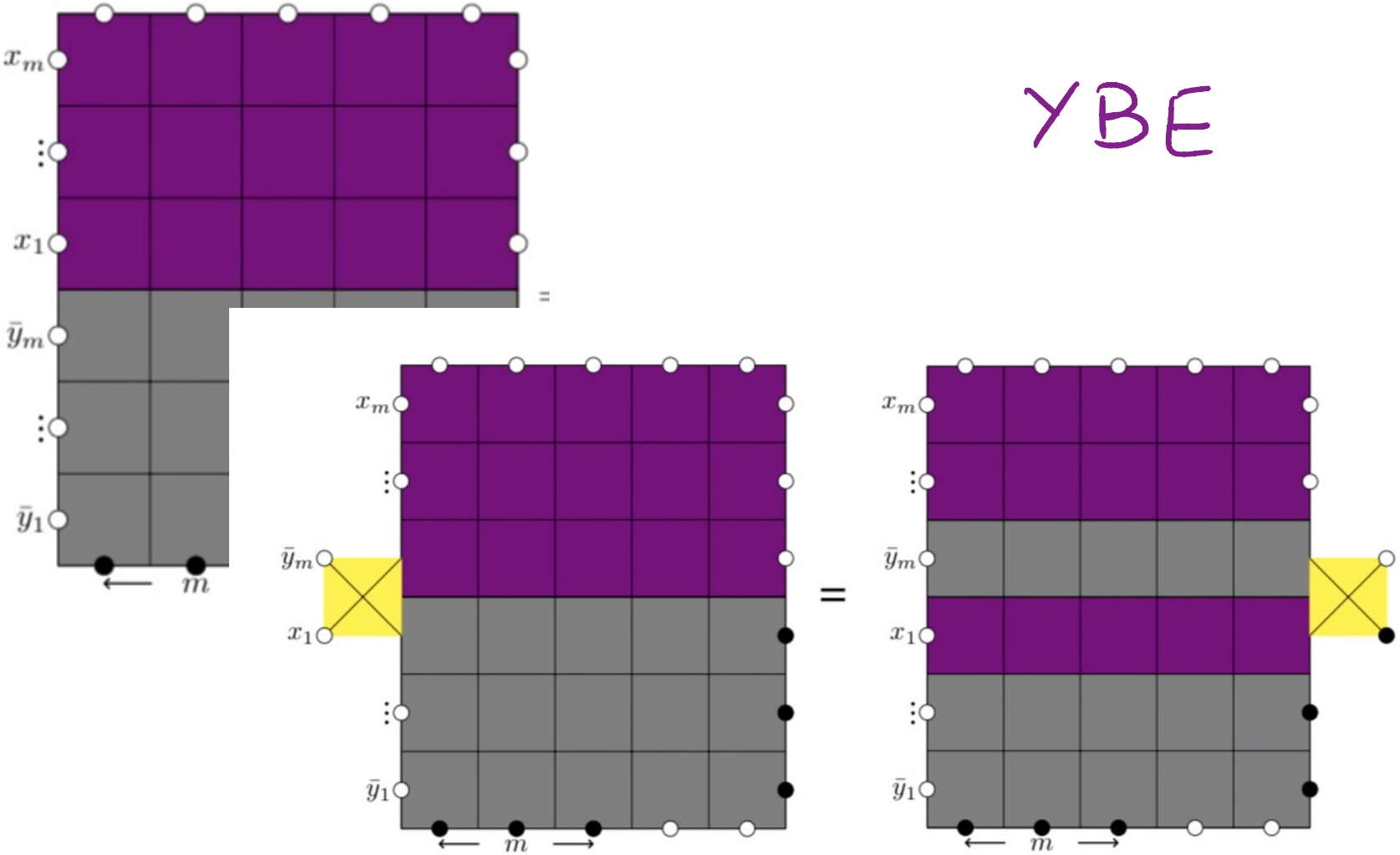
Steep tilings
(Bouttier, Chapuy, C.)
2013

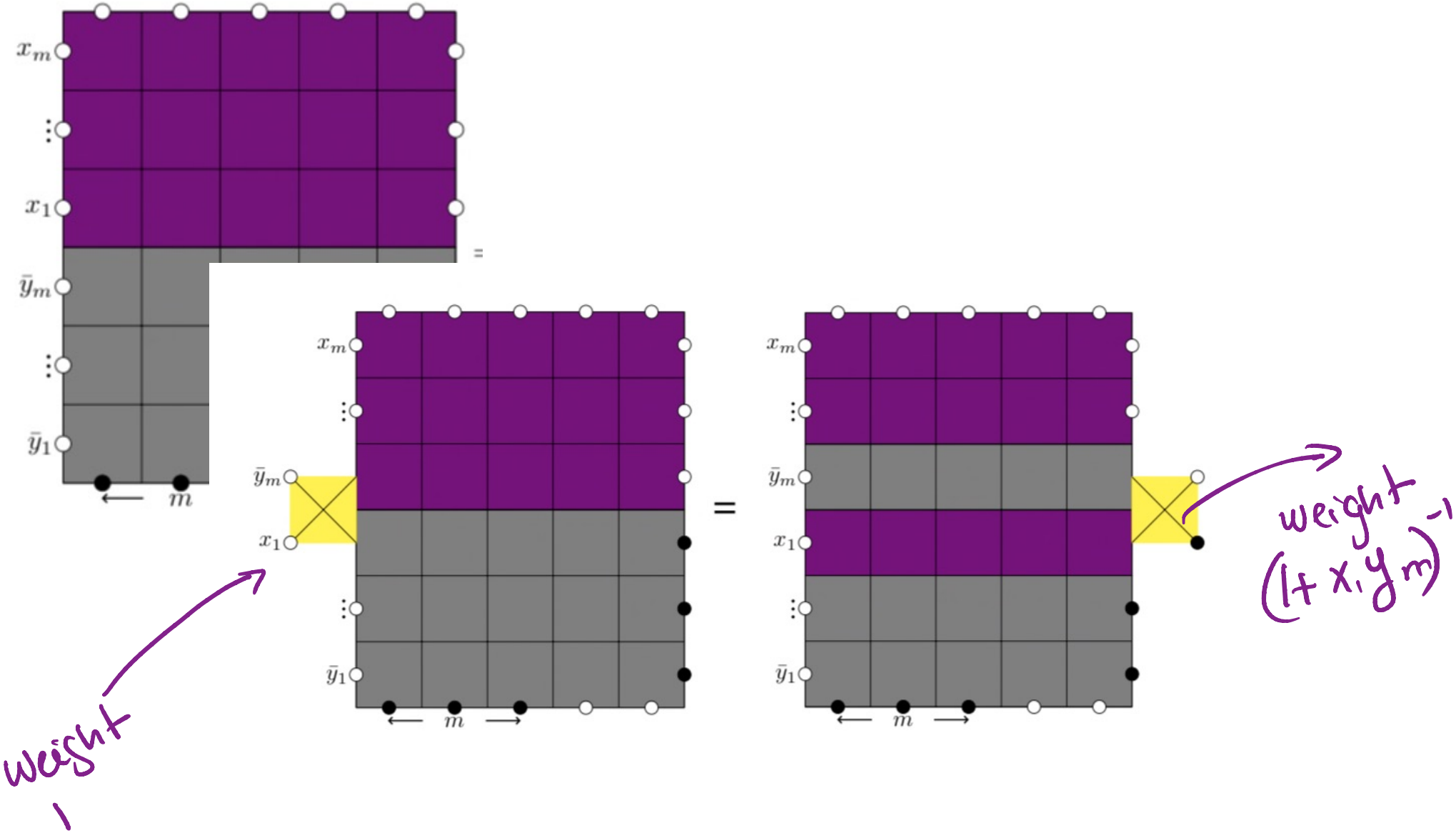


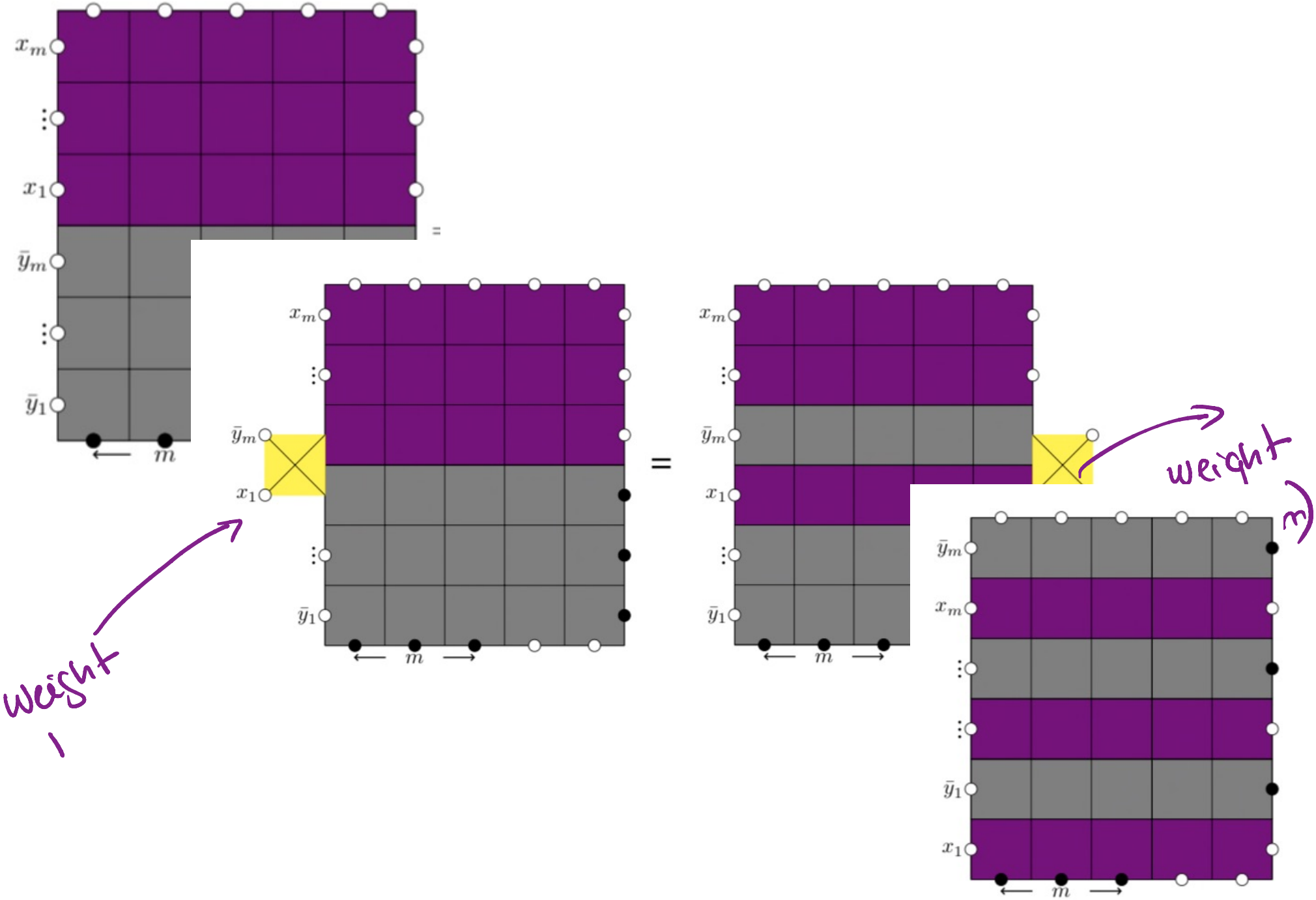


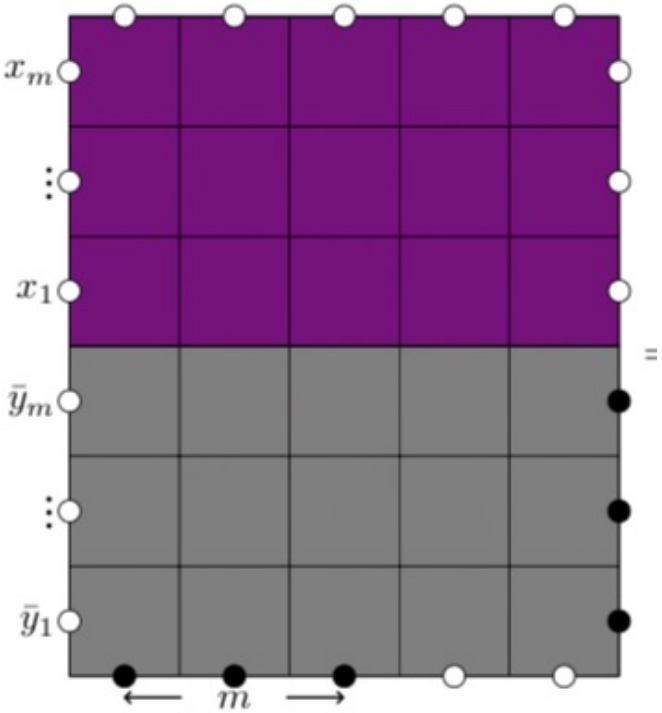

x_m
$\vdots$
x_1
$\bar{y}_m$
$\vdots$
$\bar{y}_1$
m
=

YBE

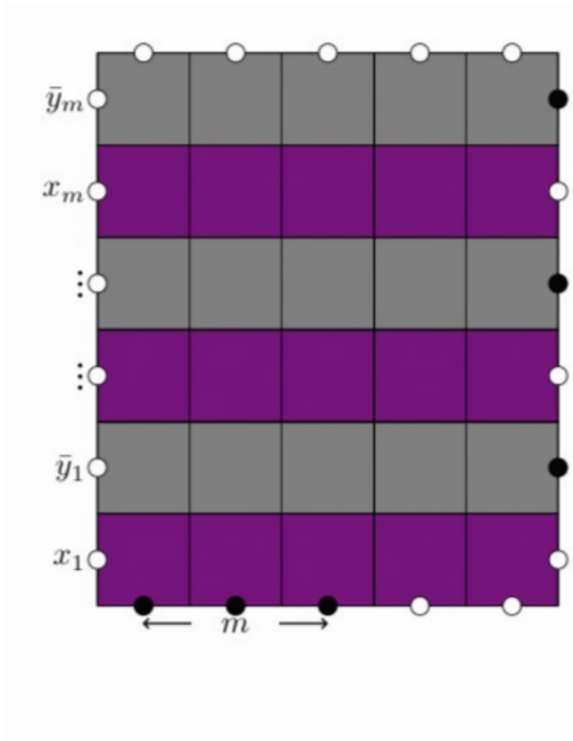



x_m
x_1
$\bar{y}_m$
$\bar{y}_1$
m
=
weight
1
weight
$(1+x_1y_m)^{-1}$


x_m
x_1
$\bar{y}_m$
$\bar{y}_1$
m
=
weight
1
weight
m

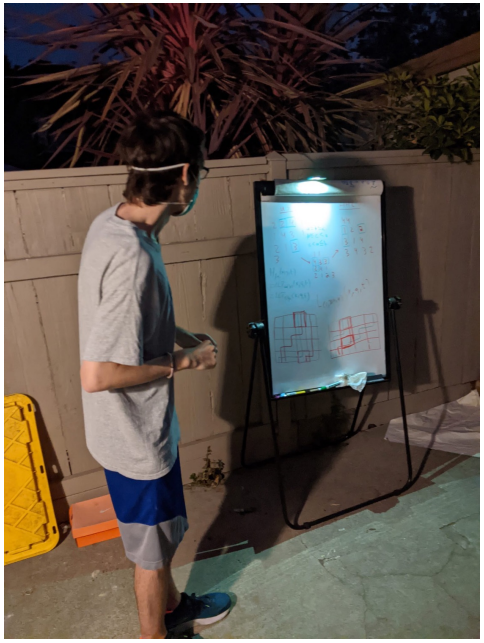


$=$

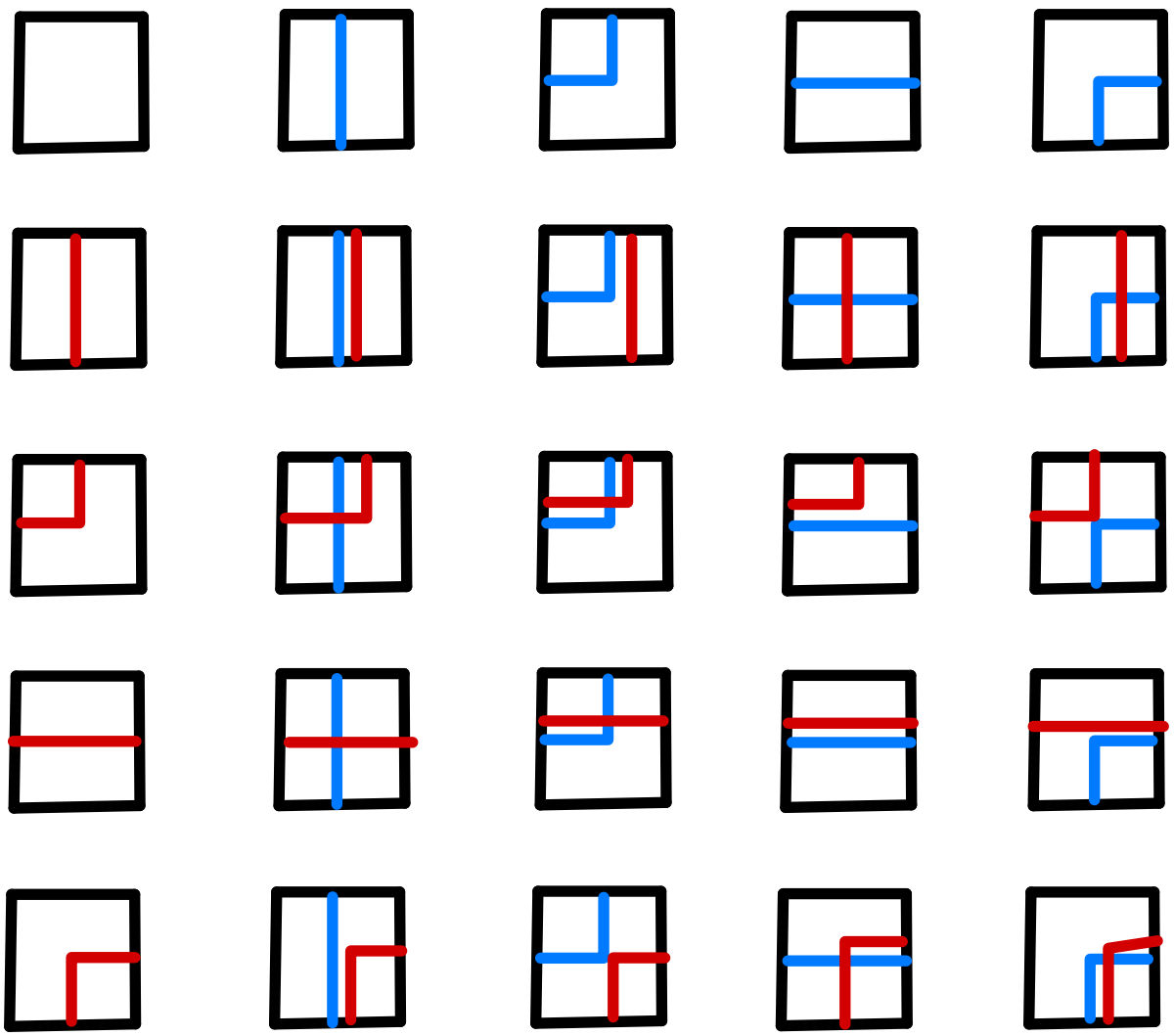


$$\times \frac{1}{\prod_{1 \le i \le j \le m} (1 + x_i y_j)}$$

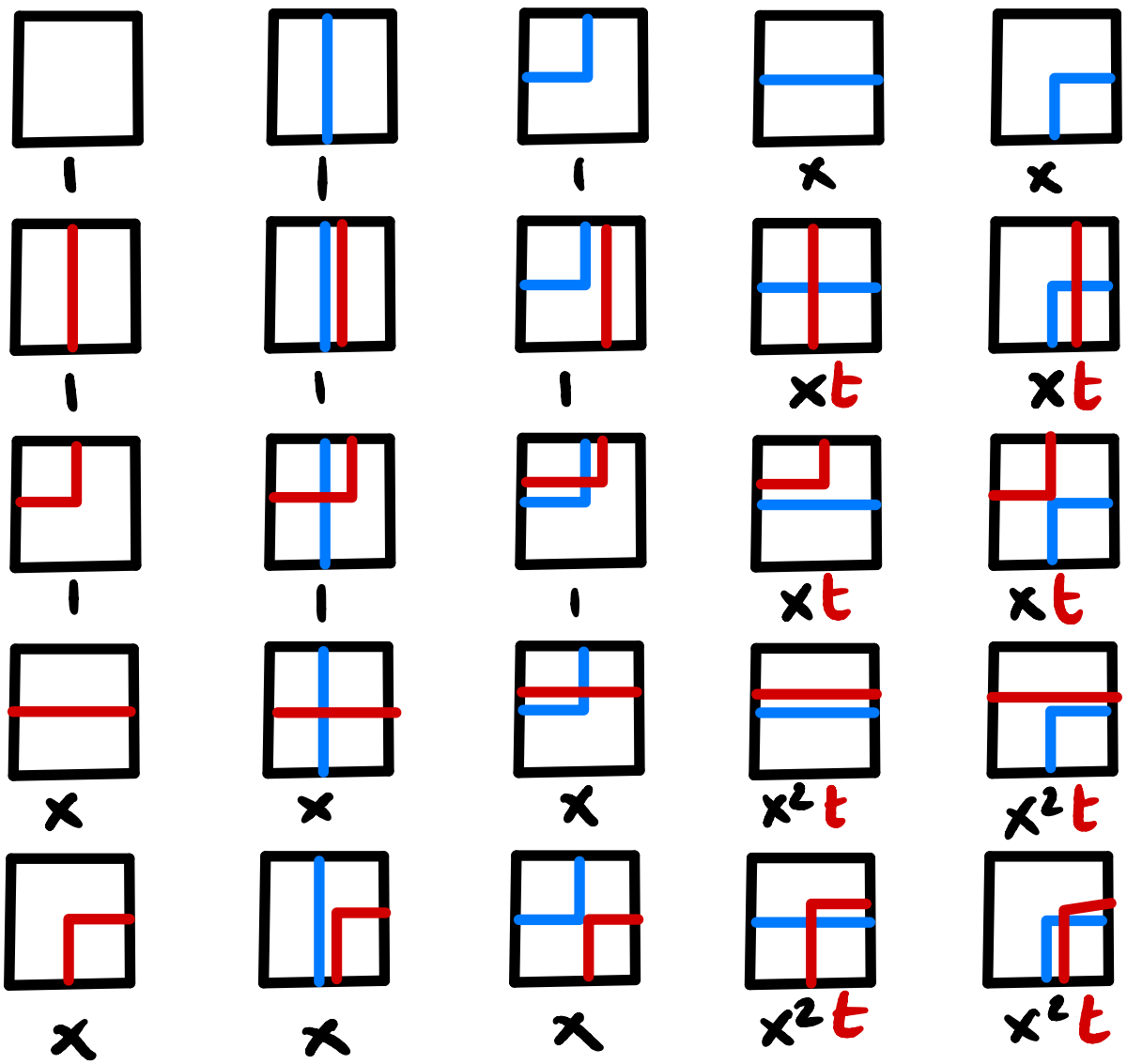
Colored five vertex models



$(\text{Five})^2$ vertex model

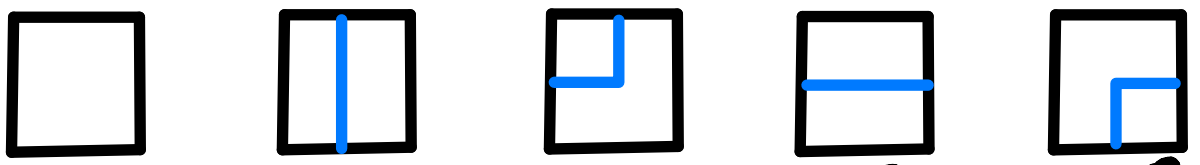


$(\text{Five})^2$ vertex model



(Five)^k vertex models

Weight for color i $\qquad i = 1, \dots, k$



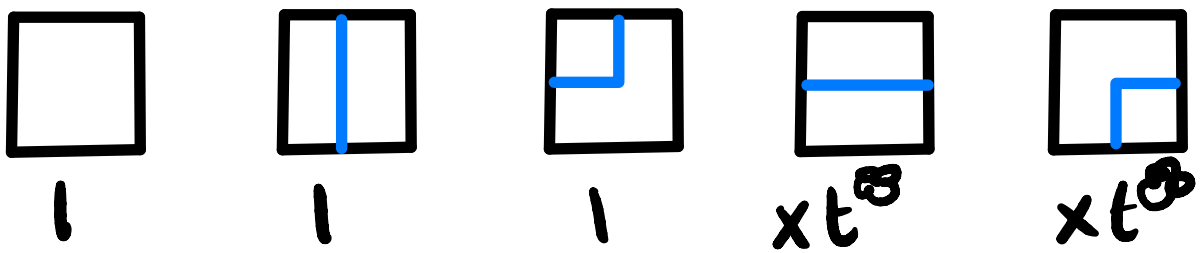
$1 \qquad 1 \qquad 1 \qquad x t^{\mathcal{B}} \qquad x t^{\mathcal{B}}$

$\mathcal{B}$: # larger colors that are present

Weight of a vertex $= \prod_{i=1}^{k}$ weight of color i

(Five)^k vertex models

Weight for color i

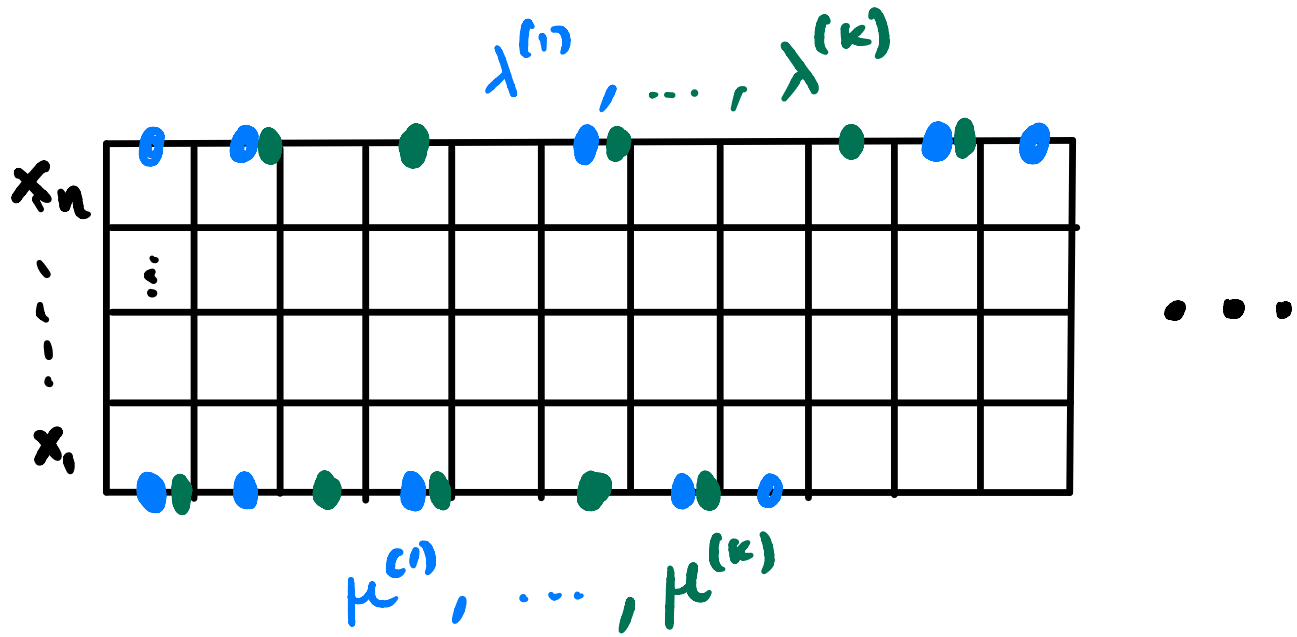


$\circledast$: # larger colors that are present

R < B < G

Weight $x^2 t^2$ $\qquad$ $x^2 t^{2+1}$

Partition function



$$Z_{\lambda/\mu}(x_1, x_2, \dots, x_n) = \sum_{\substack{c \text{ configurations} \\ \text{from } \mu \text{ to } \lambda}} wt(c)$$

Theorem (C., Gitlin, Keating & Meza 2020)

$Z_{\lambda,\mu}(x_1,\dots,x_n;t)$ is equal to the LLT polynomial $\mathcal{L}_{\lambda/\mu}(x_1,\dots,x_n;t)$

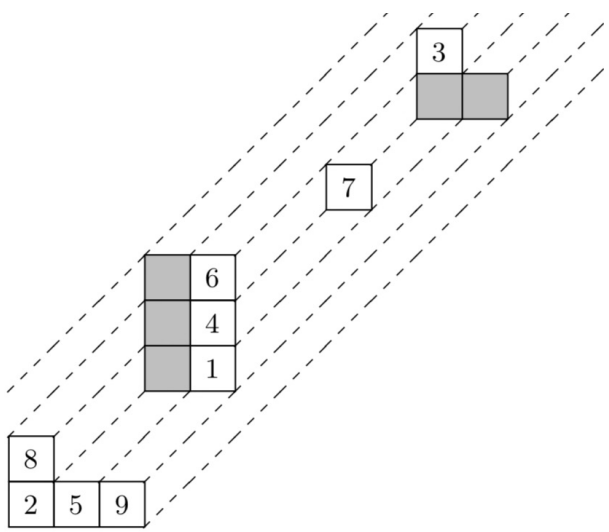
$\lambda = (\lambda^{(1)},\dots,\lambda^{(k)})$ $\quad$ $\mu = (\mu^{(1)},\dots,\mu^{(k)})$

Theorem (C., Gitlin, Keating & Meza)

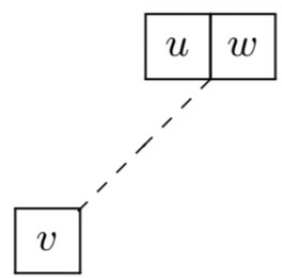
$Z_{\lambda,\mu}(x_1,\dots,x_n;t)$ is equal to the LLT polynomial $\mathcal{L}_{\lambda/\mu}(x_1,\dots,x_n;t)$

$\lambda = (\lambda^{(1)},\dots,\lambda^{(k)})$ $\qquad$ $\mu = (\mu^{(1)},\dots,\mu^{(k)})$

LLT polynomials (Haglund, Haiman, Loehr)



$$\mathcal{L}_{\lambda/\mu} = \sum_{\substack{T \text{ sequence} \\ \text{of SSYT}}} x^T t^{\mathrm{coinv}(T)}$$

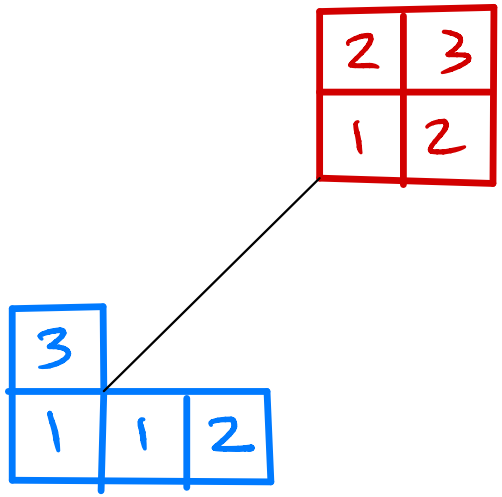


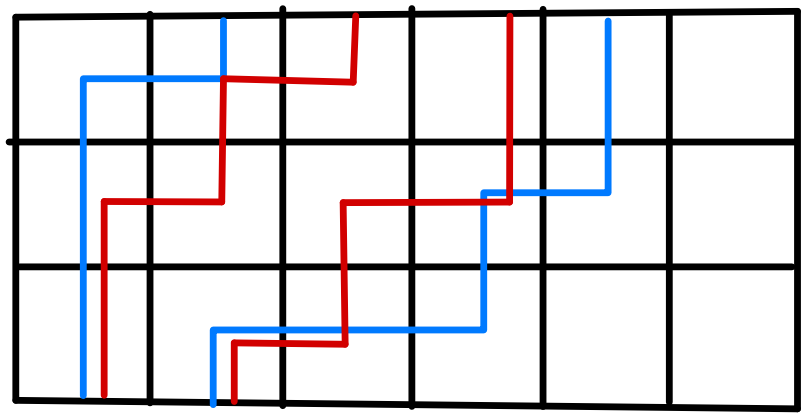
coinversion

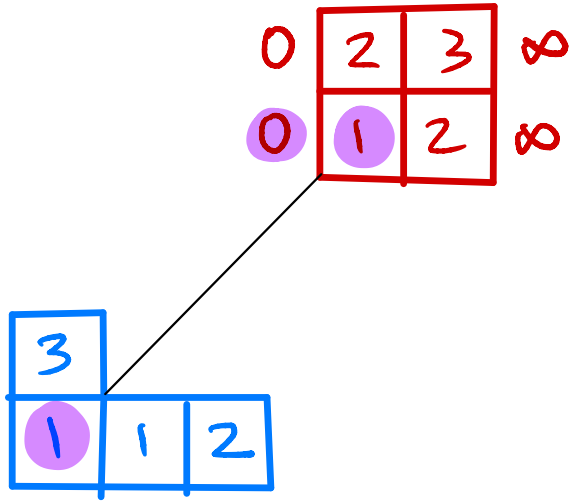
$u \leq v \leq w$

2	3
1	2

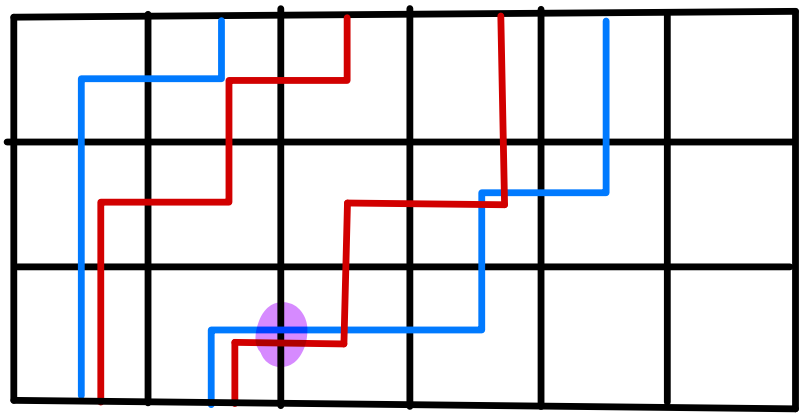
3		
1	1	2

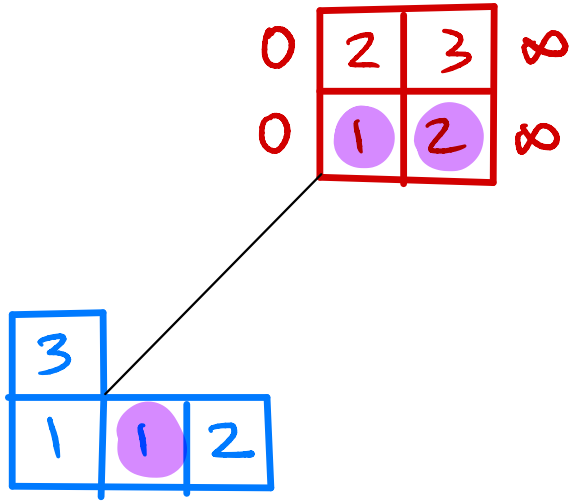

2 3
1 2
3
1 1 2

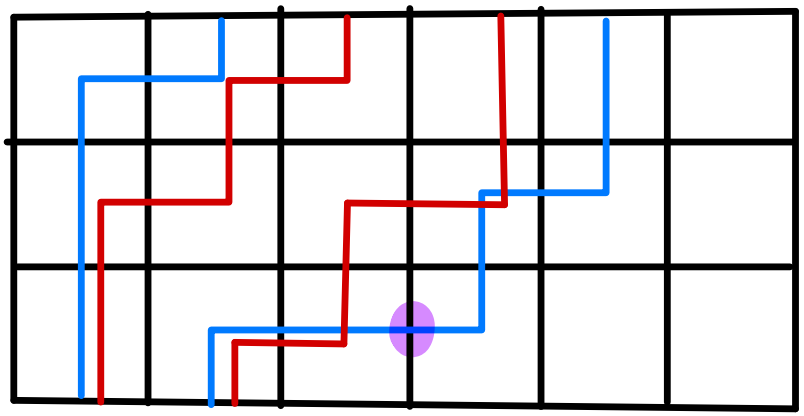


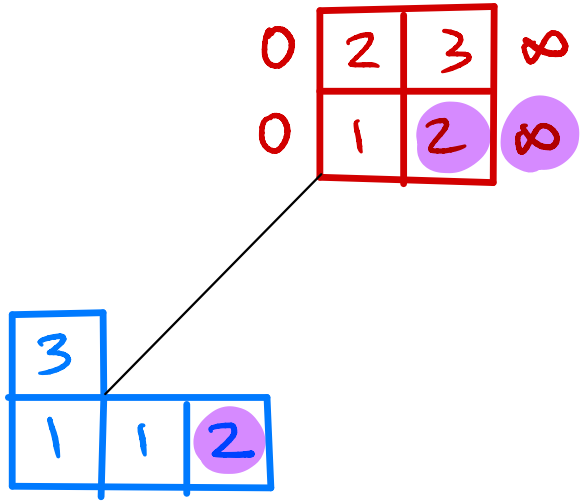


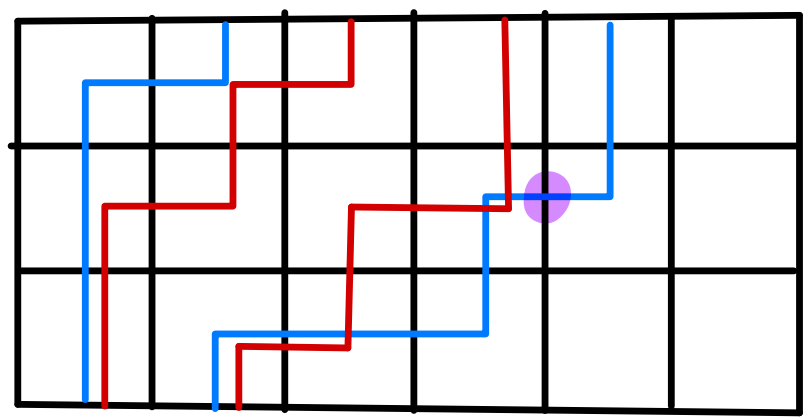
B < R




0 2 3 ∞
0 1 2 ∞
3
1 1 2

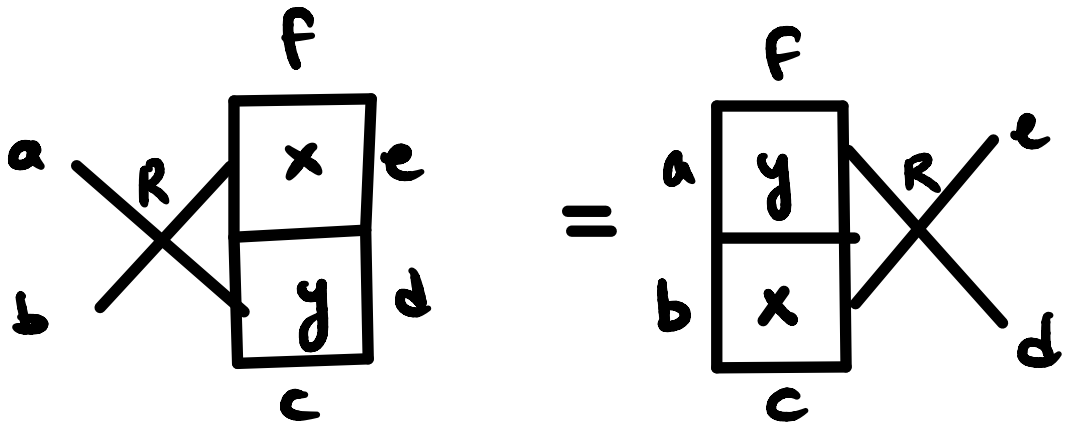






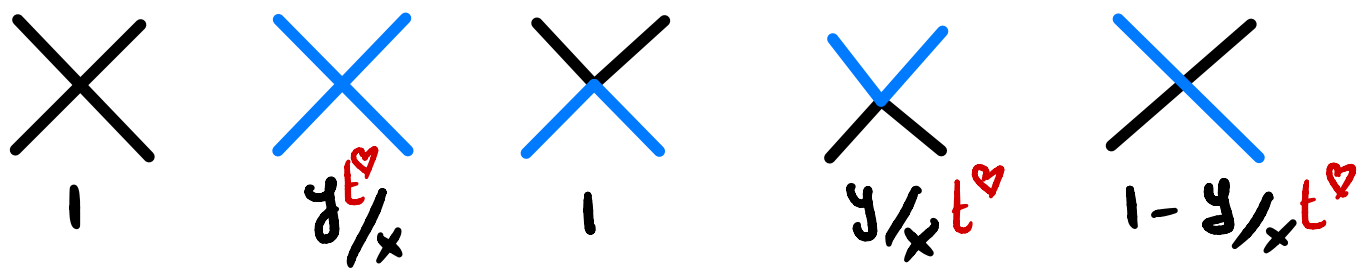
Weight $x_1^3 x_2^3 x_3^2 t^3$

Yang Baxter equation



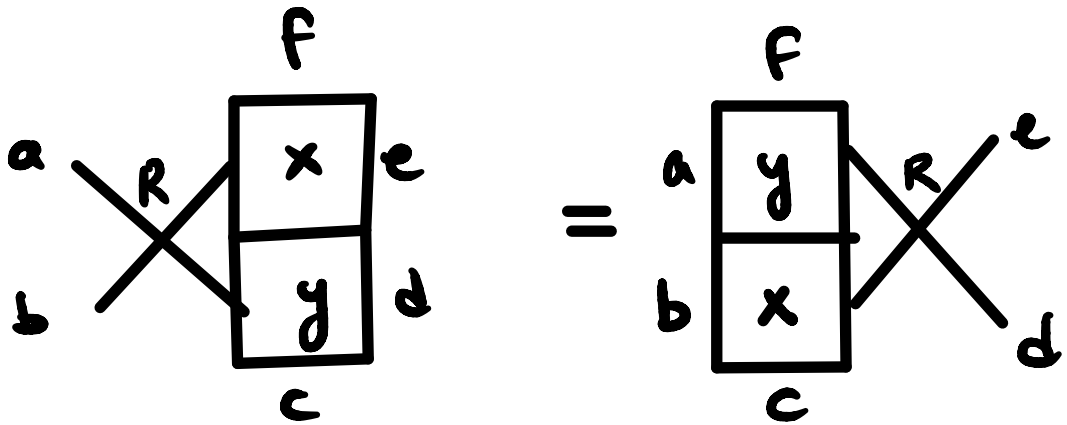
a, b, c, d, e, f fixed $\in \{\bullet, \circ\}^k$

color i R



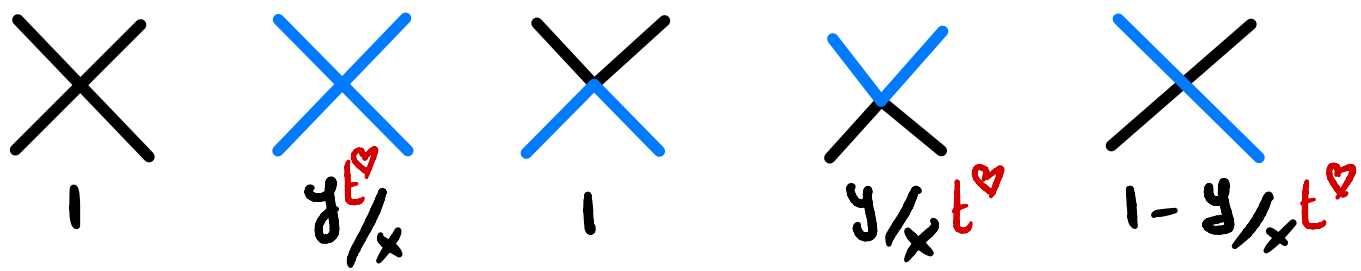
$\heartsuit$: # larger colors

Yang Baxter equation



a, b, c, d, e, f fixed $\in \{\bullet, \circ\}^k$

color i R



Proofs

Combinatorics (C., Gitlin, Keating, Meza)

Representation theory / Fusion (Aggarwal, Borodin, Wheeler)

Corollaries of the Yang Baxter equation

① $\mathcal{L}_{\lambda/\mu}(x_1, \dots, x_n; t)$ is a symmetric polynomial

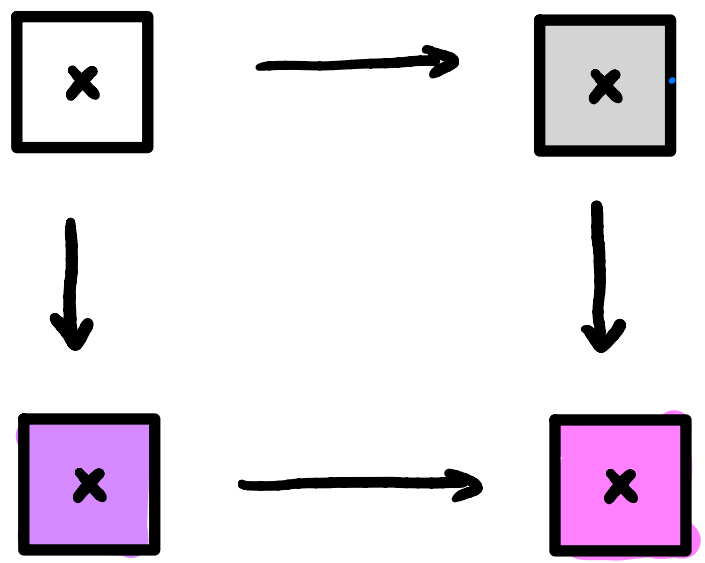
② <u>Cauchy identity</u> (Generalization of Lam OS, Aggarwal 21, Borodin Wheeler)
C., GKM 2020

$$\prod_{i=0}^{k-1} (1 - xyt^i) \sum_{\lambda} t^{d(\lambda)} \mathcal{L}_{\lambda/\mu}(x) \mathcal{L}_{\lambda/\nu}(y)$$

$$= \sum_{\lambda} t^{d(\lambda)} \mathcal{L}_{\mu/\lambda}(y) \mathcal{L}_{\nu/\lambda}(x)$$

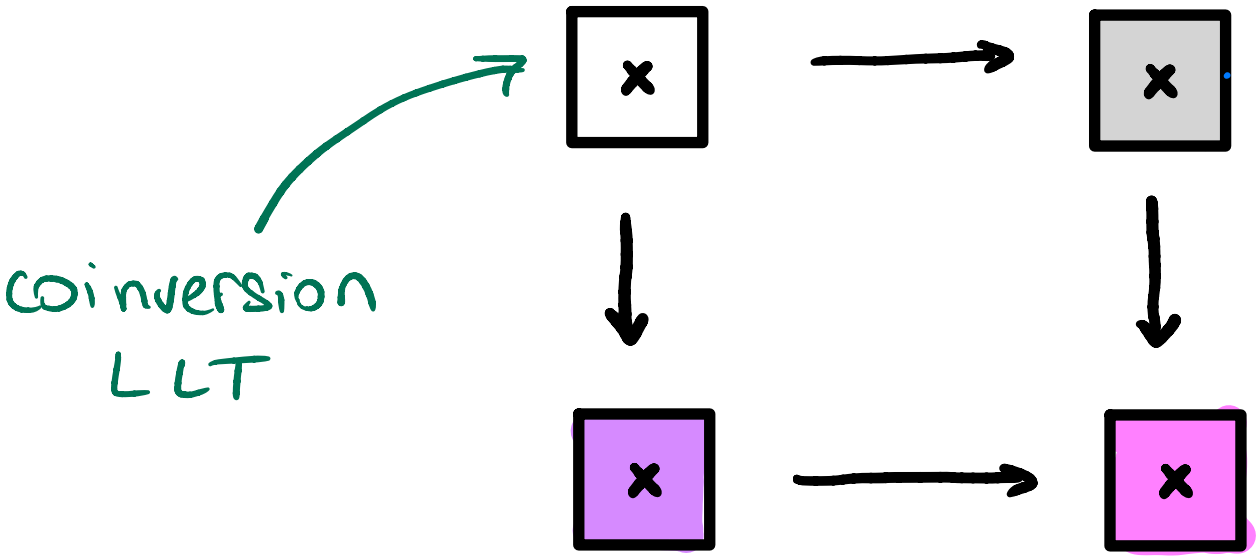
Generalisation to the four (five)^{k} vertex models

Supersymmetric LLT polynomials
(Gitlin & Keating 2021)



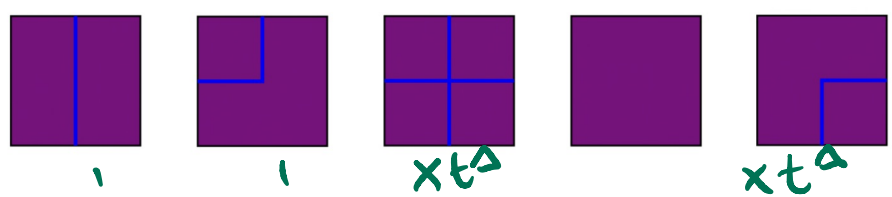
Generalisation to the four (five)$^{\kappa}$ vertex models

Supersymmetric LLT polynomials
(Gillin & Keating 2021)



coinversion LLT

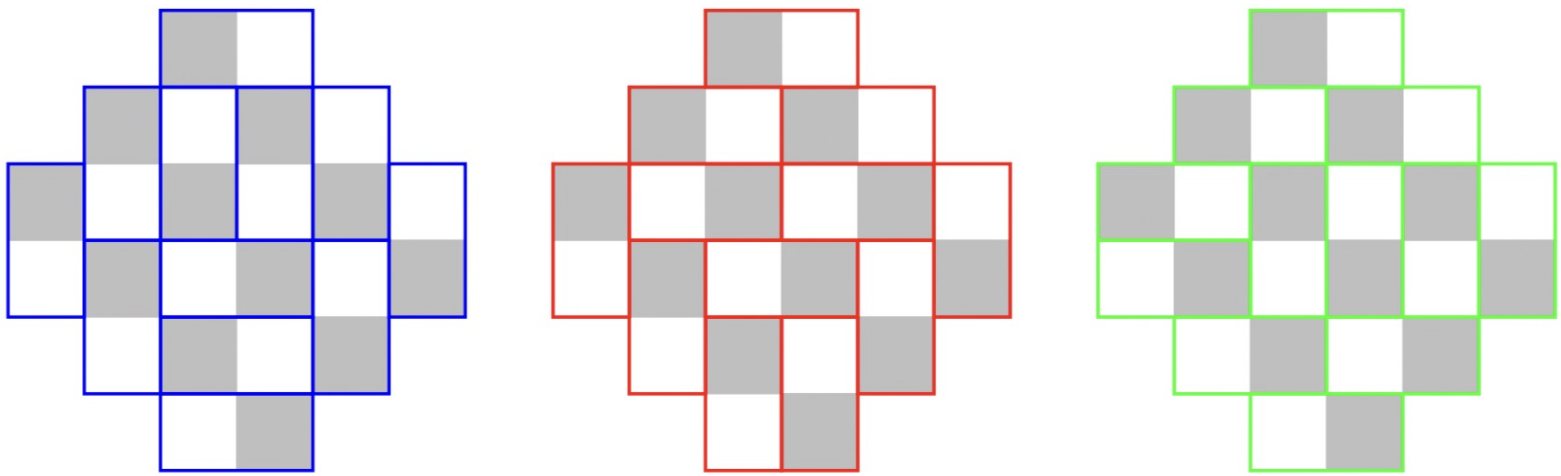
inversion LLT



Δ = number of lagercolor $\square$

k-tilings of the Aztec diamond

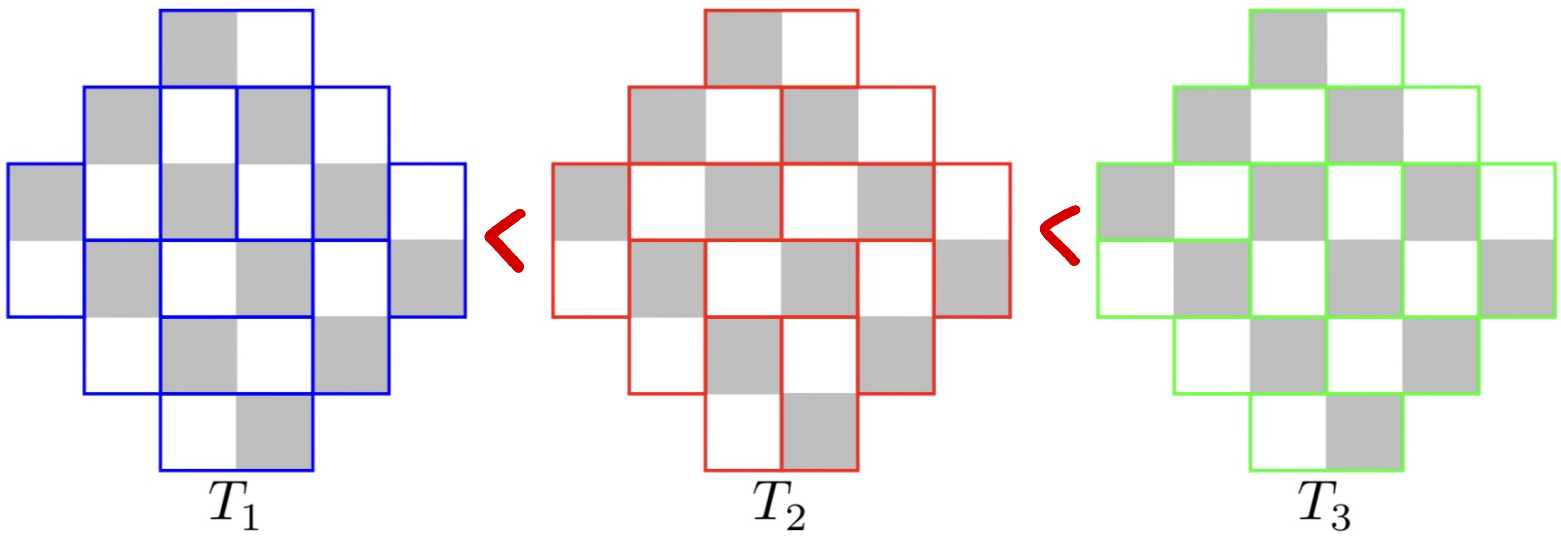
$k = 3$

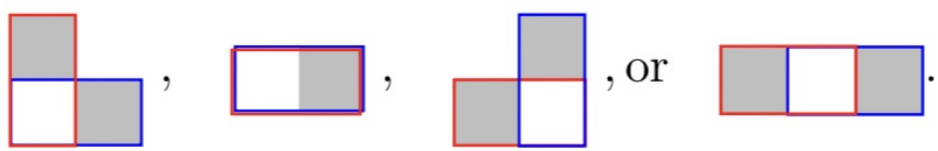


T_1 T_2 T_3

k-tilings of the Aztec diamond

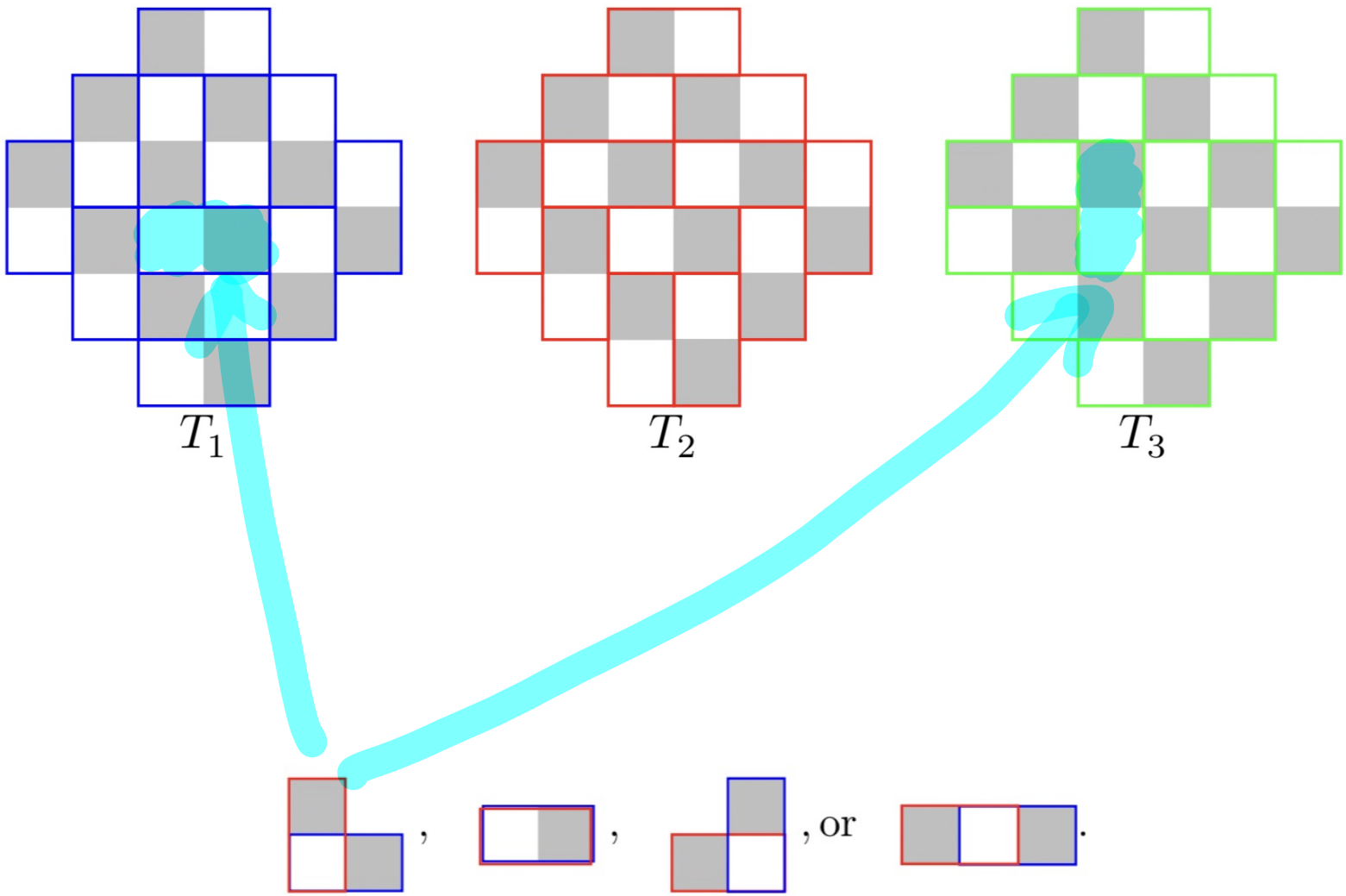
$k = 3$





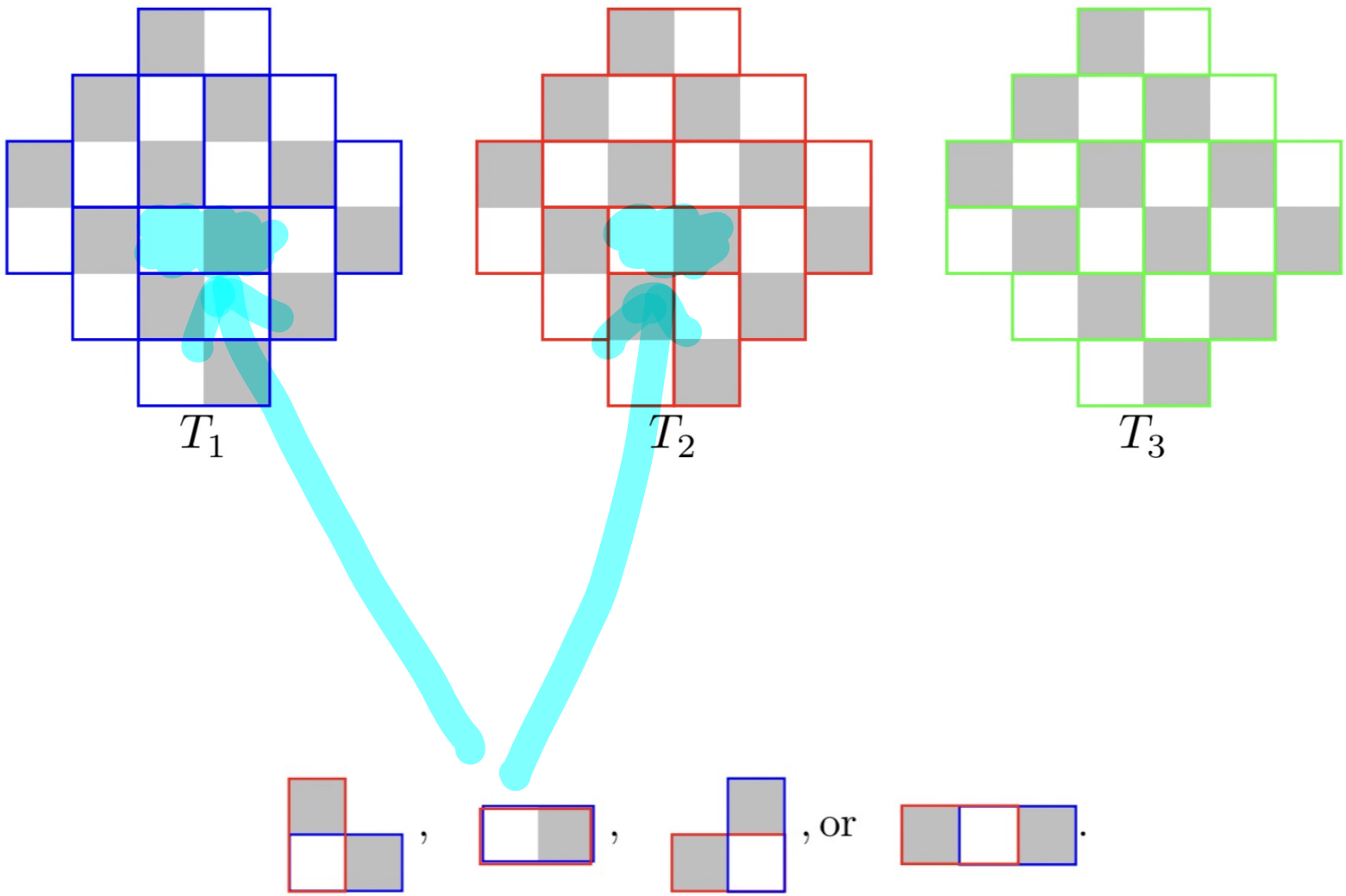
k-tilings of the Aztec diamond

$k = 3$



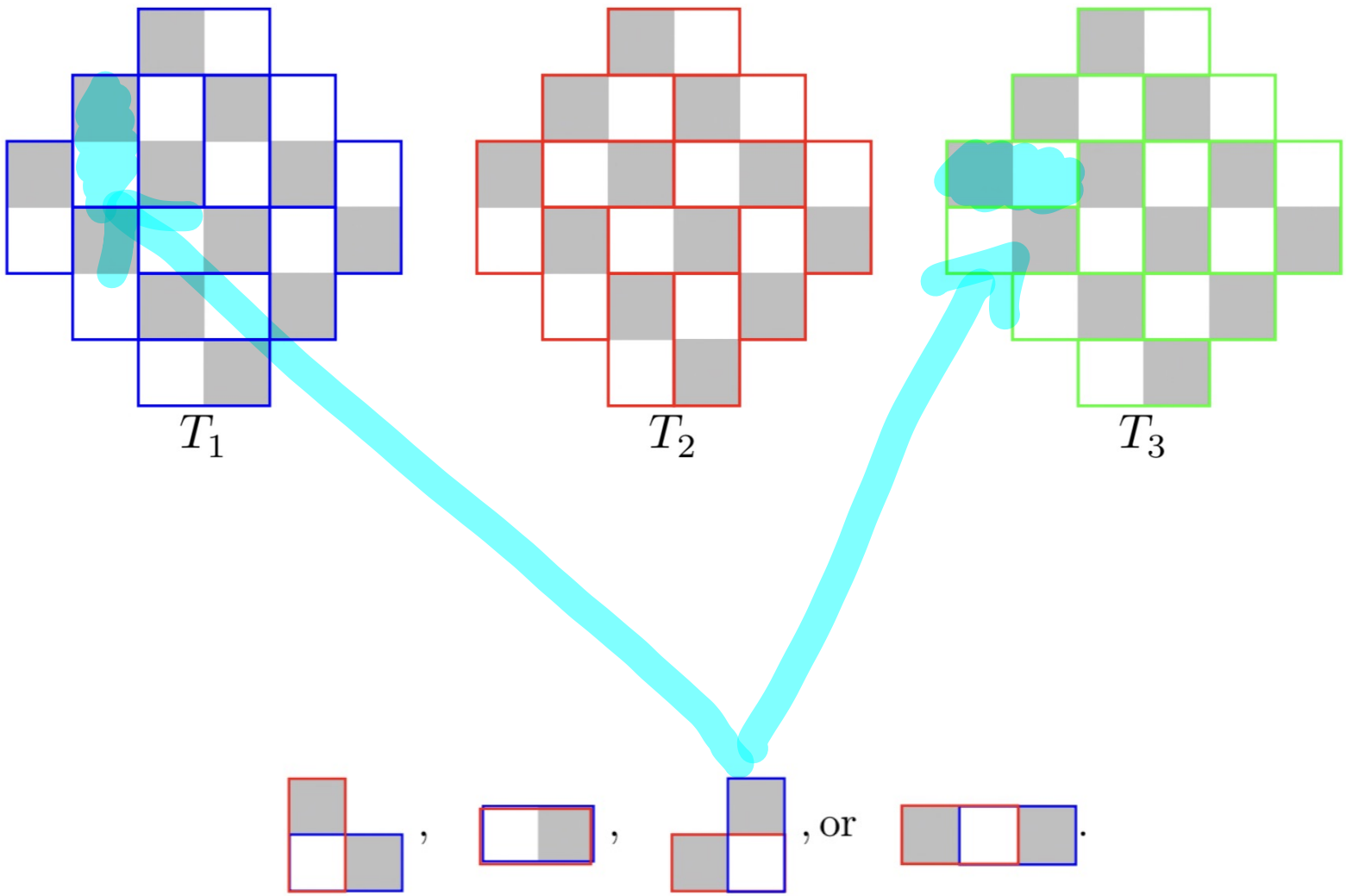
k-tilings of the Aztec diamond

$k = 3$



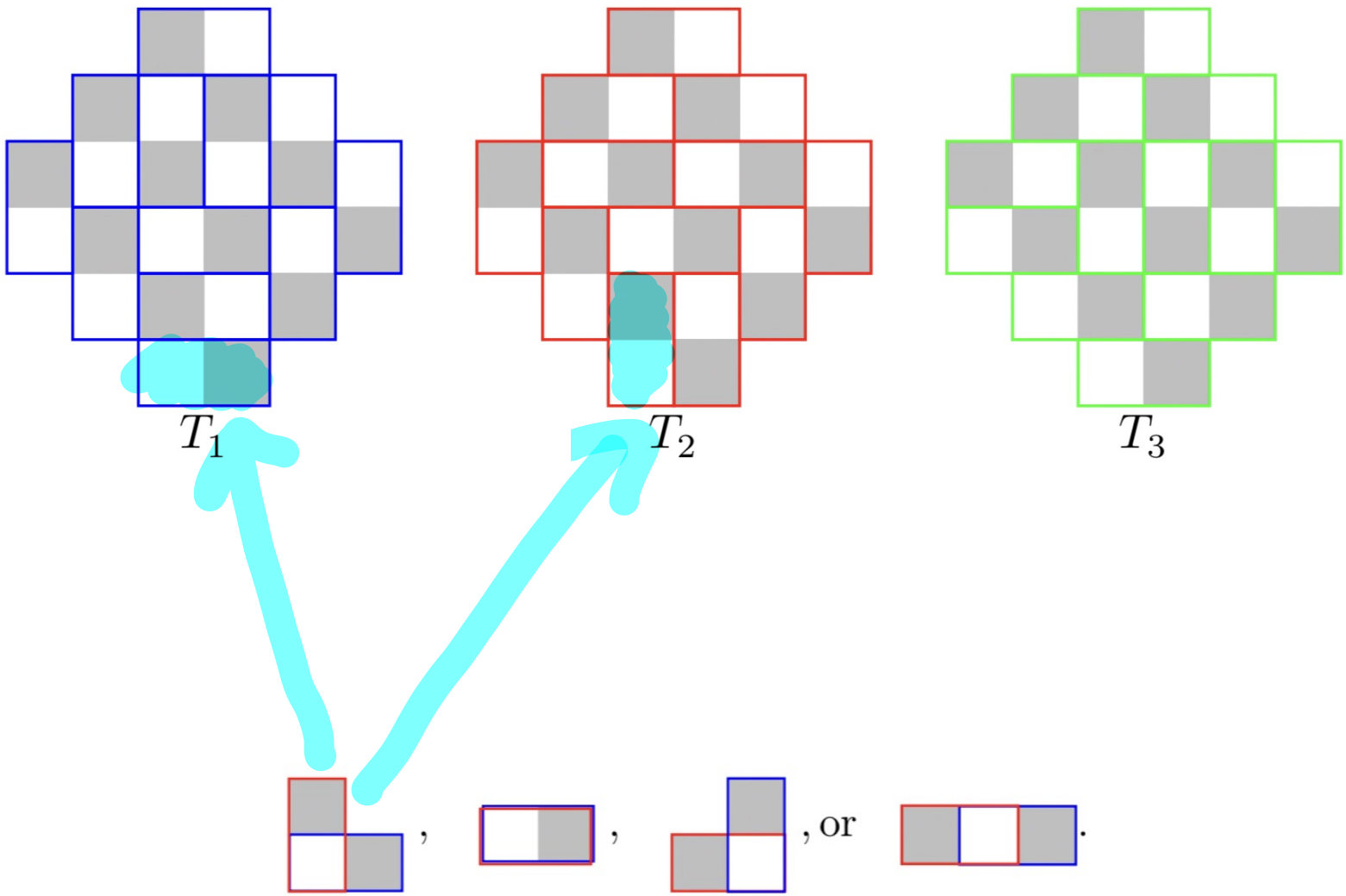
k-tilings of the Aztec diamond

$k = 3$



k-tilings of the Aztec diamond

$k = 3$



11 interactions

$$Z_m^{(k)}(x, y, t) = \sum_{\substack{k\text{-tilings} \\ \text{of the} \\ m\text{-diamond}}} t^{\#\text{inversions}} \prod_{i=1}^{k} \text{weight}^{(i)}(x,y)$$

$$Z_m^{(k)}(x,y,t) = \sum_{k\text{-tilings}} t^{\#\text{inversions}} \prod_{i=1}^{k} \text{weight}^{(i)}(x,y)$$

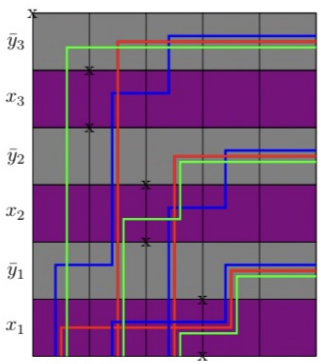
<u>Theorem</u> (C., Gitlin, Keating 2022)

$$Z_m^{(k)}(x,y,t) = \prod_{\ell=0}^{k-1} \prod_{1\le i\le j\le m} (1+t^{\ell} x_i y_j)$$

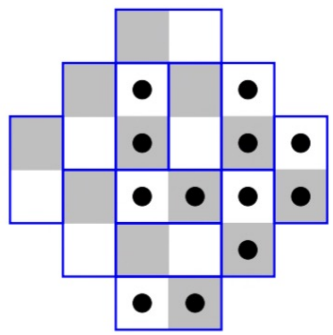
$$Z_m^{(k)}(x,y,t) = \sum_{k\text{-tilings}} t^{\#\text{inversions}} \prod_{i=1}^{k} \text{weight}^{(i)}(x,y)$$

<u>Theorem</u> (C., Gitlin, Keating 2022)

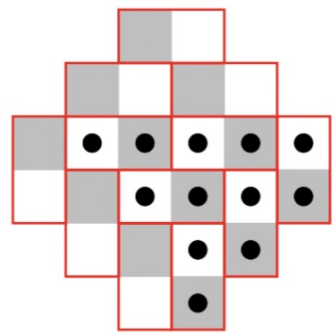
$$Z_m^{(k)}(x,y,t) = \prod_{\ell=0}^{k-1} \prod_{i\le j} (1+t^{\ell} x_i y_j)$$



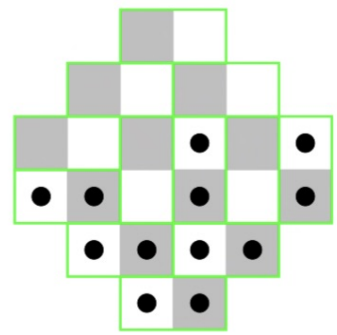
$(y_1^2 y_2^1 y_3^0)^3 t^{\binom{3}{2}\binom{3}{2}} (x_1^6 x_2^2 x_3)(y_1^2 y_2^4 y_3^3) t^{11}$



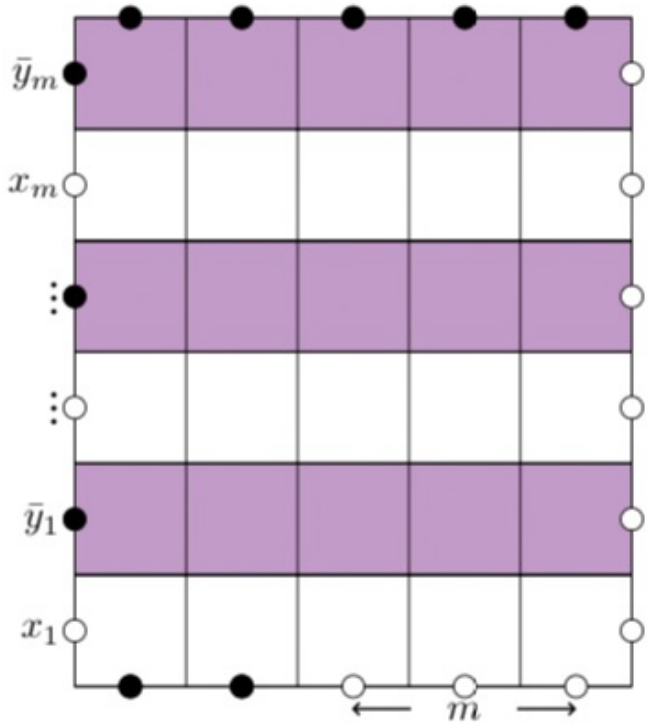
T_1

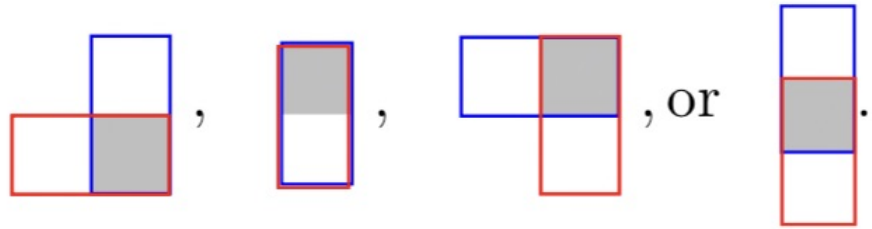


T_2

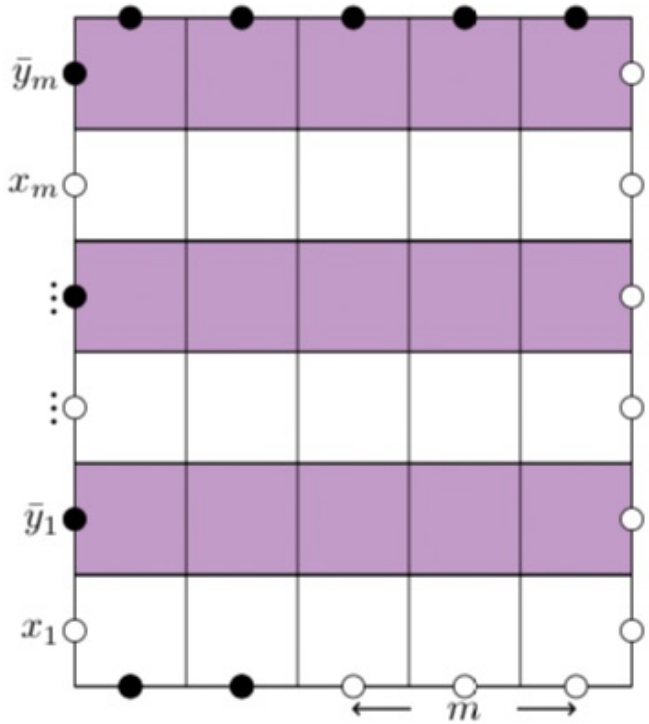


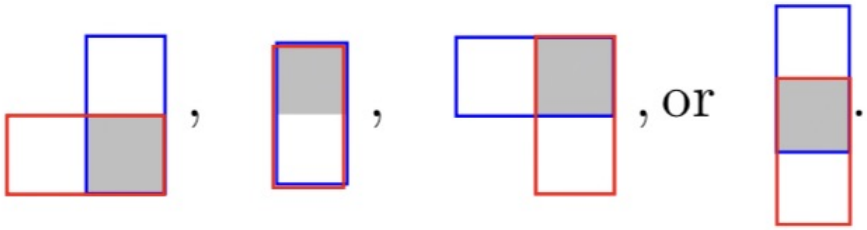
T_3





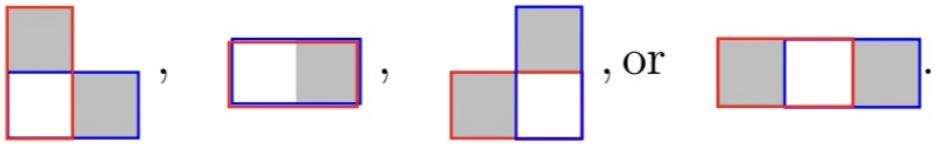
Same partition function





Same partition function

?



$\underline{t = 0}$

$$Z_m^{(k)}(x, y, 0) = \prod_{1 \leq i \leq j \leq m} (1 + x_i y_j)$$

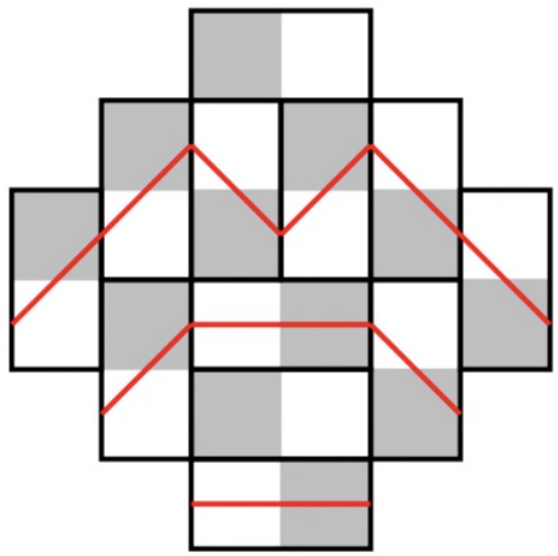
1- tilings

$\underline{t = 0}$

$$Z_m^{(k)}(x, y, 0) = \prod_{i \leq j} (1 + x_i \cdot y_j)$$

$$= Z_m(x, y)$$

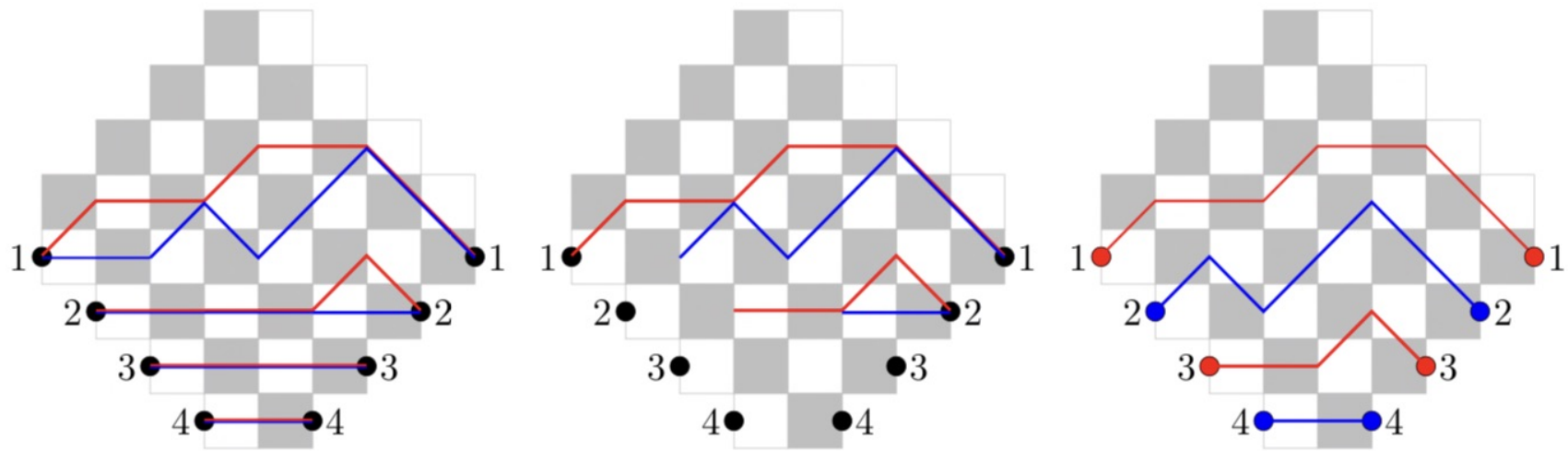
Tilings $\rightarrow$ Paths



$\underline{t = 0}$

$$Z_m^{(k)}(x, y, 0) = \prod_{i \leq j} (1 + x_i \cdot y_j)$$

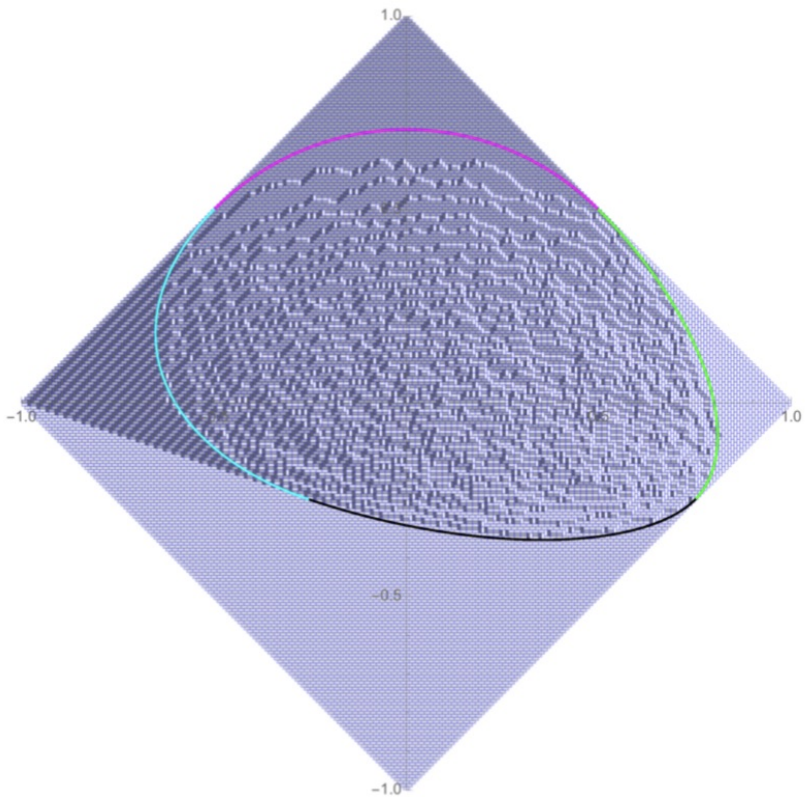
$$= Z_m(x, y)$$

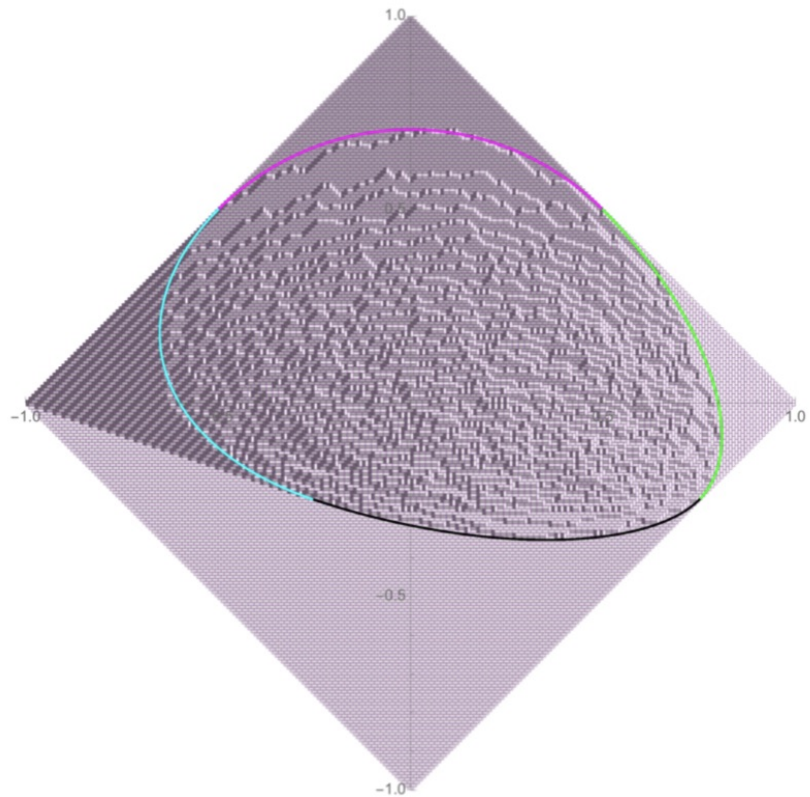


$\underline{t = 0}$

$$Z_m^{(k)}(x, y, 0) = \prod_{i \leq j} (1 + x_i \cdot y_j)$$

$$= Z_m(x, y)$$





Questions

0. k - Pyramid partitions?
1. RSK ?
2. Domino shuffling?
   (Keating & Nicoletti)
3. Kuo condensation?
4. Asymptotics $t \neq 0, 1, \infty$

   Limit shape? Height function?