Posets Arising as 1-skeleta of Simple Polytopes, the Nonrevisiting Path Conjecture & Poset Topology

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- with thanks to Karola Mészáros for fruitful discussions early in the project.

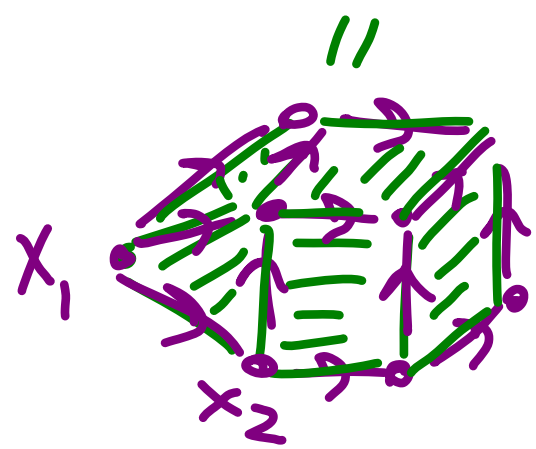
Linear Programming

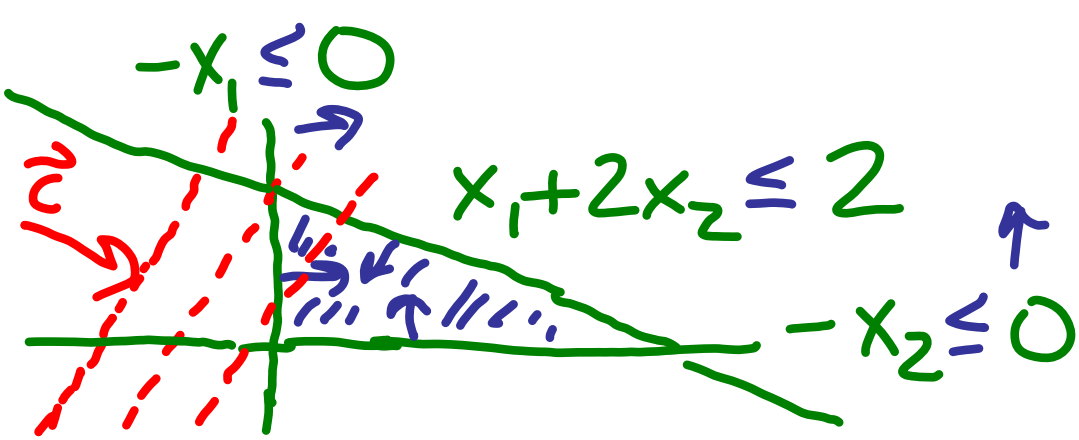
- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\}$

- $\{\vec{x} \mid A\vec{x} \le \vec{b}\}$ is polytope P if set is bounded

e.g.

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} \le \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{\vec{b}}$$





Solving Linear Programs via Simplex Method

- Define $G(P,\vec{c}) :=$ directed graph on 1-skeleton of P, i.e. on vertex-edge graph of P, with

$$x_1 \to x_2 \iff \vec{c}\cdot\vec{x}_1 < \vec{c}\cdot\vec{x}_2$$

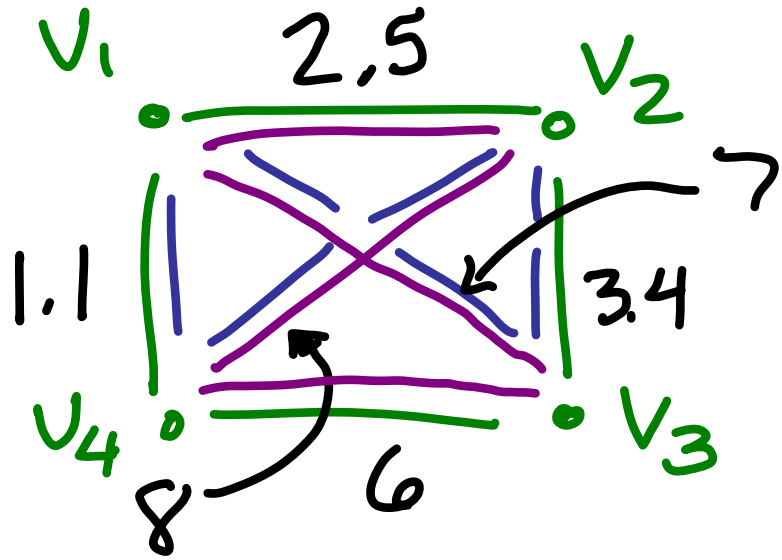
- $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\} =$ sink of $G(P,\vec{c})$

Simplex Method: walk from some vertex $v \in G(P,\vec{c})$ following arrows $v \to v_1 \to v_2 \to \dots \to s$ to sink s

- also may walk backwards to source of $G(P,\vec{c})$ to minimize $\vec{c}\cdot\vec{x}$

An Example: Traveling Salesman Problem

$G =$



Polytope Vertices:

$(1,0,1,1,0,1)$, $(1,1,0,0,1,1)$,

$(0,1,1,1,1,0)$

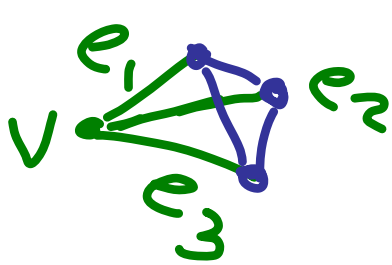
e_{12} e_{14} e_{23} e_{34}

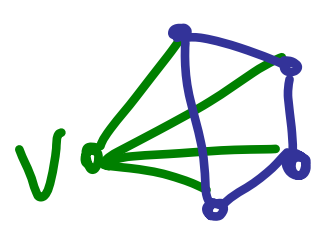
Cost Vector:

$$\vec{c} = (2.5, 7, 1.1, 3.4, 8, 6)$$

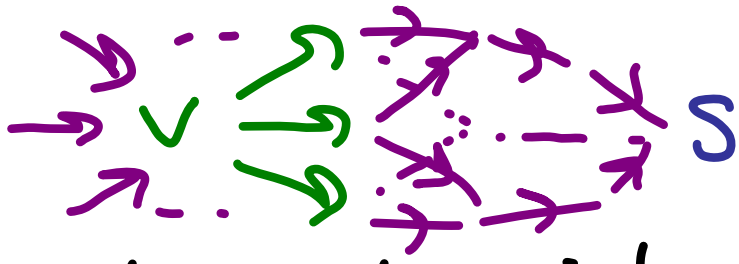
Quick Background on Polytopes

- A polytope in $\mathbb{R}^d$ is convex hull of finite # vertices, or equiv'ly a bounded set that is an intersection of half spaces.
- Polytope is simple if for each vertex v and each collection $e_1, e_2, \ldots, e_r$ of edges emanating out from v, there is r-dim'l face containing all these edges.

e.g. 

but <u>not</u> 

Pivot Rule: method to choose which out arrow



to follow from v towards sink s.

Key Questions:

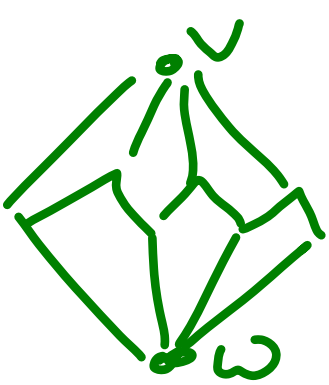
1. what is typical complexity of simplex method (path length)?

2. What is worst case? (i.e. diameter of $G(P, \vec{c})$)

Hirsch Conjecture: For d-dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies $\Delta(d,n) \leq n-d$.

Francisco Santos: After many decades eluding many people, he constructed counterexamples

("spindles" := polytopes with vertices v, w s.t. each facet includes v or w.)



43-dim's, 86 facets, diam ≥ 44

Nonrevisiting path conjecture.

For each u, v in polytope P, there is path u to v not revisiting any facet it has left.

Non-Revis. Path Conj $\Rightarrow$ Hirsch Conj.

- nonrevisiting path leaves a facet at each step & still belongs to d facets at its conclusion

Strong Monotone Path Conjecture:

there exists directed path of length $\leq n-d$ from any vertex to vertex v maximizing $\vec{c} \cdot v$ with cost increasing each step.

Our Plan

Impose further conditions on P and $\vec{c}$ that will imply a corollary of the following which we hope might also hold:

> For each $u, v \in P$, each directed path from u to v never revisits any facet it has left.

This property would make <u>all</u> pivot rules efficient for P and $\vec{c}$.

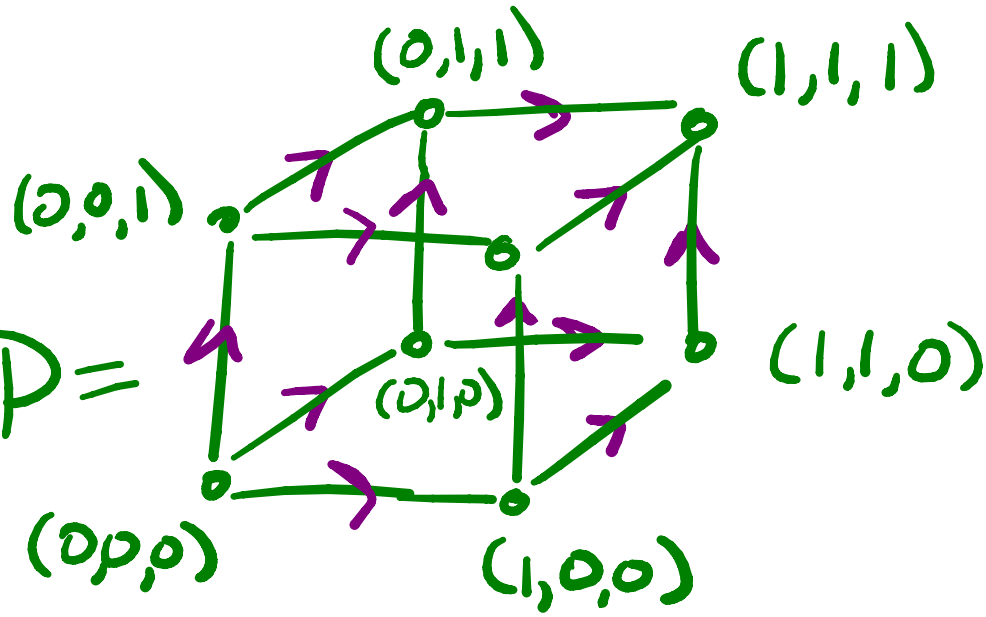
Def'n (H.): $G(P, \vec{c})$ has the **Hasse diagram property** if it is Hasse diagram of finite poset, i.e. $v_1 \to v_2 \to \cdots \to v_r$ for $r \geq 3$ directed path precludes $v_1 \to v_r$

Note: precludes d-simplices as faces for $d \geq 2$

Note: Equivalent to non-revisiting of 1-dim'l faces

Note: Poset often not "atomic"

An Example: Hypercube Polytope

$P =$ 

$\vec{c} = (1,2,3)$

Poset with Hasse diagram above

$\parallel$

"Boolean lattice" B_3 or $B_{\{1,2,3\}}$

$\parallel$

Poset of subsets of $\{1,2,3\}$ ordered by containment

Important Non-Examples:

"Klee-Minty Cubes"

e.g. $n=3$

$\vec{c}=(0,0,1)$

- path visits all vertices!
- first polytopes exhibiting inefficiency of simplex method

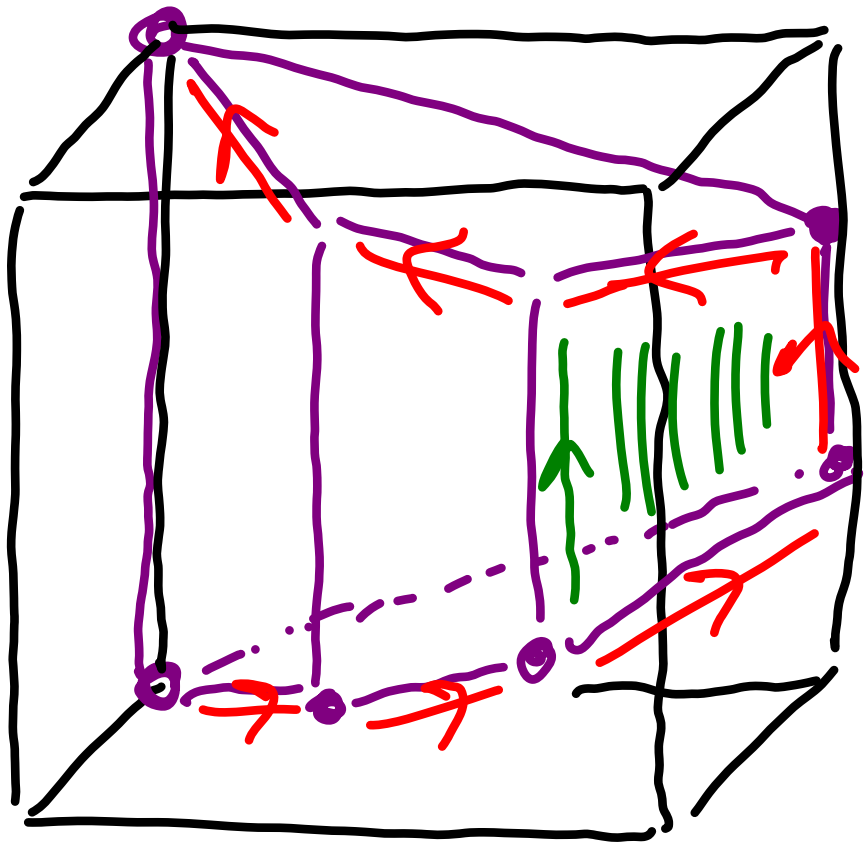


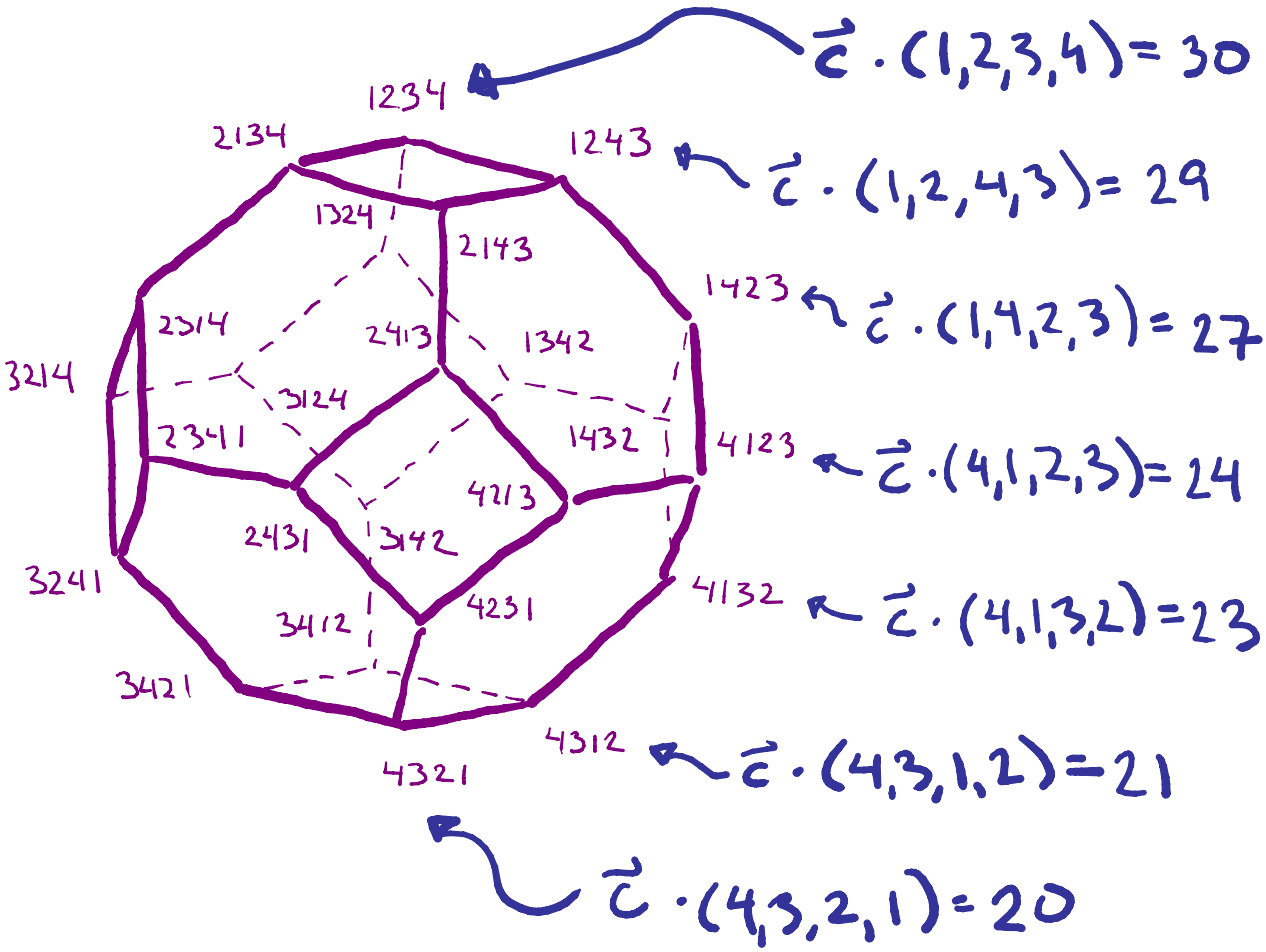
Figure modelled after one in Gärtner-Henk-Ziegler paper

n-dimensional Klee-Minty cube

$$:= \left\{ (x_1,\dots,x_n) \,\middle|\, 0 \le x_1 \le 1 \;\&\; \begin{array}{l} \varepsilon x_{i-1} < x_i < 1-\varepsilon x_{i-1} \\ \text{for } i>1 \;\&\; 0<\varepsilon<\frac{1}{2} \end{array} \right\}$$

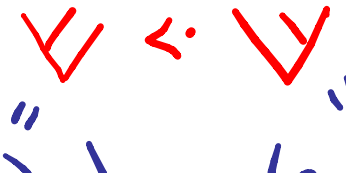
Permutahedron as Weak Order

- cost vector $\vec{c}$ any strictly ascending vector such as $\vec{c} = (1,2,3,4)$.

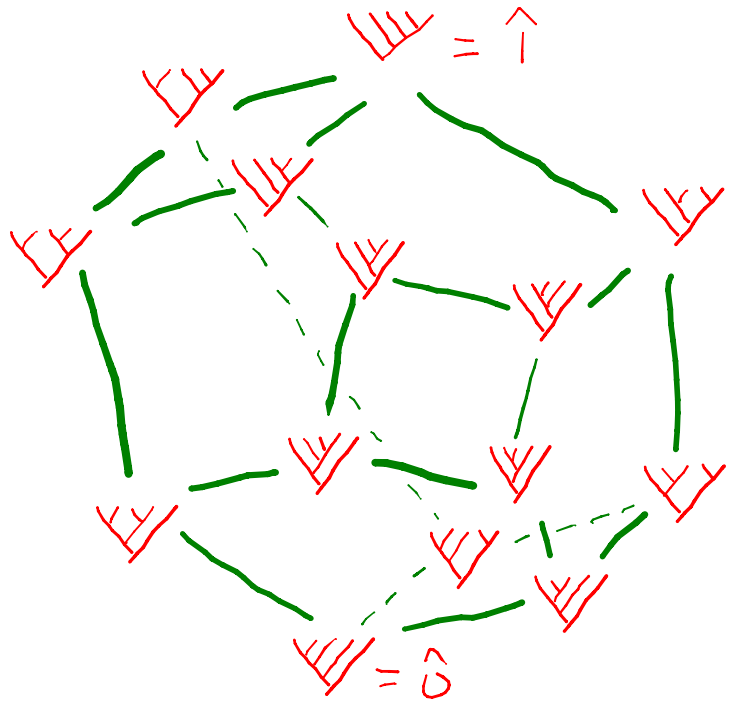


- Homotopy type 1st due to Edelman (type A) & Björner

Associahedron as Tamari Lattice

- Use Loday's realization
- Poset of binary trees with cover relations :  $\lessdot$

$((a,b),c)$ $\qquad$ $(a,(b,c))$

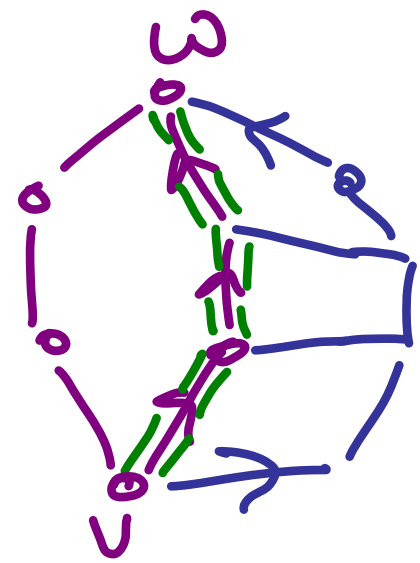


- Homotopy type 1st due to Björner & Wachs via nonpure lexicographic shellability

Note: Klee-Minty cubes violate Hasse diagram property in way that seems to be at the heart of what leads to existence of "long" directed path (visiting all 2^n vertices) in it

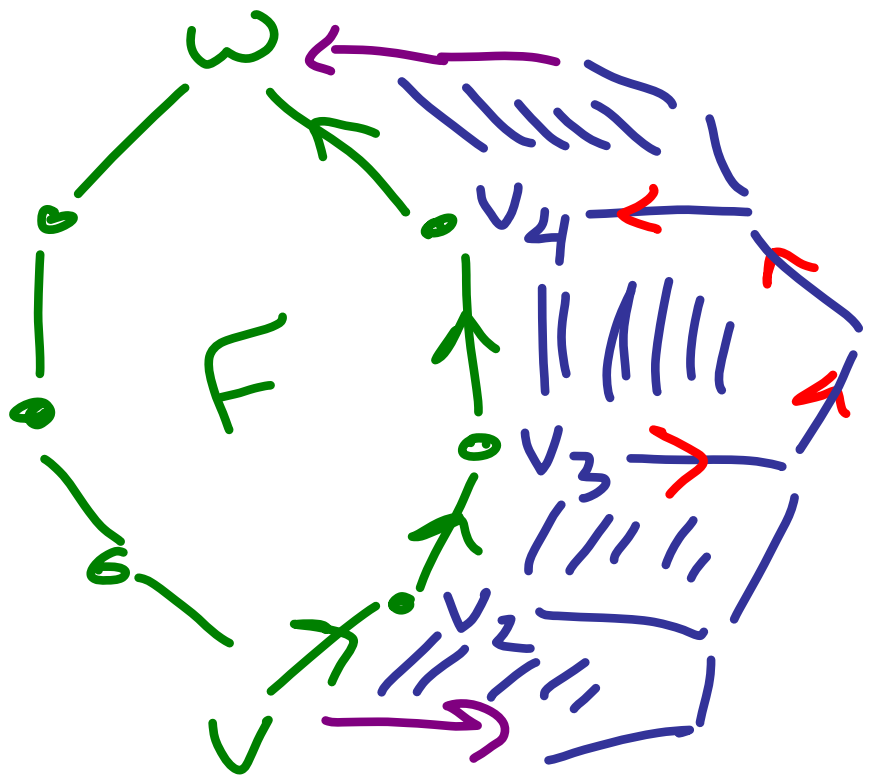
Our hope: Hasse diagram property precludes such issues.

Lemma (H.): For $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope P & generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path P_F from v to w in F,



outward oriented edge from v to $G \setminus F$ and inward oriented edge $G \setminus F$ to w.

Corollary: Monotonicity of out-degrees & partic. outward directions.

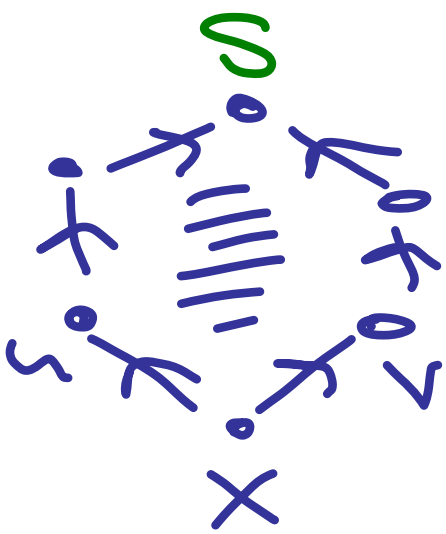


violate Hasse diagram (at switch from out to in)

Corollary: for each face $F \subseteq P$ with $\hat{0} \in F$ or $\hat{1} \in F$, directed paths cannot revisit F after departing from it.

Recall: A poset L is a lattice if for each $u, v \in L$ there exists unique least upper bound ("join") for u and v, denoted $u \vee v$, and unique greatest lower bound for u and v ("meet"), $u \wedge v$.

Note: for P simple & $G(P, \vec{c})$ Hasse diagram, an upper bound for u, v both covering x is sink s of unique 2-face containing x, u, v.



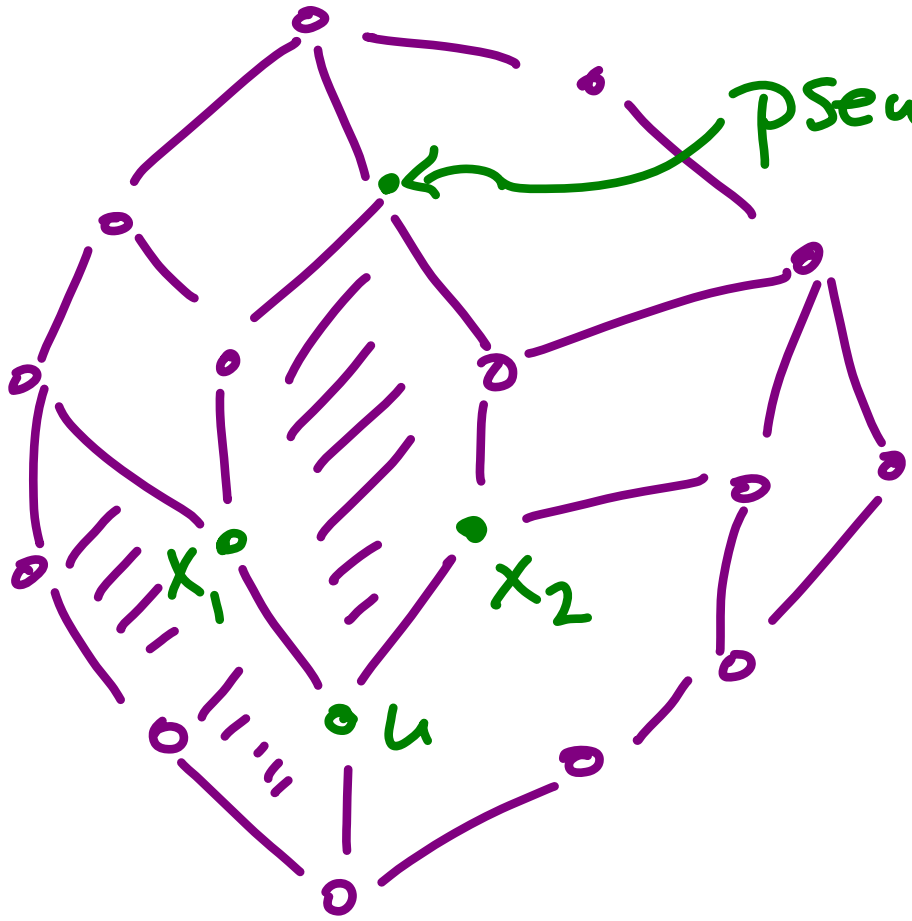
"Pseudo-joins" in a Polytope

Let P be simple polytope w/ generic cost vector $\vec{c}$ such that $G(P, \vec{c})$ is Hasse diagram of poset L with $x_1, x_2, \ldots, x_r \in L$ s.t. there exists $u \in L$ with $u \lessdot x_i$ for $i = 1, 2, \ldots, r$.

Recall:

- $u \lessdot v$ means $u < v$ with no possible z s.t. $u < z < v$ "cover rel'n"
- a_i is atom of $[u, v]$ for $u \lessdot a_i$

Def'n (H.): The pseudo-join of $x_1, x_2, \ldots, x_r$ is the sink of the unique r-face of simple polytope P containing



pseudojoin of x_1 & x_2 $(r=2)$

Lemma (H.): For P a simple polytope & $\vec{c}$ generic cost vector s.t. $G(P,\vec{c})$ is Hasse diagram of poset L, let $S,T \subseteq \{a_1,\ldots,a_n\}$ be distinct sets of atoms. Then $\text{pseudojoin}(S) \neq \text{pseudojoin}(T)$.

For L a lattice, this also holds for atoms in each interval $[u,v] \subseteq L$.

Cor: subposet of pseudojoins is isomorphic to Boolean lattice of subsets of $\{a_1,\ldots,a_n\}$.

Idea for Distinctness of Pseudo-Joins

(1) Reduce to $S \subsetneq T$ with $|T| = |S| + 1$

- $S_1 \subsetneq S_2 \subsetneq S_3$ with $psj(S_1) = psj(S_3)$
  $\Rightarrow psj(S_2) = psj(S_3)$
- $S_1 \not\subseteq S_2$ & $S_2 \not\subseteq S_1$ then use
  $S_1 \cap S_2 \subsetneq S_1$ with $psj(S_1 \cap S_2) = psj(S_1)$

(2) Use codim one nonrevisiting lemma

(3) For $[u,v]$, use that v is an upper bound for the atoms of $[u,v]$ w/ l.u.b. in $[u,v]$

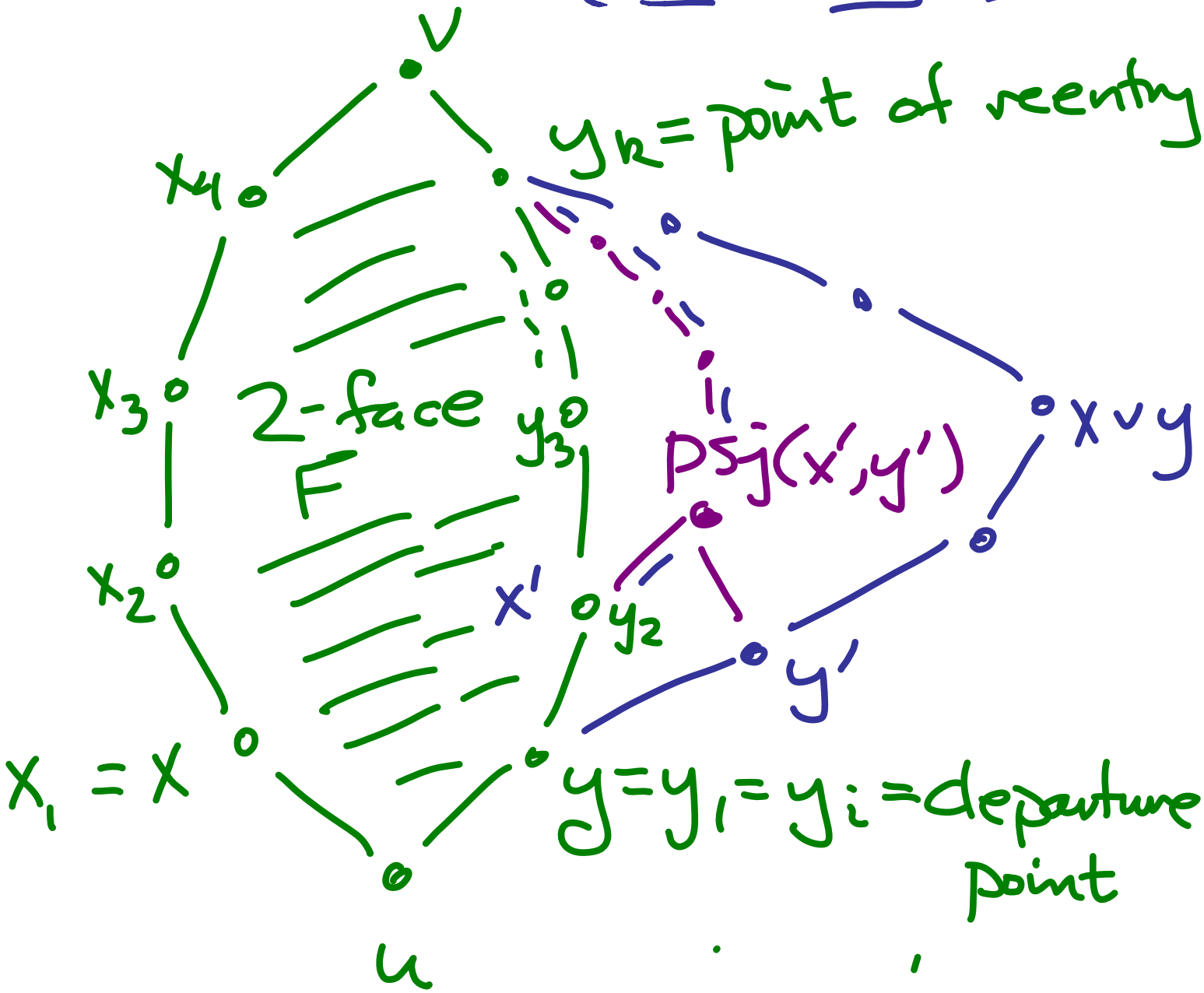
Note: Since pseudo-join of $x_1, \ldots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \cdots \vee x_r$ to pseudo-join$(x_1, \ldots, x_r)$

Thm (H.): Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then

pseudo-join$(x_1, x_2, \ldots, x_r) = x_1 \vee \cdots \vee x_r$

Pf: induction on r with $r=2$ base case especially tricky part.

Idea for "Join = Pseudo-Join" ($r=2$ case)



- $y' \notin F \Rightarrow psj(x', y') \notin F$

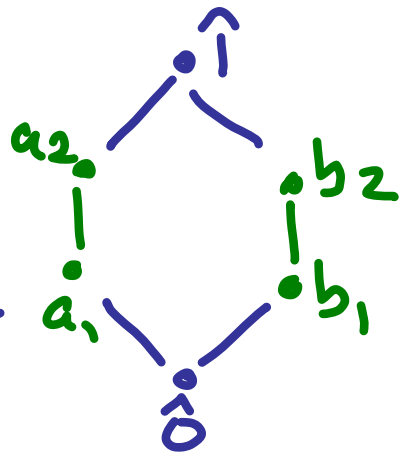
$$psj(x', y') = join(x', y') \geq y_k \in F$$

by induction on length longest path to $\hat{1}$

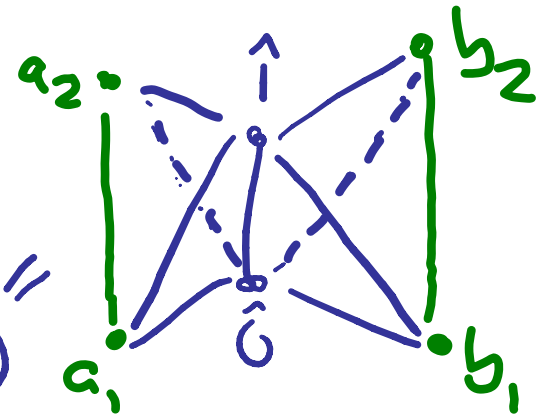
strict inequality $\Rightarrow \exists$ smaller $k-i \Rightarrow \leq$

Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \dots < v_i$ in P.

e.g. $P =$



$\Delta(P) =$



Thm (Hall; Popularized by Rota):

$$\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{u,v}_{\text{subposet } \{z \in P \mid u < z < v\}}))$$

Fact: K regular CW complex $\Rightarrow$

$$\Delta(F(K) \setminus \{\hat{0}\}) = sd(K) \cong K$$

Quillen Fiber Lemma: Given a (a.k.a. Quillen Theorem A) poset map $f: P \to Q$ s.t. $q \in Q$ implies $\Delta(f^{-1}_{\leq}(q)) = \Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$.

Remark: Used extensively in finite group theory (to characterize groups G via subgroup lattice $L(G)$) & in topological combinatorics.

e.g. **Thm (Shareshian):**
G solvable $\iff L(G)$ shellable

Thm (Stanley):
G supersolv. $\iff L(G)$ supersolv.

Thm (H.): For P a simple polytope & $\vec{c}$ a generic cost vector such that $G(P,\vec{c})$ is the Hasse diagram of a finite lattice L,

$$\Delta(\underbrace{u,v}_{=\{z\in L \mid u<z<v\}}) \simeq \text{ball or sphere } S^{|A(u,v)|-2}$$

$$A(u,v) := \{a \in L \mid u \lessdot a < v\}$$

"atoms" of $[u,v]$

Idea: Use poset map

$$f(x) = \bigvee_{a_i \in A(u,v)} a_i = \text{pseudojoin of } \{a_i \mid a_i \in A(u,v)\}$$

by "join = pseudo-join" theorem

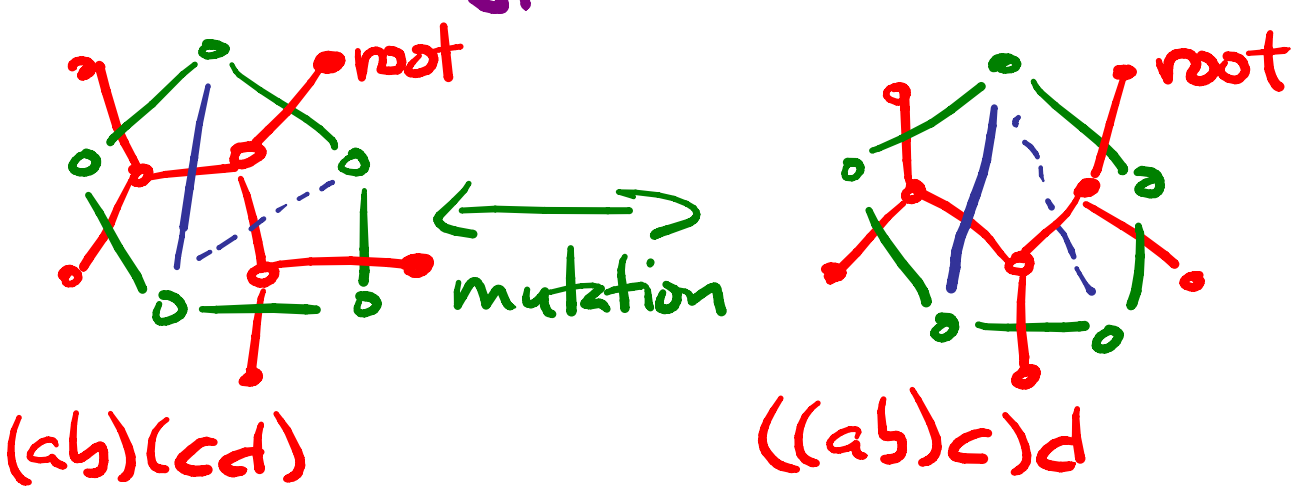
then use distinctness of pseudojoins so image = Boolean lattice w/ or without maximal element

Applications: Unified proof for

- permutahedra $\rightsquigarrow$ weak order

- associahedra $\rightsquigarrow$ Tamari lattice
  (a.k.a Stasheff polytopes)

- generalized associahedra $\rightsquigarrow$ Cambrian lattices
  (related to cluster algebras of finite type & mutation)



Some Further Questions

Qn 1: Does P simple + $G(P, \vec{c})$ Hasse diagram of lattice $\Rightarrow$ no directed path can revisit face it has departed? (If not, variations?)

Qn 2: Variations on hypotheses? Non-simple polytopes? Non-lattices?

Qn 3: Structure of posets of joins/pseudo-joins for non-simple polytopes?

Qn 4: Additional interesting examples?

- graph associahedra? (nonrevisiting proven by Barnard-McConville & related to nested set complexes e.g. in work of Feichtner-Yuzvinsky)
- classes of MV polytopes?

Qn 5: Natural/nice homotopy equiv. between our order complexes & complexes of Billera-Kapranov-Sturmfels and others?

Qn 5b: Do order complexes detect nonrevisiting, or always spheres too?

Thank you!

Idea for Inductive Step:

Induct on $|S|$ with $|S| = 2$ base case as just discussed.

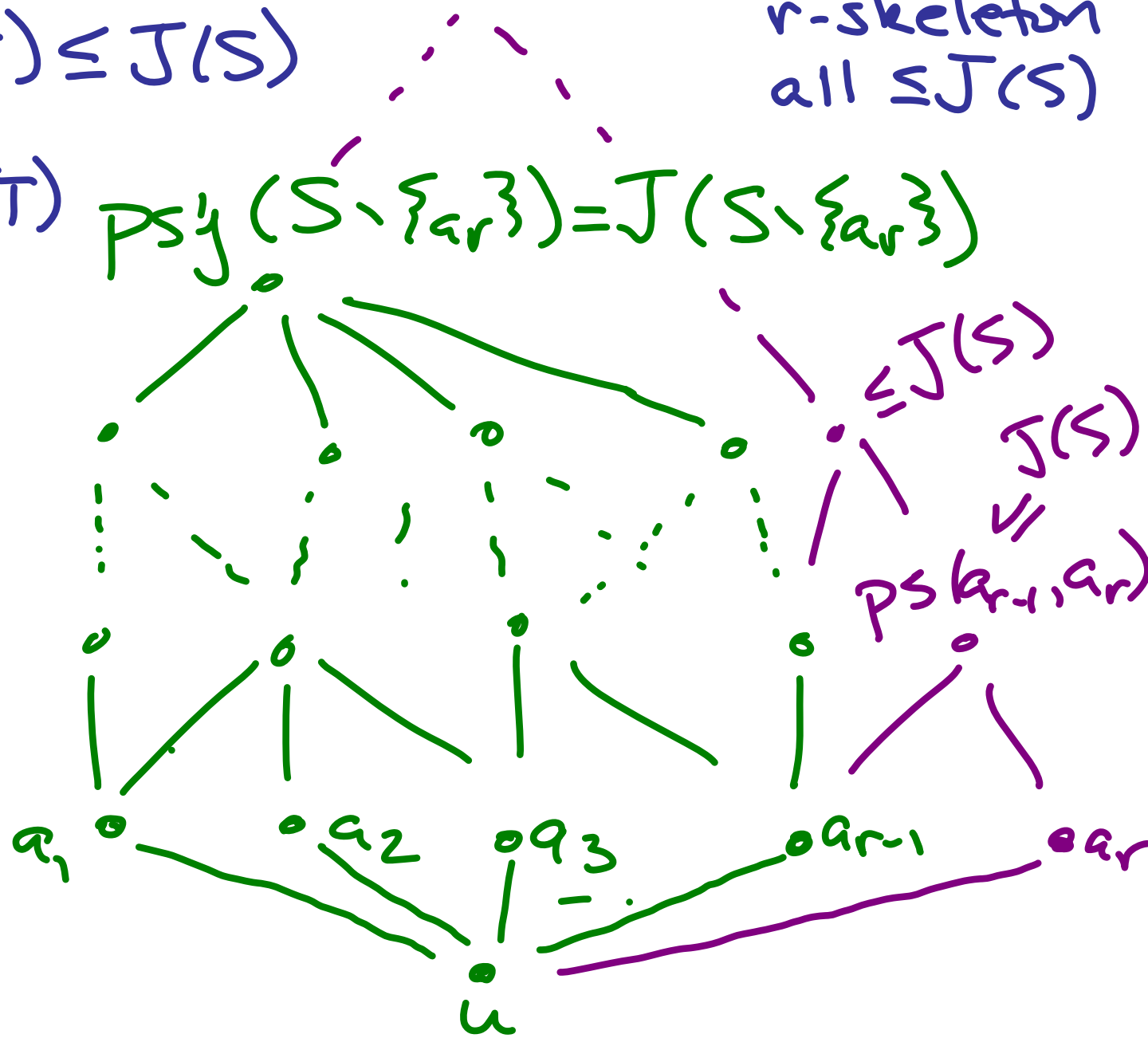
$T \subsetneq S$

$\Downarrow$

$J(T) \leq J(S)$

$\parallel$

$PS(T)$

$J(S)$

- progress upward; r-skeleton all $\leq J(S)$