

The wonderful geometry of matroids

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These notes were developed for a graduate course at the University of Oregon. The course entails 28 classes, each 50 minutes long, most of which consist of a short lecture along with some exercises to be completed (or at least started) in class. The main objectives are to cover the following results:

- The Poincaré polynomial of the complement of a complex hyperplane arrangement is determined by the characteristic polynomial of the associated matroid (Theorem 9.2), and the cohomology ring is isomorphic to the Orlik–Solomon algebra (Theorem 12.3).
- The coefficients of the reduced characteristic polynomial of a matroid form a log concave sequence (Theorem 14.4).
- The correlation constant of a matroid is bounded above by 2 (Theorem 15.6).

The second and third results are proved by studying the Chow ring of a matroid, which coincides for a matroid that is realizable over the complex numbers with the cohomology ring of the wonderful compactification of the projective arrangement complement. For the last two topics, connections with geometry are emphasized but often not proved. We avoid the language of fans (and sometimes regret it).

1 Definition and basic examples

Definition 1.1. A **matroid** is a pair $(E, \mathcal{I})$, where E is a finite set (called the **ground set**) and $\mathcal{I}$ is a collection of subsets of E (called **independent sets**) satisfying the following axioms:

- (i) $\emptyset \in \mathcal{I}$
- (ii) If $S \subset T \in \mathcal{I}$, then $S \in \mathcal{I}$.
- (iii) If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is an element $e \in T \setminus (S \cap T)$ such that $S \cup \{e\} \in \mathcal{I}$.

The most important examples of matroids come from taking the independent sets to be all linear independent subsets of a finite multiset of vectors in a vector space. More precisely, suppose that E is a finite set V is a vector space, and $\varphi : E \rightarrow V$ is any map. For any $S \subset E$, let $V_S := \text{Span}\{\varphi(e) \mid e \in S\} \subset V$. Let

$$\mathcal{I}_\varphi := \{S \subset E \mid \dim V_S = |S|\}.$$

Then $M_\varphi := (E, \mathcal{I}_\varphi)$ is a matroid. Indeed, (i) and (ii) are clear. For (iii), our hypotheses imply that there exists $e \in T$ such that $\varphi(e) \notin V_S$, and it follows that $e \notin S$ and $S \cup \{e\} \in \mathcal{I}_\varphi$. A matroid of this form is called **realizable**.

There are many equivalent ways to encode the data of a matroid, some of which we record here.

Definition 1.2. Let $(E, \mathcal{I})$ be a matroid.

- A set is **dependent** if it is not independent.
- A **basis** is a maximal independent set. These all have the same cardinality by (iii).
- A **circuit** is a minimal dependent set.
- The **rank** of a subset $S \subset E$ is the cardinality of a maximal independent subset $T \subset S$. (In the realizable case, this is equal to the dimension of V_S .)
- A **flat** is a subset $F \subset E$ such that, for all $e \in E \setminus F$, $\text{rk } F < \text{rk } F \cup \{e\}$.

Any matroid can be recovered from its ground set along with its dependent sets, bases, circuits, rank function, or flats.

Example 1.3. Let $V = \mathbb{R}^3$, let $E = \{a, b, c, d, e, f\}$, and let $\varphi : E \rightarrow V$ be given by the columns of the following matrix:

$$\begin{array}{cccccc} a & b & c & d & e & f \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Let M_φ be the associated realizable matroid.

- The bases of M_φ are abc, abd, abe, acd, ace .
- The circuits of M_φ are f, de, bcd, bce .
- The flats of M_φ are

$$abcdef \quad (\text{rank } 3)$$

$$abf, acf, adf, bcdf \quad (\text{rank } 2)$$

$$af, bf, cf, def \quad (\text{rank } 1)$$

$$f \quad (\text{rank } 0)$$

More generally, there is a bijection $F \mapsto V_F$ between flats and subspaces of V spanned by subsets of the vectors $\varphi(E)$. This bijection takes flats of rank k to subspaces of dimension k .

Definition 1.4. The rank of a matroid M is equal to the rank of the ground set E , or equivalently to the cardinality of any basis.

Graphs also provide a good source of examples of matroids. Let $\Gamma = (Q, E)$ be a graph with vertex set Q and edge set E , and let $\mathcal{I}_\Gamma$ be the collection of subsets $S \subset E$ such that (Q, S) is a forest (that is, contains no cycles). Then $M_\Gamma := (E, \mathcal{I}_\Gamma)$ is a matroid. Again, (i) and (ii) are clear. For (iii), we begin by noting that, for any $S \in \mathcal{I}_\Gamma$, the number of connected components of (Q, S) is equal to $|Q| - |S|$. Since $|T| > |S|$, (Q, S) has more connected components than (Q, T) , so there must be at least one edge $e \in T$ with endpoints in two different connected components of (Q, S) . Then $S \cup \{e\} \in \mathcal{I}_\Gamma$. A matroid of this form is called **graphical**.

It turns out that graphical matroids are a subclass of realizable matroids. Fix any field $\mathbb{F}$, and consider the vector space $\mathbb{F}^Q := \mathbb{F}\{x_v \mid v \in Q\}$. Choose an orientation of each edge, so that we have head and tail functions $h, t : E \rightarrow Q$. Define $\varphi : E \rightarrow \mathbb{F}^Q$ by putting $\varphi(e) = x_{h(e)} - x_{t(e)}$. Then $\mathcal{I}_\varphi = \mathcal{I}_\Gamma$. (Note that $\mathcal{I}_\varphi$ does not depend on the choices of orientations of the edges.)

Exercise 1.5. Write down a graph Γ with edge set $\{a, b, c, d, e, f\}$ such that M_Γ is equal to the matroid M_φ from Example 1.3.

Exercise 1.6. Let C_4 be the 4-cycle. Write down the flats of M_{C_4} . How does this generalize to n -cycles?

Exercise 1.7. Let K_4 be the complete graph on 4 vertices. Write down the flats of M_{K_4} . How does this generalize to n vertices?

Exercise 1.8. Describe the rank function and the flats of M_Γ for an arbitrary graph Γ . In particular, what is the rank of M_Γ ?

Exercise 1.9. Two matroids $(E, \mathcal{I})$ and $(E', \mathcal{I}')$ are **isomorphic** if there exists a bijection from E to E' taking $\mathcal{I}$ to $\mathcal{I}'$. Can you find two non-isomorphic graph Γ and Γ' with the property that M_Γ is isomorphic to $M_{\Gamma'}$? How about with the additional hypothesis that Γ and Γ' are both connected? How about with the hypothesis that they are both 2-vertex-connected, meaning that you cannot disconnect either of them by deleting a single vertex?

2 Deletion, contraction, and dualization

The goal of this section is to introduce some basic operations on matroids. Let $M = (E, \mathcal{I})$ be a matroid, and let $e \in E$ be an element of the ground set.

- The **deletion** $M \setminus e$ is the pair $(E \setminus \{e\}, \mathcal{I}_e)$, where $\mathcal{I}_e := \{S \subset E \setminus \{e\} \mid S \in \mathcal{I}\}$.
- The **contraction** M/e is the pair $(E \setminus \{e\}, \mathcal{I}^e)$, where $\mathcal{I}^e := \{S \subset E \setminus \{e\} \mid S \cup \{e\} \in \mathcal{I}\}$.

It is trivial to check that the deletion is a matroid. The contraction is a little bit more subtle, and in fact it is not true as written that the contraction is a matroid! If the set $\{e\}$ is independent, then everything is fine. However, if the set $\{e\}$ is dependent, then $\mathcal{I}^e = \emptyset$, so M/e does not satisfy the first matroid axiom. In this case, we simply define $M/e := M \setminus e$.

Remark 2.1. The names deletion and contraction derive from the graphical case, where they correspond to deleting or contracting an edge. If $\{e\}$ is dependent, we say that e is a **loop**, again motivated by the graphical case. The discussion above says that something fishy happens when you try to contract a loop, but we simply define contracting a loop to be the same as deleting a loop.

Example 2.2. Suppose that V is a vector space over a field $\mathbb{F}$, $\varphi : E \rightarrow V$ is a map of sets, and $M = M_\varphi$. Then $M \setminus e$ is the matroid associated with $\varphi : E \setminus \{e\} \rightarrow V$, and M/e is the matroid associated with $\bar{\varphi} : E \setminus \{e\} \rightarrow V/\mathbb{F}\varphi(e)$. (Note that they are the same if $\varphi(e) = 0$.)

Let M be a matroid, and let $\mathcal{B}$ be the set of bases. Define $\mathcal{B}^* := \{S \mid E \setminus S \in \mathcal{B}\}$, and let

$$\begin{aligned} \mathcal{I}^* &:= \{S \mid \text{there exists } T \text{ such that } S \subset T \in \mathcal{B}^*\} \\ &= \{S \mid \text{there exists } B \in \mathcal{B} \text{ such that } S \cap B = \emptyset\}. \end{aligned}$$

Proposition 2.3. *The pair $M^* := (E, \mathcal{I}^*)$ is a matroid, called the **dual** of M .*

Before giving the proof of Proposition 2.3, we formulate the following lemma.

Lemma 2.4. *Suppose that S is independent and $S \subset T$. Then S can be extended to a maximal independent subset $U \subset T$ with $S \subset U$.*

Proof. Let M' be the matroid obtained from M by deleting every element of $E \setminus T$. Then S is independent for M' , and is therefore contained in a basis U for M' . $\square$

Proof of 2.3. As usual, the first and second axioms are trivial. Suppose that $S, T \in \mathcal{I}^*$ with $|S| < |T|$. Choose a basis B for M that is disjoint from T . Applying Lemma 2.4 to the pair of sets $B \setminus S \subset E \setminus S$, we can produce a maximal independent set $B' \subset E \setminus S$ with $B \setminus S \subset B'$. Next, note that $E \setminus S$ contains a basis for M , therefore any maximal independent subset of $E \setminus S$ is a basis for M . In particular, B' is a basis for M .

We claim that $T \setminus S$ is not contained in B' . If it were, we would have

$$\begin{aligned}
|B| &= |B \cap S| + |B \setminus S| \\
&\leq |S \setminus T| + |B \setminus S| && (\text{because } B \subset E \setminus T) \\
&< |T \setminus S| + |B \setminus S| && (\text{because } |S| < |T|) \\
&\leq |B'| && (\text{because } T \setminus S \subset B', B \setminus S \subset B', \text{ and } T \cap B = \emptyset),
\end{aligned}$$

contradicting the fact that every basis has the same cardinality. We have now shown that there exists an element $e \in (T \setminus S) \setminus B'$. Then $S \cup \{e\}$ is disjoint from B' , so $S \cup \{e\} \in \mathcal{I}^*$. $\square$

We now explain how to think of duality for realizable matroids. Suppose that V is a vector space over the field $\mathbb{F}$ and we are given a map $\varphi : E \rightarrow V$, and that $V_E = V$. This induces a surjection $\mathbb{F}^E \rightarrow V$, which we'll also call φ . Let W be the kernel, and let $\psi : W \rightarrow \mathbb{F}^E$ be the inclusion, so that we have

$$0 \longrightarrow W \xrightarrow{\psi} \mathbb{F}^E \xrightarrow{\varphi} V \longrightarrow 0.$$

Now dualize the exact sequence:

$$0 \longleftarrow W^* \xleftarrow{\psi^*} \mathbb{F}^E \xleftarrow{\varphi^*} V^* \longleftarrow 0.$$

Proposition 2.5. *We have $M_{\psi^*} = M_{\varphi^*}^*$.*

Proof. We need to show that B is a basis for M_{φ} if and only if $E \setminus B$ is a basis for M_{ψ^*} . Indeed,

$$\begin{aligned}
B \text{ is a basis for } M_{\varphi} &\Leftrightarrow \mathbb{F}^B \hookrightarrow \mathbb{F}^E \xrightarrow{\varphi} V \text{ is an isomorphism} \\
&\Leftrightarrow \mathbb{F}^B \hookleftarrow \mathbb{F}^E \xleftarrow{\varphi^*} V^* \text{ is an isomorphism} \\
&\Leftrightarrow V^* \cap \mathbb{F}^{E \setminus B} = 0 \\
&\Leftrightarrow W^* \xleftarrow{\psi^*} \mathbb{F}^E \hookleftarrow \mathbb{F}^{E \setminus B} \text{ is an isomorphism} \\
&\Leftrightarrow E \setminus B \text{ is a basis for } M_{\psi^*}.
\end{aligned}$$

This completes the proof. $\square$

Exercise 2.6. Show that $(M/e)^* = M^* \setminus e$.

Remark 2.7. We say that an element $e \in E$ is a **coloop** for M if it is contained in every basis, or equivalently if it is a loop for M^* . It is easy to check that deleting a coloop is the same as contracting a coloop. By Exercise 2.6, this is equivalent to the statement that contracting a loop is the same as deleting a loop, which is true by definition. (If you didn't like the definition of deleting a loop, you can take this as justification.)

Exercise 2.8. What is the relationship between the ranks of M , M^* , $M \setminus e$, and M/e ? Be careful about the cases where e is a loop or a coloop.

Exercise 2.9. Show that $F \subset E \setminus \{e\}$ is a flat of M/e if and only if $F \cup \{e\}$ is a flat of M , and it is a flat of $M \setminus e$ if and only if either F or $F \cup \{e\}$ (or possibly both) is a flat of M .

Exercise 2.10. Let Γ be a planar graph, and let Γ^* be its planar dual (a vertex in every face of Γ and an edge crossing every edge of Γ). Note that the edge set of Γ^* is canonically in bijection with the edge set of Γ . Convince yourself that $M_{\Gamma^*} = M_{\Gamma}^*$.

Exercise 2.11. Exercise 2.10 implies that the dual of the matroid associated with a planar graph is again graphical. One can still define the dual of a matroid associated with a non-planar graph, but it will not be graphical. For example, verify that $(M_{K_{3,3}})^*$ is not isomorphic to M_{Γ} for any Γ .

Exercise 2.12. For any graph Γ , show that M_{Γ}^* naturally admits a realization via vectors in the vector space $H^1(\Gamma; \mathbb{F})$. That is, if $\varphi : \mathbb{F}^E \rightarrow V$ is your favorite realization of M_{Γ} , show that we have a canonical isomorphism $\ker \varphi \cong H_1(\Gamma; \mathbb{F})$.

3 Direct sum, truncation, and free extension

Suppose we are given matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$.

Definition 3.1. We define the **direct sum** $M_1 \oplus M_2 := (E_1 \sqcup E_2, \mathcal{I})$, where

$$\mathcal{I} := \{S \subset E_1 \sqcup E_2 \mid S \cap E_1 \in \mathcal{I}_1 \text{ and } S \cap E_2 \in \mathcal{I}_2\}.$$

It is easy to check that $M_1 \oplus M_2$ satisfies the matroid axioms, and that $\text{rk } M_1 \oplus M_2 = \text{rk } M_1 + \text{rk } M_2$.

Example 3.2. Suppose we are given finite sets E_1 and E_2 , vector spaces V_1 and V_2 over the same field $\mathbb{F}$, and maps $\varphi_1 : E_1 \rightarrow V_1$ and $\varphi_2 : E_2 \rightarrow V_2$. Define

$$\varphi : E_1 \sqcup E_2 \rightarrow V_1 \oplus V_2$$

in the obvious way: $\varphi(e_1) = (\varphi_1(e_1), 0)$ for all $e_1 \in E_1$, and similarly for E_2 . Then $M_\varphi = M_{\varphi_1} \oplus M_{\varphi_2}$.

Example 3.3. Suppose that Γ_1 and Γ_2 are graphs. Then $M_{\Gamma_1} \oplus M_{\Gamma_2} = M_{\Gamma_1 \sqcup \Gamma_2}$. It is also M_Γ , where Γ is a graph obtained by gluing Γ_1 and Γ_2 at a single vertex. (I say “a graph” because there are many such, depending on the choices of vertices.)

Let M be the matroid consisting of a single coloop. Alternatively, it is the matroid associated with a single nonzero vector in any vector space, or the matroid associated with the graph consisting of a single edge with distinct endpoints. We define the **Boolean matroid of rank n**

$$\text{Boo}_n := M \oplus \cdots \oplus M.$$

Explicitly, it is the matroid on the ground set $[n]$ for which the only basis is $[n]$ itself. It is also isomorphic to M_Γ for any tree Γ with n edges.

Definition 3.4. Let $M = (E, \mathcal{I})$ be a matroid. We define the **truncation** $\text{tr } M := (E, \text{tr } \mathcal{I})$, where

$$\text{tr } \mathcal{I} := \{S \in \mathcal{I} \mid |S| < \text{rk } M\}.$$

It is clear that $\text{tr } M$ is a matroid of rank one smaller than the rank of M .

Example 3.5. Suppose that we are given $\varphi : E \rightarrow V$ such that $V_E = V$, along with a *generic* line $L \subset V$. By generic, we mean that there is no subset $S \subset E$ with $L \subset V_S \subsetneq V$. Let $\bar{V} := V/L$, and let $\bar{\varphi} : E \rightarrow \bar{V}$ be the map obtained by composing φ with the projection from V to V/L . Then

$$\begin{aligned} S \in \text{tr } \mathcal{I}_\varphi &\Leftrightarrow \dim V_S = |S| \text{ and } V_S \not\subset V \\ &\Leftrightarrow \dim \bar{V}_S = |S| \\ &\Leftrightarrow S \in \mathcal{I}_{\bar{\varphi}}. \end{aligned}$$

Thus $\text{tr } M_\varphi = M_{\bar{\varphi}}$.

Definition 3.6. We define the **uniform matroid** $U_{d,n} := \text{tr}^{n-d} \text{Boo}_n$ to be the matroid on the ground set $[n]$ whose independent sets consists of all subsets of cardinality at most d . This corresponds to a collection of n vectors in $\mathbb{F}^d$ in general position (which can always be achieved if $\mathbb{F}$ is infinite).

Definition 3.7. Let $M = (E, \mathcal{I})$ be a matroid of rank d . The **free extension** of M is the matroid $F(M) := (E', \mathcal{I}')$, where $E' := E \sqcup \{0\}$ and $S \in \mathcal{I}'$ if one of the following two properties hold:

- $0 \notin S$ and $S \in \mathcal{I}$,
- $0 \in S$, $S \setminus \{0\} \in \mathcal{I}$, and $|S| < d$.

Then $F(M) \setminus 0 = M$, and $F(M)$ is the most generic matroid of rank d with this property.¹

Example 3.8. Let V be a vector space over the field $\mathbb{F}$, E a finite set, and $\varphi : E \rightarrow V$ a map. If $\mathbb{F}$ is infinite, then it is always possible to extend φ to a map $\varphi' : E \sqcup \{0\} \rightarrow V$ in such a way that $M_{\varphi'} = F(M)$. If $\mathbb{F}$ is finite, however, this may not be possible. For example, if $\dim V > 0$, $E = V$, and φ is the identity map, then no such φ' exists.

Proposition 3.9. *For any matroid M , we have the following identities:*

- $\text{tr } M = F(M)/0$
- $F(M) = \text{tr}(M \oplus \text{Boo}_1)$

Proof. The first statement is equivalent to Example 3.5, while the second follows immediately from the definitions. □

Exercise 3.10. *Describe the circuits and flats of $U_{d,n}$.*

Exercise 3.11. *For which values of d and n is $U_{d,n}$ graphical?*

Exercise 3.12. *For which values of d and n is $U_{d,n}$ isomorphic to M_φ for some $\varphi : [n] \rightarrow \mathbb{F}_2^d$?*

Exercise 3.13. *For which values of q and n is $U_{2,n}$ isomorphic to M_φ for some $\varphi : [n] \rightarrow \mathbb{F}_q^2$?*

Exercise 3.14. *For which values of q is $U_{3,5}$ isomorphic to M_φ for some $\varphi : [5] \rightarrow \mathbb{F}_q^3$?*

Exercise 3.15. *Show that $\text{tr } M_{K_5}$ is not graphical.*

¹By this we mean that $F(M)$ has the most independent sets of any matroid with this property.

4 Realizability

Let $M = (E, \mathcal{I})$ be a matroid. A **realization** of M over a field $\mathbb{F}$ is a vector space V over $\mathbb{F}$ and a map $\varphi : E \rightarrow V$ such that $M = M_\varphi$.

Definition 4.1. We say that M is **realizable** if there exists a realization over some field.

Remark 4.2. Exercises 3.12, 3.13, and 3.14 are all about realizability of uniform matroids over various fields. The general problem of determining the set of q for which $U_{d,n}$ is realizable over $\mathbb{F}_q$ is still open! There is a conjectural answer due to Segré.

Example 4.3. The **non-Pappus matroid** is a matroid of rank 3 on 9 elements that is not realizable over $\mathbb{R}$ (by Pappus's theorem). In fact, it is not realizable over any field.

Definition 4.4. We say that M is **regular** if there exists a realization over any field. So we have

$$\{\text{graphical matroids}\} \subset \{\text{regular matroids}\} \subset \{\text{realizable matroids}\} \subset \{\text{matroids}\}.$$

Let's show that each of these containments is strict.

Example 4.5. By Exercise 2.11, $(M_{K_{3,3}})^*$ is not graphical. However, Proposition 2.5 says that dualization preserves realizability (over any field), so $(M_{K_{3,3}})^*$ is regular.

Example 4.6. Consider the matroid that is realized by the columns of the following matrix:

$$\begin{array}{cccccc} a & b & c & d & e & f & g \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

This is actually not enough information to determine the matroid, because we have not yet specified the field $\mathbb{F}$. Specifically, the set $\{d, e, f\}$ is independent if and only if the characteristic of $\mathbb{F}$ is not equal to 2. If $\text{char } \mathbb{F} = 2$, this matroid is called the **Fano matroid** and denoted F_7 . If $\text{char } \mathbb{F} \neq 2$, it is called the **non-Fano matroid** and denoted F_7^- . With some work, one can see that F_7 is *only* realizable over fields of characteristic 2, and F_7^- is *only* realizable over fields of characteristic different from 2 (see Exercise 4.9). In particular, these two matroids are realizable but not regular.

Example 4.7. Suppose that $M = M_1 \oplus M_2$. If M is realizable over $\mathbb{F}$, then so are M_1 and M_2 , as each can be recovered from M by repeated deletions. This means that $F_7 \oplus F_7^-$ cannot be realizable over any field, since there is no field over which both F_7 and F_7^- are realizable.

Remark 4.8. A theorem of Nelson says that almost all matroids are non-realizable. That is, if $f(n)$ is equal to the number of realizable matroids on the set $[n]$ divided by the total number of matroids on the set $[n]$, then we have $\lim_{n \rightarrow \infty} f(n) = 0$.

Exercise 4.9. Suppose that we have a realization of F_7 over a field $\mathbb{F}$. Without loss of generality, we can take $V = \mathbb{F}^3$, $\varphi(a) = (1, 0, 0)$, $\varphi(b) = (0, 1, 0)$, and $\varphi(c) = (0, 0, 1)$. We know that $\varphi(d)$ is in the span of $\varphi(a)$ and $\varphi(b)$, but is not proportional to either one. After scaling, we may assume that $\varphi(d) = (1, \delta, 0)$ for some nonzero $\delta \in \mathbb{F}$. Similarly, we can assume that $\varphi(e) = (0, 1, \epsilon)$ and $\varphi(f) = (1, 0, \sigma)$. Since $\varphi(g)$ is in the span of $\varphi(a)$ and $\varphi(e)$, but not proportional to either one, we can assume that $\varphi(g) = (1, \gamma, \gamma\epsilon)$ for some nonzero $\gamma \in \mathbb{F}$. In other words, our realization comes from a matrix of the form

$$\begin{array}{cccccc} & a & b & c & d & e & f & g \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \delta & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 & \epsilon & \sigma & \gamma\epsilon \end{bmatrix} \end{array}.$$

- (i) Use the fact that $\{b, f, g\}$ is dependent to conclude that $\sigma = \gamma\epsilon$.
- (ii) Use the fact that $\{c, d, g\}$ is dependent to conclude that $\gamma = \delta$.
- (iii) Use the fact that $\{d, e, f\}$ is dependent to conclude that $\sigma + \delta\epsilon = 0$.
- (iv) Use parts (i)-(iii) to derive a contradiction if $\text{char } \mathbb{F} \neq 2$.

Exercise 4.10. Modify the above argument to show that F_7^- is not realizable over $\mathbb{F}$ if $\text{char}(\mathbb{F}) = 2$.

Exercise 4.11. Show that every deletion of F_7 is isomorphic to M_{K_4} .

Exercise 4.12. If you did not do Exercises 3.12, 3.13, and 3.14 last time, do them now!

5 The lattice of flats

Definition 5.1. A **lattice** is a poset P with the property that, for all $x, y \in P$, there exists:

- a unique maximal element $x \wedge y$ below both x and y , called the **meet** of x and y
- a unique minimal element $x \vee y$ above both x and y , called the **join** of x and y .

Let $M = (E, \mathcal{I})$ be a matroid. Recall that a flat of M is a subset $F \subset E$ that is maximal within its rank, and a circuit of M is a minimal dependent subset of E . The proof of the following lemma is left as an exercise, but you should at least think through why it is true for realizable matroids (see Example 1.3) and graphical matroids (Exercise 1.8).

Lemma 5.2. *A subset $F \subset E$ is a flat if and only if the following condition holds: For all circuits $C \subset E$, if F contains all but one element of C , then F contains C . That is, $|F \cap C| \neq |C| - 1$.*

Corollary 5.3. *If F and G are flats, so is $F \cap G$.*

Corollary 5.4. *For any $S \subset E$, there exists a smallest flat containing S , namely*

$$\bar{S} := \bigcap_{S \subset F} F.$$

Corollary 5.5. *The poset $\mathcal{L}(M)$ of flats of M is a lattice, with $F \wedge G = F \cap G$ and $F \vee G = \overline{F \cup G}$.*

Example 5.6. The matroid Boo_n has no circuits, so every subset is a flat. The lattice $\mathcal{L}(\text{Boo}_n)$ is called the **Boolean lattice**, with meet given by intersection and join given by union.

Example 5.7. Consider the graphical matroid M_{K_4} , which is also realizable. By Exercise 1.7, the elements of $\mathcal{L}(M_{K_4})$ may be identified with partitions of $[4]$, with the meet $\wedge$ given by refining and the join $\vee$ given by coarsening. For example, we have

$$(12, 34) \wedge (13, 24) = (1, 2, 3, 4) \quad \text{and} \quad (12, 34) \vee (13, 24) = (1234).$$

Suppose we are given a finite set E , a vector space V , and a map $\varphi : E \rightarrow V$. In Example 1.3, we explained that there is a bijection $F \rightarrow V_F$ from $\mathcal{L}(M_\varphi)$ to the set of subspaces of V spanned by subsets of $\varphi(E)$, and this bijection takes flats of rank k to subspaces of dimension k . We have $V_{F \vee G} = V_F + V_G$. We also have $V_{F \wedge G} \subset V_F \cap V_G$, but the containment might be strict, as $V_F \cap V_G$ might not be spanned by any subset of $\varphi(E)$.

Example 5.8. Let $E = \{1, 2, 3, 4\}$ and define $\varphi : E \rightarrow \mathbb{R}^3$ as follows:

$$\varphi(1) = (1, 0, 0), \quad \varphi(2) = (0, 1, 0), \quad \varphi(3) = (0, 0, 1), \quad \text{and} \quad \varphi(4) = (1, 1, 1).$$

Then $M_\varphi = U_{3,4}$. We have

$$\{1, 2\} \wedge \{3, 4\} = \emptyset \quad \text{and} \quad \{1, 2\} \vee \{3, 4\} = \{1, 2, 3, 4\}.$$

The subspaces $V_{\{1,2\}}$ and $V_{\{3,4\}}$ are both planes, and they intersect in the line through $(1, 1, 0)$. However, this line is not spanned by any element of $\varphi(E)$, which is why

$$V_{\{1,2\} \vee \{3,4\}} = V_{\{1,2,3,4\}} = 0.$$

Proposition 5.9. *For any flats F and G , we have $\text{rk } F + \text{rk } G \geq \text{rk } F \wedge G + \text{rk } F \vee G$.*

Proposition 5.9 says that the lattice $\mathcal{L}(M)$ is **semimodular**. If we always had equality, it would be called **modular**, but Examples 5.7 and 5.8 both illustrate that this need not be the case. In each example, we have a pair of rank 2 flats whose join has rank 3 and whose meet has rank 0.

Proof. Let S be a maximal independent subset of $F \wedge G$. By Lemma 2.4, we can choose a maximal independent subset $T \subset F \cup G$ with $S \subset T$. Then

$$\begin{aligned} \text{rk } F \vee G &= |T| \\ &= |T \cap F| + |T \cap G| - |T \cap (F \cap G)| \\ &= |T \cap F| + |T \cap G| - |S| \\ &\leq \text{rk } F + \text{rk } G - \text{rk}(F \wedge G), \end{aligned}$$

where the last inequality follows from the fact that $T \cap F$ is an independent subset of F and $T \cap G$ is an independent subset of G , so $|T \cap F| \leq \text{rk } F$ and $|T \cap G| \leq \text{rk } G$. $\square$

Remark 5.10. A ranked lattice is called **geometric** if it is graded (rank of an element is the length of a maximal chain from the minimum element), **atomic** (every element is a join of rank 1 elements), and semimodular. Proposition 5.9 implies that $\mathcal{L}(M)$ is a geometric lattice. Conversely, every geometric lattice is isomorphic to $\mathcal{L}(M)$ for some matroid M .

A matroid is called **simple** if every subset of size at most 2 is independent. Given a matroid M , there is a canonical **simplification** $\bar{M}$ given by deleting loops and identifying two elements if and only if they span the same rank 1 flat. Then two matroids M and N have isomorphic lattices of flats if and only if they have isomorphic simplifications. That is, isomorphism classes of geometric lattices are in canonical bijection with isomorphism classes of simple matroids.

Exercise 5.11. Let $\mathcal{L}$ be a finite lattice. Let $A(\mathcal{L})$ be the ring with $\mathbb{Z}$ -basis $\{\sigma_x \mid x \in \mathcal{L}\}$ and multiplication

$$\sigma_x \cdot \sigma_y = \delta_{xy} \sigma_x.$$

In other words, this (boring!) algebra is just a direct sum of copies of $\mathbb{Z}$, one for each element of $\mathcal{L}$. Let $B(\mathcal{L})$ be the ring with $\mathbb{Z}$ -basis $\{\epsilon_x \mid x \in \mathcal{L}\}$ and multiplication

$$\epsilon_x \cdot \epsilon_y = \epsilon_{x \vee y}.$$

This ring, which seems much more interesting, is called the **Möbius ring** of $\mathcal{L}$. Show that the Möbius ring $B(\mathcal{L})$ is in fact isomorphic to the boring ring $A(\mathcal{L})$. Hint: Write down an explicit homomorphism from $B(\mathcal{L})$ to $A(\mathcal{L})$ by specifying where each ϵ_x should go.

Exercise 5.12. Suppose that $\mathcal{L}$ is geometric, and define $B^i(\mathcal{L}) := \mathbb{Z}\{\epsilon_x \mid \text{rk } x \leq i\} \subset B(\mathcal{L})$. Show that $B^i(\mathcal{L}) \cdot B^j(\mathcal{L}) \subset B^{i+j}(\mathcal{L})$, i.e. that this defines a filtration of the ring $B(\mathcal{L})$. Describe the associated graded ring in terms of generators and relations. (This ring plays a major role in the proof of the Dowling–Wilson conjecture.)

Exercise 5.13. Prove Lemma 5.2: A subset $F \subset E$ is a flat if and only if, for all circuits $C \subset E$, if F contains all but one element of C , then F contains C .

6 The Möbius function

Let P be a finite poset (think of the lattice of flats).

Definition 6.1. We define the **incidence algebra**

$$I(P) := \{f : P \times P \rightarrow \mathbb{Z} \mid f(x, y) \neq 0 \Rightarrow x \leq y\}$$

with multiplication

$$(f * g)(x, z) := \sum_{x \leq y \leq z} f(x, y) g(y, z).$$

Note that this is just matrix multiplication! If we pick a refinement of our partial order and use that to order the rows and columns of our matrices, then the matrices will be upper triangular, and they will have zeros in certain spots corresponding to incomparable elements. The following basic properties are immediate from this interpretation:

- Multiplication is associative.
- The multiplicative identity is the Kronecker delta function δ .
- An element $f \in I(P)$ is invertible if and only if $f(x, x) = \pm 1$ for all $x \in P$.

Let $\zeta(x, y) = 1$ for all $x \leq y$, and let $\mu = \zeta^{-1}$; this is called the **Möbius function** of P . We have $\mu\zeta = \delta$, therefore $\mu(x, x) = 1$ for all $x \in P$, and if $x < z$, then

$$0 = \delta(x, z) = \sum_{x \leq y \leq z} \mu(x, y) \zeta(y, z) = \sum_{x \leq y < z} \mu(x, y),$$

and therefore

$$\mu(x, z) = - \sum_{x \leq y < z} \mu(x, y).$$

This gives a recursive way to compute $\mu(x, z)$ for all $x < z$.

Example 6.2. If we take P to be the lattice of flats of $U_{2,3}$, then $\mu(\emptyset, \{e\}) = -1$ for $e \in \{1, 2, 3\}$, and $\mu(\emptyset, \{1, 2, 3\}) = 2$.

Example 6.3. If we take P to be the Boolean lattice, then $\mu(S, T) = (-1)^{|T \setminus S|}$. The fact that this satisfies the relevant recursion follows from the binomial theorem.

Let $\mathcal{L}$ be a finite lattice, and recall the algebras $A(\mathcal{L})$ and $B(\mathcal{L})$ from Exercise 5.11. The hope is that you found the following isomorphism:

$$\begin{aligned} \varphi : B(\mathcal{L}) &\rightarrow A(\mathcal{L}) \\ \epsilon_x &\mapsto \sum_{x \leq y} \sigma_y = \sum_{x \leq y} \zeta(x, y) \sigma_y. \end{aligned}$$

One can prove that this is an isomorphism without explicitly constructing the inverse, but now we have the tool to construct it. Define

$$\begin{aligned}\psi : A(\mathcal{L}) &\rightarrow B(\mathcal{L}) \\ \sigma_y &\mapsto \sum_{y \leq z} \mu(y, z) \epsilon_z.\end{aligned}$$

Then we have

$$\begin{aligned}\psi \circ \varphi(\epsilon_x) &= \sum_{x \leq y \leq z} \zeta(x, y) \mu(y, z) \epsilon_z \\ &= \sum_z \epsilon_z \sum_{x \leq y \leq z} \zeta(x, y) \mu(y, z) \\ &= \sum_z \epsilon_z \delta(x, z) \\ &= \epsilon_x,\end{aligned}$$

and similarly $\varphi \circ \psi(\sigma_y) = \sigma_y$.

Theorem 6.4 (Cross-cut theorem). *Let M be a simple matroid and μ the Möbius function on $\mathcal{L}(M)$. We have*

$$\mu(\emptyset, E) = \sum_{\bar{S}=E} (-1)^{|\bar{S}|}.$$

Proof. For each $e \in E$, we have

$$\varphi(\epsilon_\emptyset - \epsilon_{\{e\}}) = \sum_F \sigma_F - \sum_{e \in F} \sigma_F = \sum_{e \notin F} \sigma_F.$$

Taking the product over all elements of E , we have

$$\prod_{e \in E} \varphi(\epsilon_\emptyset - \epsilon_{\{e\}}) = \prod_{e \in E} \left(\sum_{e \notin F} \sigma_F \right) = \sigma_\emptyset.$$

Applying ψ to both sides, we obtain the equation

$$\prod_{e \in E} (\epsilon_\emptyset - \epsilon_{\{e\}}) = \sum_F \mu(\emptyset, F) \epsilon_F.$$

Now let's compare coefficients of ϵ_E , and note that $\prod_{e \in S} \epsilon_{\{e\}} = \epsilon_{\bar{S}}$. □

Remark 6.5. Theorem 6.4 works for arbitrary loopless matroids; we just need to replace elements of E with rank 1 flats wherever they appear in the proof. It definitely fails for matroids with loops, even if you replace $\emptyset$ with the minimal flat, as the sum on the right would be zero. More generally,

for any flat F in a loopless matroid M , we have

$$\mu(\emptyset, F) = \sum_{\bar{S}=F} (-1)^{|\bar{S}|}.$$

Theorem 6.6 (Weisner's theorem). *Let $M = (E, \mathcal{I})$ be a simple matroid and $F \neq \emptyset$ a nonempty flat. Then*

$$\sum_{F \vee G = E} \mu(\emptyset, G) = 0.$$

Proof. First, we note that

$$\varphi(\epsilon_F \psi(\sigma_\emptyset)) = \sum_{F \leq G} \sigma_G \sigma_\emptyset = 0,$$

so $\epsilon_F \psi(\sigma_\emptyset) = 0$. Thus

$$0 = \epsilon_F \psi(\sigma_\emptyset) = \epsilon_F \sum_G \mu(\emptyset, G) \epsilon_G = \sum_G \mu(\emptyset, G) \epsilon_{F \vee G}.$$

Looking at the coefficient of ϵ_E gives us the theorem. □

Exercise 6.7. *Draw the lattice of flats of M_{K_4} and compute $\mu(\emptyset, F)$ for each flat F . You can do this by hand or using the cross-cut theorem. Try both!*

Exercise 6.8. *Let $M = (E, \mathcal{I})$ be a matroid, and let μ be the Möbius function on $\mathcal{L}(M)$. If $\bar{S} = E$, we say that S is a **spanning set**. Show that the number of spanning sets is equal to*

$$\sum_{F \in \mathcal{L}(M)} \mu(F, E) 2^{|F|}.$$

Hint: Consider the following two functions on the lattice of flats: $s(F) := |\{S \subset E \mid \bar{S} = F\}|$, and $t(F) := 2^{|F|}$. Express the relationship between these two functions algebraically. If you struggle, skip ahead to Theorem 8.1.

7 The characteristic polynomial

Let $M = (E, \mathcal{I})$ be a matroid. For any subset $S \subset E$, we define the **corank** $\text{crk } S := \text{rk } M - \text{rk } S$. We then define the **characteristic polynomial**

$$\chi_M(t) := \sum_{S \subset E} (-1)^{|S|} t^{\text{crk } S}.$$

Note that, if M has a loop $e \in E$, then $\text{crk } S = \text{crk } S \cup \{e\}$ for any $S \subset E \setminus \{e\}$, and therefore $\chi_M(t) = 0$.

Proposition 7.1. *If M is loopless, then*

$$\chi_M(t) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F) t^{\text{crk } F}.$$

Proof. This follows from Remark 6.5 and the fact that $\text{crk } S = \text{crk } \bar{S}$ for any $S \subset E$. $\square$

If $E \neq \emptyset$, the binomial theorem tells us that $\chi_M(1) = 0$, and therefore that $\chi_M(t)$ is a multiple of $t - 1$. We therefore define the **reduced characteristic polynomial**

$$\bar{\chi}_M(t) := \chi_M(t)/(t - 1).$$

Let's see how the characteristic polynomial and reduced characteristic polynomial behave with respect to various matroid operations that we have studied. This first proposition is immediate from the definition.

Proposition 7.2. *For any matroids M_1 and M_2 , we have $\chi_{M_1 \oplus M_2}(t) = \chi_{M_1}(t) \chi_{M_2}(t)$.*

Example 7.3. We have $\chi_{\text{Boo}_n}(t) = (t - 1)^n$. This can be seen either using Proposition 7.2 or directly from the definition using the binomial theorem.

Proposition 7.4. *If $\text{rk } M \geq 1$, then*

$$\bar{\chi}_{\text{tr}(M)}(t) = \frac{\bar{\chi}_M(t) - \bar{\chi}_M(0)}{t}.$$

Before proving Proposition 7.4, let's do an example.

Example 7.5. Let $M = U_{4,5} = \text{tr Boo}_5$. Then

$$\bar{\chi}_M(t) = \frac{(t - 1)^4 - (-1)^4}{t} = t^3 - 4t^2 + 6t - 4.$$

Proof of Proposition 7.4. We first note that

$$\text{rk}_{\text{tr}(M)} S = \begin{cases} \text{rk}_M S & \text{if } \text{rk}_M S < \text{rk } M \\ \text{rk } M - 1 & \text{if } \text{rk}_M S = \text{rk } M, \end{cases}$$

and therefore

$$\text{crk}_{\text{tr}(M)} S = \begin{cases} \text{crk}_M S - 1 & \text{if } \text{crk}_M S > 0 \\ 0 & \text{if } \text{crk}_M S = 0. \end{cases}$$

This easily implies that

$$\chi_{\text{tr}(M)}(t) = \frac{\chi_M(t) - \chi_M(0)}{t} + c$$

for some constant $c \in \mathbb{Z}$. Evaluating at $t = 1$ lets us conclude that $c = \chi_M(0)$. The proposition now follows from a little bit of algebraic manipulation. $\square$

We have already observed that, if $e \in E$ is a loop, then $\chi_M(t) = 0$. If $e \in E$ is a coloop, then $M = \text{Boo}_1 \oplus M/e$, so Proposition 7.2 implies that $\chi_M(t) = (t - 1)\chi_{M/e}(t)$. These observations, combined with the next proposition, provide a useful inductive procedure for computing $\chi_M(t)$.

Proposition 7.6. *If e is not a coloop, then*

$$\chi_M(t) = \chi_{M \setminus e}(t) - \chi_{M/e}(t).$$

Proof. We have

$$\begin{aligned} \chi_M(t) &= \sum_{S \subseteq E} (-1)^{|S|} t^{\text{crk } S} \\ &= \sum_{S \subseteq E \setminus \{e\}} (-1)^{|S|} t^{\text{crk } S} - \sum_{S \subseteq E \setminus \{e\}} (-1)^{|S|} t^{\text{crk } S \cup \{e\}} \\ &= \chi_{M \setminus e}(t) - \chi_{M/e}(t). \end{aligned}$$

Note that we used the fact that e is not a coloop in the last equality, because we needed to know that $\text{rk } M \setminus e = \text{rk } M$. $\square$

Exercise 7.7. *Two elements e and f of the ground set of a matroid M are said to be **parallel** if $\{e\}$ and $\{f\}$ are independent but $\{e, f\}$ is dependent. If this is the case, show that $\chi_M(t) = \chi_{M \setminus e}(t)$. Thus we can always reduce to a simple matroid when computing the characteristic polynomial.*

Exercise 7.8. *Compute $\chi_{M_{K_4}}(t)$.*

Exercise 7.9. *Compute $\chi_{M_{C_n}}(t)$.*

Exercise 7.10. *Prove that, if M has no loops, the coefficients of $\chi_M(t)$ alternate in sign. Equivalently, the coefficients of $(-1)^{\text{rk } M} \chi_{M_G}(-t)$ are all positive. Do the same for $\bar{\chi}_M(t)$ (with the opposite sign).*

8 Möbius inversion and the characteristic polynomial

Theorem 8.1 (Möbius inversion). *Let P be a poset, and let $w, v : P \rightarrow \mathbb{Z}$ be functions.*

- *If $w(x) = \sum_{x \leq y} v(y)$, then $v(x) = \sum_{x \leq y} \mu(x, y)w(y)$.*
- *If $w(y) = \sum_{x \leq y} v(x)$, then $v(y) = \sum_{x \leq y} w(x)\mu(x, y)$.*

We will use only the first of these two statements in this section, but the second statement is the one needed for Exercise 6.8.

Proof. We prove only the first statement; the second is similar. We have

$$\begin{aligned}
 \sum_{x \leq y} \mu(x, y)w(y) &= \sum_{x \leq y} \mu(x, y) \sum_{y \leq z} v(z) \\
 &= \sum_z v(z) \sum_{x \leq y \leq z} \mu(x, y) \\
 &= \sum_z v(z) \delta(x, z) \\
 &= v(x).
 \end{aligned}$$

This completes the proof. □

Remark 8.2. An equivalent formulation of the proof of Theorem 8.1 is to say that the abelian group of functions from P to $\mathbb{Z}$ is a left-module over $I(P)$ and $w = \zeta \cdot v$, so $\mu \cdot w = \mu \cdot \zeta \cdot v = \delta \cdot v = v$. The second statement is similar, but uses a right-module structure, instead.

Let V be a vector space, E a finite set, and $\varphi : E \rightarrow V$ a map such that $V_E = V$. For all $e \in E$, let

$$H_e := \{\alpha \in V^* \mid \langle \alpha, \varphi(e) \rangle = 0\}.$$

The letter H stands for *hyperplane*, and if $\varphi(e) \neq 0$, then H_e is the hyperplane in V^* perpendicular to the line in V spanned by $\varphi(e)$. But we also allow for the degenerate possibility that $\varphi(e) = 0$, in which case $H_e = V^*$. Before we state any theorems, it should already seem natural to think about hyperplanes rather than vectors in the context of matroid theory, because the hyperplanes don't change when we scale the vectors by nonzero constants.

For any subset $S \subset E$, we have a linear subspace $V_S = \text{Span}\{\varphi(e) \mid e \in S\} \subset V$, and we also define the perpendicular space

$$H_S := V_S^\perp = \bigcap_{e \in S} H_e.$$

We have already noted in Example 1.3 that there is a bijection between $F \mapsto V_F$ between flats of M_φ and subspaces of V spanned by subsets of the vectors $\varphi(E)$, and that it takes flats of rank k to subspaces of dimension k . Similarly, there is a bijection $F \mapsto H_F$ between flats of M_φ and

subspaces of V^* obtained by intersecting some subset of the hyperplanes $\{H_e \mid e \in E\}$, and it takes flats of rank k to subspaces of codimension k .

Proposition 8.3. *Suppose that our field $\mathbb{F}$ is a finite field of cardinality q . Then*

$$\chi_{M_\varphi}(q) = \left| V^* \setminus \bigcup_{e \in E} H_e \right|.$$

Proof. If there is some e such that $\varphi(e) = 0$, then both sides are equal to zero. Thus we may assume that this is not the case. Define two functions on the lattice of flats by putting

$$v(F) := \left| H_F \setminus \bigcup_{F < G} H_G \right| \quad \text{and} \quad w(F) := |H_F| = q^{\text{crk } F}.$$

Then

$$w(F) = \sum_{F \leq G} v(G),$$

so Theorem 8.1 says that

$$v(F) = \sum_{F \leq G} \mu(F, G) w(G) = \sum_{F \leq G} \mu(F, G) q^{\text{crk } G}.$$

The proposition now follows from Proposition 7.1 by evaluating at $F = \emptyset$. $\square$

Example 8.4. Suppose that $M_\varphi = U_{2,3}$, which means that H_1 , H_2 , and H_3 are three distinct lines in a 2-dimensional vector space over $\mathbb{F}_q$. We start with q^2 points, remove three hyperplanes, and then correct for the fact that we removed the origin three times. This tells us that the total number of points on the complement of the hyperplanes is $q^2 - 3q + 2 = \chi_{U_{2,3}}(q)$.

Remark 8.5. If we drop the condition that $V_E = V$, then we will have $\dim H_F = \text{crk } F + \dim H_E$, and therefore

$$q^{\dim H_E} \chi_{M_\varphi}(q) = \left| V^* \setminus \bigcup_{e \in E} H_e \right|.$$

Example 8.6. Let's compute the characteristic polynomial of M_{K_n} . We know that it is realized by the vectors $\{x_i - x_j \mid 1 \leq i < j \leq n\} \subset \mathbb{F}_q^n$. The perpendicular hyperplanes are

$$H_{ij} = \{z \in \mathbb{F}_q^n \mid z_i = z_j\},$$

and H_E is the line through the point $(1, \dots, 1)$. The complement of these hyperplanes can be regarded as the space of n distinct elements of $\mathbb{F}_q$, so we have

$$q \chi_{M_{K_n}}(q) = q(q-1)(q-2) \dots (q-n+1).$$

Since this is true for every prime power q , we must have

$$\chi_{M_{K_n}}(t) = (t-1)(t-2) \dots (t-n+1).$$

For any graph Γ , let $\xi_\Gamma(t)$ be the **chromatic polynomial**. Explicitly, if $t \geq 0$, $\xi_\Gamma(t)$ is the number of way to color the vertices of Γ with t colors such that no two adjacent vertices get the same color. The following proposition generalizes Example 8.6.

Proposition 8.7. *If Γ is a graph with c connected components, then $\xi_\Gamma(t) = t^c \chi_{M_\Gamma}(t)$.*

Proof. Label the vertices with the set $[n]$. Then M_Γ is realized by the vectors

$$\{x_i - x_j \mid i \leq j \text{ connected by an edge}\} \subset \mathbb{F}_q^n.$$

This should be regarded as a multiset (as the graph can have multiple edges between the same vertices), and it may include copies of the zero vector (one for each loop). These vectors do not span $\mathbb{F}_q^n$; the codimension of the span is equal to c . A q -coloring of the vertices with adjacent vertices getting different colors is precisely an element of the complement of the corresponding hyperplanes in $\mathbb{F}_q^n$. Thus Remark 8.5 tells us that $q^c \chi_{M_\Gamma}(q) = \xi_\Gamma(q)$. Since this is true for any prime power q , we must have $t^c \chi_{M_\Gamma}(t) = \xi_\Gamma(t)$. $\square$

Example 8.8. In Proposition 7.4, we showed that

$$\bar{\chi}_{U_{n-1,n}}(t) = \bar{\chi}_{\text{tr}(\text{Boon})}(t) = \frac{(t-1)^{n-1} - (-1)^{n-1}}{t}.$$

Thus

$$\xi_{C_n}(t) = t \chi_{U_{n-1,n}}(t) = t(t-1) \bar{\chi}_{U_{n-1,n}}(t) = (t-1)^n + (-1)^n(t-1).$$

Exercise 8.9. *If Γ is a graph and e is an edge, interpret the deletion/contraction formula for $\chi_{M_\Gamma}(t)$ in terms of graph colorings. Remember to consider the cases where e is a loop or a coloop separately!*

Exercise 8.10. *Let Γ be the graph obtained from K_4 by deleting a single edge. Compute $\xi_\Gamma(t)$ in four different ways:*

- *by adding the edge back to get K_4 ,*
- *by deleting an edge to get C_4 ,*
- *by computing the lattice of flats of M_Γ ,*
- *directly from the definition in terms of graph colorings.*

Exercise 8.11. *What is the analogue of Proposition 8.3 for the reduced characteristic polynomial?*

9 The Poincaré polynomial

Let V be a vector space over a field $\mathbb{F}$, E a finite set, and $\varphi : E \rightarrow V$ a map. Let

$$U_\varphi := V^* \setminus \bigcup_{e \in E} H_e \quad \text{and} \quad \bar{U}_\varphi := \mathbb{P}(V^*) \setminus \bigcup_{e \in E} \mathbb{P}(H_e) = U_\varphi / \mathbb{F}^\times.$$

Proposition 8.3 says that, if $|\mathbb{F}| = q$, then $\chi_{M_\varphi}(q) = q^{\dim H_E} |U_\varphi|$, and therefore $\bar{\chi}_{M_\varphi}(q) = q^{\dim H_E} |\bar{U}_\varphi|$ provided that $E \neq \emptyset$. The next few lectures are devoted to making an analogous statement when $\mathbb{F} = \mathbb{C}$ is the field of complex numbers.

We begin with the following lemma.

Lemma 9.1. *If $E \neq \emptyset$, we have an isomorphism $U_\varphi \cong \bar{U}_\varphi \times \mathbb{F}^\times$.*

Proof. Choose an element $e \in E$. If e is a loop, then both sides are empty, so we may assume that it is not. We have a map

$$\rho : U_\varphi \rightarrow \bar{U}_\varphi \times \mathbb{F}^\times$$

given by the formula

$$\rho(\alpha) := ([\alpha], \langle \varphi(e), \alpha \rangle).$$

We also have a map

$$\sigma : \bar{U}_\varphi \times \mathbb{F}^\times \rightarrow U_\varphi$$

given by the formula

$$\sigma([\alpha], \lambda) = \lambda \alpha$$

for all $\alpha \in U_\varphi$ with $\langle \varphi(e), \alpha \rangle = 1$. It's straightforward to check that these are inverse isomorphisms. $\square$

Assume now that $\mathbb{F} = \mathbb{C}$. Define the **Poincaré polynomial**

$$\pi_\varphi(t) := \sum_{i \geq 0} t^i \dim H^i(U_\varphi; \mathbb{C}).$$

If $E \neq \emptyset$, define also the **reduced Poincaré polynomial**

$$\bar{\pi}_\varphi(t) := \sum_{i \geq 0} t^i \dim H^i(\bar{U}_\varphi; \mathbb{C}).$$

By Lemma 9.1 and the Künneth isomorphism, we have $\pi_\varphi(t) = \bar{\pi}_\varphi(t)(1+t)$. The following theorem, due to Orlik and Solomon, is the first major result in this course.

Theorem 9.2. *The (reduced) characteristic polynomial is the (reduced) Poincaré polynomial backward, with signs. That is, we have*

$$\chi_{M_\varphi}(t) = t^{\text{rk } M_\varphi} \pi_\varphi(-t^{-1}) \quad \text{and} \quad \bar{\chi}_{M_\varphi}(t) = t^{\text{rk } M_\varphi - 1} \bar{\pi}_\varphi(-t^{-1}),$$

or, equivalently,

$$\pi_\varphi(t) = (-t)^{\text{rk } M_\varphi} \chi_{M_\varphi}(-t^{-1}) \quad \text{and} \quad \bar{\pi}_\varphi(t) = (-t)^{\text{rk } M_\varphi - 1} \bar{\chi}_{M_\varphi}(-t^{-1}).$$

Remark 9.3. It is straightforward to check that any one of the equations in Theorem 9.2 implies the other three.

We will delay the proof of Theorem 9.2 and focus in this lecture on a couple of examples.

Example 9.4. Let's consider the analogue of Example 8.4 over the complex numbers. Suppose that $M_\varphi = U_{2,3}$, which means that H_1 , H_2 , and H_3 are three distinct lines in $\mathbb{C}^2$. Then $\bar{U}_\varphi$ is equal to the complement of three points in $\mathbb{CP}^1$, so

$$\bar{\pi}_\varphi(t) = 1 + 2t.$$

On the other hand, we have $\bar{\chi}_{U_{2,3}}(t) = t - 2 = t \bar{\pi}_\varphi(-t^{-1})$.

We conclude this lecture with one extended example. Consider our usual realization of the matroid M_{K_n} by the vectors

$$\{x_i - x_j \mid i < j\} \subset \mathbb{C}^n,$$

and let

$$U_n := \{(z_1, \dots, z_n \mid z_i \neq z_j \text{ for all } i \neq j\} \subset \mathbb{C}^n$$

be the corresponding hyperplane complement. We have a projection $p_n : U_n \rightarrow U_{n-1}$ given by forgetting the last coordinate; this is a fiber bundle with fibers homeomorphic to the space

$$F_{n-1} := \mathbb{C} \setminus \{1, 2, \dots, n-1\}.$$

Lemma 9.5. *There exists an isomorphism*

$$H^*(U_n; \mathbb{C}) \cong H^*(U_{n-1}; \mathbb{C}) \otimes H^*(F_{n-1}; \mathbb{C})$$

of graded vector spaces.

Proof. Consider the inclusion $\iota : F_{n-1} \rightarrow U_n$ as the fiber of p_n over the point $(1, 2, \dots, n-1) \in U_{n-1}$. By the Leray–Hirsch theorem, it is sufficient to prove that the restriction map

$$\iota^* : H^*(U_n; \mathbb{C}) \rightarrow H^*(F_{n-1}; \mathbb{C})$$

is surjective. For all $1 \leq i \leq n-1$, define $f_i : U_n \rightarrow \mathbb{C}^\times$ by putting $f_i(z) = z_n - z_i$. Let $\theta \in H^1(\mathbb{C}^\times; \mathbb{C})$ be the cohomology class that evaluates to 1 on the homology class of the positively oriented unit circle, and let $\theta_i := f_i^* \theta \in H^1(U_n; \mathbb{C})$. We can see that $\theta_1, \dots, \theta_{n-1}$ restricts to a basis for $H^1(F_{n-1}; \mathbb{C})$. More explicitly, if $C_i \subset F_{n-1}$ is a positively oriented circle of radius $1/2$ around the point i , then

$$\langle \theta_i, [C_j] \rangle = \langle \theta, (f_i)_*[C_j] \rangle = \delta_{ij}.$$

Hence ι^* is surjective. □

Let $\pi_n(t)$ be the Poincaré polynomial of U_n . By Lemma 9.5, we have

$$\pi_n(t) = \pi_{n-1}(t)(1 + (n-1)t) = \cdots = (1+t)(1+2t)\cdots(1+(n-1)t),$$

which agrees with the formula predicted by Example 8.6 and Theorem 9.2.

Remark 9.6. Let's say a word about the relationship between Proposition 8.3 and Theorem 9.2. Since $M = M_\varphi$ is realizable over $\mathbb{C}$, it is also realizable over $\mathbb{F}_q$ for some prime power q . We'll denote that realization by ψ , so we have a hyperplane complement U_ψ over $\mathbb{F}_q$. We want there to be some relationship between φ and ψ ; the precise statement involves the phrase *spreading out*. The situation that you should have in mind is where the vectors $\varphi(e)$ have integer coordinates, and we reduce modulo a large prime without changing which subsets of vectors are independent.

Choose a prime l not dividing q , and consider the l -adic étale cohomology of $U_\psi(\bar{\mathbb{F}}_q)$. By general nonsense, the dimension of $H_{\text{ét}}^i(U_\psi(\bar{\mathbb{F}}_q); \bar{\mathbb{Q}}_l)$ is equal to that of $H^i(U_\varphi; \mathbb{C})$, and same for compactly supported cohomology. Let us assume that our vectors $\psi(E)$ span the vector space, and let d be the rank of M . One can show that the Frobenius automorphism acts on $H_{\text{ét},c}^i(U_\psi(\bar{\mathbb{F}}_q); \bar{\mathbb{Q}}_l)$ as multiplication by q^{i-d} .² Then, by the Grothendieck–Lefschetz fixed point formula, we have

$$\begin{aligned} \chi_M(q) = |U_\psi| &= \sum_{i \geq 0} (-1)^i \operatorname{tr}(\operatorname{Fr} \curvearrowright H_{\text{ét},c}^i(U_\psi(\bar{\mathbb{F}}_q); \bar{\mathbb{Q}}_l)) \\ &= q^d \dim H_{\text{ét},c}^{2d}(U_\psi(\bar{\mathbb{F}}_q); \bar{\mathbb{Q}}_l) - q^{d-1} \dim H_{\text{ét},c}^{2d-1}(U_\psi(\bar{\mathbb{F}}_q); \bar{\mathbb{Q}}_l) + \cdots \\ &= q^d \dim H_c^{2d}(U_\varphi; \mathbb{C}) - q^{d-1} \dim H_c^{2d-1}(U_\varphi; \mathbb{C}) + \cdots \\ &= q^d \dim H^0(U_\varphi; \mathbb{C}) - q^{d-1} \dim H^1(U_\varphi; \mathbb{C}) + \cdots \\ &= (-q)^d \pi_\varphi(-q^{-1}). \end{aligned}$$

Since this holds for infinitely many prime powers q , it must hold at the level of polynomials.

Exercise 9.7. Let U be the complement of a collection of 6 hyperplanes in $\mathbb{C}^4$, any four of which intersect only at the origin. Compute the Betti numbers of U .

Exercise 9.8. Let $U'_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 \neq z_2 \neq z_3 \neq \cdots \neq z_n \neq z_1\}$. (Note that U_n is an open subset of U'_n .) Compute the Betti numbers of U'_n .

²Proving this would require some of the material that we will develop in the next few sections. It amounts to showing that the cohomology of $U_\psi(\bar{\mathbb{F}}_q)$ admits a surjection from the cohomology of a torus.

10 The long exact sequence

As in the previous lecture, let V be a vector space over $\mathbb{C}$, E a finite set, $\varphi : E \rightarrow V$ a map,

$$U_\varphi := V^* \setminus \bigcup_{e \in E} H_e,$$

and

$$\pi_\varphi(t) := \sum_{i \geq 0} t^i \dim H^i(U_\varphi; \mathbb{C}).$$

The goal of this lecture is to work toward the proof of Theorem 9.2. Let

$$\pi_M(t) := (-t)^{\text{rk } M} \chi_M(-t^{-1}),$$

so that Theorem 9.2 is the statement that $\pi_\varphi(t) = \pi_{M_\varphi}(t)$. The polynomial $\pi_M(t)$ is called the **Poincaré polynomial** of M . We can use Proposition 7.6 to derive a deletion/contraction recurrence for $\pi_M(t)$. Concretely:

- If $e \in E$ is a loop, then $\chi_M(t) = 0$, so $\pi_M(t) = 0$.
- If $e \in E$ is a coloop, then $\chi_M(t) = (t - 1)\chi_{M/e}(t)$, so $\pi_M(t) = (1 + t)\pi_{M/e}(t)$.
- If $e \in E$ is neither, then $\chi_M(t) = \chi_{M \setminus e}(t) - \chi_{M/e}(t)$, so $\pi_M(t) = \pi_{M \setminus e}(t) + t\pi_{M/e}(t)$.

We now want to show that $\pi_\varphi(t)$ satisfies similar recursions. Fix an element $e \in E$, and define

$$\varphi' : E \setminus \{e\} \rightarrow V \quad \text{and} \quad \varphi'' : E \setminus \{e\} \rightarrow V/\mathbb{C}\varphi(e),$$

so that $M_{\varphi'} = M_\varphi \setminus e$ and $M_{\varphi''} = M_\varphi/e$ (Example 2.2).

- If e is a loop, then $H_e = V^*$, so $U_\varphi = \emptyset$, and therefore $\pi_\varphi(t) = 0$.
- If e is a coloop, then $U_\varphi \cong \mathbb{C}^\times \times U_{\varphi''}$, so $\pi_\varphi(t) = (1 + t)\pi_{\varphi''}(t)$.

It remains to show that, if e is neither a loop nor a coloop, we have $\pi_\varphi(t) = \pi_{\varphi'}(t) + t\pi_{\varphi''}(t)$.

Let us examine the geometric relationship between U_φ , $U_{\varphi'}$, and $U_{\varphi''}$. We have

$$U_{\varphi''} \subset (V/\mathbb{C}\varphi(e))^* \cong \varphi(e)^\perp = H_e \subset V^*,$$

so all three spaces are subspaces of V^* . Furthermore, we have

$$U_{\varphi'} = U_\varphi \sqcup U_{\varphi''},$$

with $U_\varphi \subset U_{\varphi'}$ being open and $U_{\varphi''} \subset U_{\varphi'}$ being closed. Consider the long exact sequence of the pair $(U_{\varphi'}, U_\varphi)$:

$$\cdots \rightarrow H^i(U_{\varphi'}) \rightarrow H^i(U_\varphi) \rightarrow H^{i+1}(U_{\varphi'}, U_\varphi) \rightarrow H^{i+1}(U_{\varphi'}) \rightarrow \cdots, \quad (1)$$

all with complex coefficients.

Lemma 10.1. *There is an isomorphism $H^{i+1}(U_{\varphi'}, U_{\varphi}) \cong H^{i-1}(U_{\varphi''})$.*

Proof. The normal bundle to $U_{\varphi''}$ in $U_{\varphi'}$ is the restriction to $U_{\varphi''}$ of the normal bundle to H_e in V^* . In particular, it is trivial. Thus we may choose a tubular neighborhood N of $U_{\varphi''}$ in $U_{\varphi'}$ with $N \cong U_{\varphi''} \times \mathbb{C}$. Let

$$N^* := N \setminus U_{\varphi''} = N \cap U_{\varphi} \cong U_{\varphi''} \times \mathbb{C}^{\times}.$$

By excision, we have

$$H^{i+1}(U_{\varphi'}, U_{\varphi}) \cong H^{i+1}(N, N^*) \cong H^{i+1}(U_{\varphi''} \times \mathbb{C}, U_{\varphi''} \times \mathbb{C}^{\times}).$$

By the Künneth isomorphism, this is isomorphic to $H^{i-1}(U_{\varphi''}) \otimes H^2(\mathbb{C}, \mathbb{C}^{\times}) \cong H^{i-1}(U_{\varphi''})$. $\square$

Combining Lemma 10.1 with Equation (1), we obtain a long exact sequence

$$\cdots \longrightarrow H^i(U_{\varphi'}) \xrightarrow{a_i} H^i(U_{\varphi}) \xrightarrow{b_i} H^{i-1}(U_{\varphi''}) \xrightarrow{d_i} H^{i+1}(U_{\varphi'}) \rightarrow \cdots . \quad (2)$$

To prove Theorem 9.2, it would suffice to show that the map d_i vanishes for all i . We will prove this in the next two lectures.

Exercise 10.2. *Suppose that $M_{\varphi} = U_{2,3}$. That is, suppose that we have three vectors $\varphi(1)$, $\varphi(2)$, and $\varphi(3)$ in $\mathbb{C}^2$, no two of them parallel, along with their perpendicular hyperplanes (lines) H_1 , H_2 , and H_3 . Then we have*

$$\cdots \rightarrow H^1(U_{\varphi}) \xrightarrow{b_1} H^0(U_{\varphi''}) \xrightarrow{d_1} H^2(U_{\varphi'}) \xrightarrow{a_2} H^2(U_{\varphi}) \rightarrow \cdots .$$

Show that $d_1 = 0$. (Hint: You can try to show this directly, or you can show that b_1 is surjective, or that a_2 is injective.)

Exercise 10.3. *Use the ideas in this section to give an alternative proof of Proposition 8.3.*

11 De Rham cohomology

In this lecture, we'll give a quick and dirty introduction to De Rham cohomology, containing just enough information to follow the next lecture. We will be guided by [MT97, Chapters 2 and 3], to which one can refer for more details.

Let V be a real vector space of dimension d over $\mathbb{R}$. An **alternating p -tensor** on V is defined to be a map $\omega : V^p \rightarrow \mathbb{R}$, linear in each of the p variables, with the property that $\omega(v_1, \dots, v_p) = 0$ whenever $v_i = v_j$ for some $i \neq j$. The set $\text{Alt}^p(V) = \wedge^p V^*$ of alternating p -tensors on V is itself a vector space of dimension $\binom{d}{p}$. Furthermore, it admits a graded ring structure

$$\begin{aligned} \text{Alt}^p(V) \otimes \text{Alt}^q(V) &\rightarrow \text{Alt}^{p+q}(V) \\ (\omega_1, \omega_2) &\mapsto \omega_1 \wedge \omega_2. \end{aligned}$$

When $p = q = 1$, we have

$$(\omega_1 \wedge \omega_2)(v_1, v_2) := \omega_1(v_1)\omega_2(v_2) - \omega_2(v_1)\omega_1(v_2).$$

The formula for general p and q involves an alternating sum over all shuffles; see [MT97, Definition 2.5]. The ring

$$\text{Alt}^*(V) := \bigoplus_{p=0}^d \text{Alt}^p(V) = \bigoplus_{p=0}^d \wedge^p V^*$$

is also called the exterior algebra of V^* .

Example 11.1. Suppose that $V = \mathbb{R}^d = \mathbb{R}\{x_1, \dots, x_d\}$. For all i , let $dx_i \in V^*$ be the linear functional given by the formula $dx_i(x_j) = \delta_{ij}$. Then $\text{Alt}^p(V)$ has basis

$$\left\{ dx_{i_1} \wedge \dots \wedge dx_{i_p} \mid i_1 < \dots < i_p \right\},$$

where

$$dx_{i_1} \wedge \dots \wedge dx_{i_p}(v_1, \dots, v_p) = \sum_{\sigma \in S_p} \text{sign}(\sigma) dx_{i_1}(v_{\sigma(1)}) \dots dx_{i_p}(v_{\sigma(p)}).$$

Any linear map $\varphi : V \rightarrow W$ induces a map of algebras

$$\varphi^* : \text{Alt}^*(W) \rightarrow \text{Alt}^*(V)$$

by the formula

$$(\varphi^* \omega)(v_1, \dots, v_p) := \omega(\varphi(v_1), \dots, \varphi(v_p))$$

for all $\omega \in \text{Alt}^p(W)$.

Now suppose that $U \subset V$ is an open set. We define a **differential p -form** on U to be a smooth map $\omega : U \rightarrow \text{Alt}^p(V)$, and we write $\Omega^p(U; \mathbb{R})$ to be the vector space of such forms. Note that

$\Omega^0(U; \mathbb{R})$ is simply the ring of smooth $\mathbb{R}$ -valued functions on U . The direct sum

$$\Omega^*(U; \mathbb{R}) = \bigoplus_{p=0}^d \Omega^p(U; \mathbb{R})$$

is a graded commutative ring under wedge product.

Example 11.2. Let's continue with Example 11.1. If $U \subset \mathbb{R}^d$, then an element of $\Omega^p(U; \mathbb{R})$ may be written in the form

$$\sum_{i_1 < \dots < i_p} g_{i_1, \dots, i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p},$$

where $g_{i_1, \dots, i_p}$ is a smooth function on U .

Given two open sets $U \subset V$ and $U' \subset V'$ along with a smooth map $f : U \rightarrow U'$, we obtain a graded ring homomorphism

$$f^* : \Omega^*(U'; \mathbb{R}) \rightarrow \Omega^*(U; \mathbb{R})$$

given by the formula

$$(f^*\omega)_x := df_x^* \omega_{f(x)}$$

for all $\omega \in \Omega^p(U'; \mathbb{R})$ and $x \in U$. Equivalently, we have

$$(f^*\omega)_x(v_1, \dots, v_p) = \omega_{f(x)}(df_x(v_1), \dots, df_x(v_p)).$$

In the special case where $p = 0$, f^* is the usual pullback of smooth functions.

The key extra piece of structure that we have is a differential $d : \Omega^p(U; \mathbb{R}) \rightarrow \Omega^{p+1}(U; \mathbb{R})$. A precise definition can be found in [MT97, Definition 3.2], but we will content ourselves with stating its important properties:

- If $g \in \Omega^0(U; \mathbb{R})$ is a function, then dg is the usual derivative. In particular, if $V = \mathbb{R}^d$, then

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \dots + \frac{\partial g}{\partial x_d} dx_d.$$

- We have $d \circ d = 0$.
- For any $\omega_1 \in \Omega^p(U; \mathbb{R})$ and $\omega_2 \in \Omega^q(U; \mathbb{R})$, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$.
- For any smooth map $f : U \rightarrow U'$, $d \circ f^* = f^* \circ d$. That is, f^* is a homomorphism of differential graded algebras.

Remark 11.3. The first three properties above characterize d [MT97, Theorem 3.7], and the last property characterizes f^* [MT97, Theorem 3.12].

Example 11.4. Suppose that $U \subset \mathbb{R}^3$. We have

$$\begin{aligned}\Omega^0(U) &= \{g : U \rightarrow \mathbb{R}\} \\ \Omega^1(U) &= \{g_1 dx_1 + g_2 dx_2 + g_3 dx_3\} \\ \Omega^2(U) &= \{g_{12} dx_1 dx_2 + g_{13} dx_1 dx_3 + g_{23} dx_2 dx_3\} \\ \Omega^3(U) &= \{g_{123} dx_1 dx_2 dx_3\}.\end{aligned}$$

The operator d represents grad, curl, and div:

$$\begin{aligned}dg &= \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 + \frac{\partial g}{\partial x_3} dx_3, \\ d(g_1 dx_1 + g_2 dx_2 + g_3 dx_3) &= \left(\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2 + \left(\frac{\partial g_3}{\partial x_1} - \frac{\partial g_1}{\partial x_3}\right) dx_1 dx_3 + \left(\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3}\right) dx_2 dx_3, \\ d(g_{12} dx_1 dx_2 + g_{13} dx_1 dx_3 + g_{23} dx_2 dx_3) &= \left(\frac{\partial g_{12}}{\partial x_3} - \frac{\partial g_{13}}{\partial x_2} + \frac{\partial g_{23}}{\partial x_1}\right) dx_1 dx_2 dx_3.\end{aligned}$$

In particular, the composition $d \circ d = 0$.

Definition 11.5. The **degree p De Rham cohomology** of U is the vector space

$$H_{\text{DR}}^p(U; \mathbb{R}) := \frac{\ker(d : \Omega^p(U; \mathbb{R}) \rightarrow \Omega^{p+1}(U; \mathbb{R}))}{\text{im}(d : \Omega^{p-1}(U; \mathbb{R}) \rightarrow \Omega^p(U; \mathbb{R}))}.$$

Given $\omega \in \Omega^p(U; \mathbb{R})$ with $d\omega = 0$, we will write $[\omega] \in H_{\text{DR}}^p(U; \mathbb{R})$ to denote its De Rham cohomology class. It is straightforward to check that wedge product and pullback both descend from forms to cohomology.

Theorem 11.6. *There is a canonical isomorphism of graded rings $H_{\text{DR}}^*(U; \mathbb{R}) \cong H^*(U; \mathbb{R})$ between De Rham cohomology and singular cohomology. This isomorphism is compatible with pullbacks—that is, it is a natural isomorphism of functors from open subsets of Euclidean spaces to graded commutative algebras.*

Remark 11.7. We will not prove Theorem 11.6, but let's say a few words about why it should be plausible, at least for those who are comfortable with the notion of differential forms on manifolds. If a De Rham cohomology class is supposed to be the same as a singular cohomology class, then it should be possible to pair it with a singular homology class to get a number. Any element of $H_p(U; \mathbb{R})$ can be represented by a linear combination of classes of the form $i_*[X]$, where X is a compact oriented manifold of dimension p and $i : X \rightarrow U$ is a smooth map. If $\omega \in \Omega^p(U; \mathbb{R})$ and $d\omega = 0$, the pairing between $[\omega]$ and $i_*[X]$ is equal to the integral

$$\int_X i^* \omega.$$

For this to make sense, we need to know that this integral vanishes if $\omega = d\eta$ for some $\eta \in$

$\Omega^{p-1}(U; \mathbb{R})$, or if i extends to a map $Y \rightarrow U$ for some compact oriented manifold with boundary Y such that $\partial Y = X$. In both cases, the integral vanishes by Stokes's theorem.

Remark 11.8. Up to this point, we have worked with De Rham cohomology with real coefficients. However, in the next section we will find it convenient to work with complex coefficients. There is nothing subtle here—instead of working with smooth maps from U to $\text{Alt}^p(V)$, we take smooth maps from U to $\text{Alt}^p(V) \otimes \mathbb{C}$, and the ring we end up with is isomorphic to cohomology with coefficients in $\mathbb{C}$. Our reason for working with complex coefficients is that, when $U = \mathbb{C}^\times \subset \mathbb{C} \cong \mathbb{R}^2$, it is convenient to work with the 1-form $\frac{dz}{z}$ (see Exercise 11.9). We could of course take the real and imaginary parts of this form and work exclusively with real coefficients, but that would be a little bit messier.

Exercise 11.9. Let $U = \mathbb{C}^\times \subset \mathbb{C} \cong \mathbb{R}^2$, and let

$$\omega := \frac{1}{2\pi i} \frac{dz}{z} \in \Omega^1(U; \mathbb{C}).$$

Check that $d\omega = 0$, but that there is no function $\eta \in \Omega^0(U; \mathbb{C})$ such that $d\eta = \omega$. Check that the cohomology class $[\omega] \in H^1(U; \mathbb{C})$ evaluates to 1 on the homology class of the positively oriented unit circle, thus this agrees with the class θ that appeared in the proof of Lemma 9.5.

Exercise 11.10. Continuing with Exercise 11.9, compute the real and imaginary parts of ω . One of these is equal to $d\eta$ for some $\eta \in \Omega^0(U; \mathbb{R})$, and the other is not. Which is which?

12 The Orlik–Solomon algebra

Retain all of the notation from the previous lecture, and assume for simplicity that $\varphi(e) \neq 0$ for all $e \in E$. (Otherwise, U_φ is empty and everything is trivial.) In order to complete our proof of Theorem 9.2, we will attempt to give a combinatorial presentation of the cohomology ring $H^*(U_\varphi; \mathbb{C})$. Rather than jumping right to the answer, let's try to figure it out as we go.

As in Exercise 11.9, let

$$\theta := \frac{1}{2\pi i} \frac{dz}{z} \in \Omega^1(\mathbb{C}^\times; \mathbb{C}).$$

For all $e \in E$, we may regard $\varphi(e) \in V$ as a map from V^* to $\mathbb{C}$, which restricts to a map from U_φ to $\mathbb{C}^\times$. Let

$$\theta_e := \varphi(e)^* \theta = \frac{1}{2\pi i} \frac{d\varphi(e)}{\varphi(e)} \in \Omega^1(U_\varphi; \mathbb{C}).$$

These differential forms are all in degree 1, so they anticommute. Furthermore, they satisfy the following fundamental relations.

Lemma 12.1. *Suppose that $\{e_1, \dots, e_k\} \subset E$ is a circuit in M_φ . Then*

$$\sum_{j=1}^k (-1)^j \theta_{e_1} \cdots \widehat{\theta}_{e_j} \cdots \theta_{e_k} = 0.$$

Proof. Because $\{e_1, \dots, e_k\} \subset E$ is a circuit, there exist constants $c_1, \dots, c_{k-1} \in \mathbb{C}$ such that

$$\varphi(e_k) = \sum_{j=1}^{k-1} c_j \varphi(e_j), \tag{3}$$

and therefore

$$\theta_{e_k} = \frac{1}{2\pi i} \frac{d\varphi(e_k)}{\varphi(e_k)} = \sum_{j=1}^{k-1} \frac{c_j}{2\pi i} \frac{d\varphi(e_j)}{\varphi(e_k)} = \sum_{j=1}^{k-1} c_j \frac{\varphi(e_j)}{\varphi(e_k)} \theta_{e_j}.$$

Thus

$$\begin{aligned} \sum_{j=1}^k (-1)^j \theta_{e_1} \cdots \widehat{\theta}_{e_j} \cdots \theta_{e_k} &= \sum_{j=1}^{k-1} (-1)^j \theta_{e_1} \cdots \widehat{\theta}_{e_j} \cdots \theta_{e_{k-1}} \theta_{e_k} + (-1)^k \theta_{e_1} \cdots \theta_{e_{k-1}} \\ &= \sum_{j=1}^{k-1} (-1)^j \theta_{e_1} \cdots \widehat{\theta}_{e_j} \cdots \theta_{e_{k-1}} c_j \frac{\varphi(e_j)}{\varphi(e_k)} \theta_{e_j} + (-1)^k \theta_{e_1} \cdots \theta_{e_{k-1}}. \end{aligned}$$

Note what we did here: θ_{e_k} is a sum of $k-1$ terms, but the i^{th} term is killed when we multiply it by θ_{e_i} (because $\theta_{e_i}^2 = 0$), so only the j^{th} term survives! This entire expression simplifies to

$$(-1)^k \theta_{e_1} \cdots \theta_{e_{k-1}} \left(1 - \sum c_j \frac{\varphi(e_j)}{\varphi(e_k)} \right),$$

which is equal to zero by Equation (3). □

Lemma 12.1 motivates the following definition.

Definition 12.2. Let $M = (E, \mathcal{I})$ be a matroid. Let $\Lambda_{\mathbb{C}}[x_e \mid e \in E]$ be the graded $\mathbb{C}$ -algebra freely generated by anticommuting variables in degree 1 indexed by elements of E . We define the **Orlik–Solomon ideal**

$$J(M) := \left\langle \sum_{j=1}^k (-1)^j x_{e_1} \cdots \widehat{x}_{e_j} \cdots x_{e_k} \mid \{e_1, \dots, e_k\} \text{ a circuit} \right\rangle,$$

and the **Orlik–Solomon algebra**

$$\text{OS}^*(M) := \Lambda_{\mathbb{C}}[x_e \mid e \in E] / J(M).$$

Lemma 12.1 implies that there is a graded algebra homomorphism $\xi_{\varphi} : \text{OS}^*(M_{\varphi}) \rightarrow H^*(U_{\varphi}; \mathbb{C})$ taking x_e to $[\theta_e]$. The following theorem, due to Orlik and Solomon [OS80], is our second major result.

Theorem 12.3. *The homomorphism $\xi_{\varphi} : \text{OS}^*(M_{\varphi}) \rightarrow H^*(U_{\varphi}; \mathbb{C})$ is an isomorphism.*

Remark 12.4. Lemma 12.1 tells us something much stronger than the fact that ξ_{φ} is well defined. Rather than just proving that this relation holds at the level of cohomology classes, our argument shows that it holds at the level of differential forms! This tells us that the space U_{φ} is *formal*, meaning that its cohomology ring is quasi-isomorphic to the dg -algebra of differential forms.

Remark 12.5. Though we assumed for ease of notation that $\varphi(e) \neq 0$ for all $e \in E$, the two rings are isomorphic even without this assumption. Indeed, since $\{e\}$ is a circuit, we have $\text{OS}^*(M_{\varphi}) = 0$, and we already know that $H^*(U_{\varphi}; \mathbb{C}) = 0$ because $U_{\varphi} = \emptyset$.

Remark 12.6. Suppose that φ_1 and φ_2 are two different complex realizations of the same matroid M . The topological spaces U_{φ_1} and U_{φ_2} need not be homeomorphic, or even homotopy equivalent. Indeed, Rybnikov famously showed that they can have non-isomorphic fundamental groups. However, Theorem 12.3 says that their cohomology rings are canonically isomorphic.

Remark 12.7. There is a philosophical point to Theorem 12.3 that will come up again and again in this course. Given a matroid that is realizable over the complex numbers, we can construct a topological space (the complement of the hyperplane arrangement corresponding to a choice of realization) and a ring (its cohomology ring). Given an arbitrary matroid, there is no topological space, but there is still a ring, namely the Orlik–Solomon algebra, which serves as a combinatorially stand-in for the cohomology ring even in the absence of any geometric object.

Exercise 12.8. Show that $x_{e_1} \cdots x_{e_k} \in J(M)$ whenever $\{e_1, \dots, e_k\}$ is a circuit.

Exercise 12.9. Show that

$$\sum_{j=1}^k (-1)^j x_{e_1} \cdots \widehat{x}_{e_j} \cdots x_{e_k} \in J(M)$$

whenever $\{e_1, \dots, e_k\}$ is dependent. (That is, we don't need it to be a circuit.)

Exercise 12.10. As in Lecture 9, let

$$U_n := \{(z_1, \dots, z_n) \mid z_i \neq z_j \text{ for all } i \neq j\}$$

be the configuration space of n distinct labeled points in $\mathbb{C}$. Write down an explicit presentation for the cohomology ring of U_4 . In your first attempt, you should have 6 generators and 7 relations. Try to show that 4 of these relations are in fact sufficient. More generally, try to identify a minimal presentation for the cohomology ring of U_n .

13 Proving Theorems 9.2 and 12.3

We are now ready to give the proofs of our first two significant theorems, which will proceed by induction on the cardinality of the ground set E .

Let $M = (E, \mathcal{I})$ be a matroid, and let $e \in E$ be an element of the ground set. Since circuits of $M \setminus e$ are simply circuits of M that do not contain the element e , we have a natural inclusion of graded $\mathbb{C}$ -algebras

$$\rho_e : \text{OS}^*(M \setminus e) \rightarrow \text{OS}^*(M).$$

Consider the map of $\mathbb{C}$ -vector spaces

$$\partial_e : \Lambda_{\mathbb{C}}[x_f \mid f \in E] \rightarrow \Lambda_{\mathbb{C}}[x_f \mid f \in E \setminus \{e\}]$$

that is given by “differentiation by x_e ”. That is, we write

$$\Lambda_{\mathbb{C}}[x_f \mid f \in E] = \Lambda_{\mathbb{C}}[x_f \mid f \in E \setminus \{e\}] \oplus x_e \cdot \Lambda_{\mathbb{C}}[x_f \mid f \in E \setminus \{e\}],$$

and define ∂_e to be the unique map of right $\Lambda_{\mathbb{C}}[x_f \mid f \in E \setminus \{e\}]$ -modules with the property that $\partial_e(1) = 0$ and $\partial_e(x_e) = 1$. You will show in Exercise 13.4 that ∂_e descends to a map from $\text{OS}^*(M)$ to $\text{OS}^*(M/e)$, which we will also denote ∂_e . This is clearly a surjection, and it lowers degree by 1. Furthermore, it is clear that $\partial_e \circ \rho_e = 0$, so we have a complex

$$0 \longrightarrow \text{OS}^*(M \setminus e) \xrightarrow{\rho_e} \text{OS}^*(M) \xrightarrow{\partial_e} \text{OS}^*(M/e) \longrightarrow 0. \quad (4)$$

Lemma 13.1. *The complex (4) is exact.*

Remark 13.2. Lemma 13.1 is one of the two parts of the proof of Theorems 9.2 and 12.3 for which we will not give a full argument. The strategy is to show that the generators we gave for the Orlik–Solomon ideal form a Gröbner basis, which gives us a basis for $\text{OS}^*(M)$ consisting of classes called **nbc monomials**. The map ρ_e takes nbc monomials to nbc monomials, and the map ∂_e takes the nbc monomials that are not in the image of ρ_e bijectively to the nbc monomials for M/e . For details, see [OT92, Theorems 3.43 and 3.65] or [Lev, Lemmas 4.2 and 4.3].

Recall from Lecture 10 that we defined the polynomial $\pi_M(t) := (-t)^{\text{rk } M} \chi_M(-t^{-1})$ and derived a recursion for $\pi_M(t)$ based on deletion and contraction.

Proposition 13.3. *We have*

$$\pi_M(t) = \sum_{i \geq 0} t^i \dim \text{OS}^i(M).$$

Proof. We proceed by induction on the cardinality of the ground set. If $e \in E$ is a loop, then $\text{OS}^*(M) = 0$ and $\pi_M(t) = 0$. If $e \in E$ is a coloop, then $\text{OS}^*(M) \cong \Lambda_{\mathbb{C}}[x_e] \otimes \text{OS}^*(M/e)$ and $\pi_M(t) = (1 + t)\pi_{M/e}(t)$, so the Proposition follows from the inductive hypothesis applied to the contraction M/e . Finally, if e is neither a loop nor a coloop, then Lemma 13.1 implies that there

exists an isomorphism of graded vector spaces

$$\mathrm{OS}^*(M) \cong \mathrm{OS}^*(M \setminus e) \oplus \mathrm{OS}^*(M/e)[-1].$$

On the other hand, we showed in Lecture 10 that $\pi_M(t) = \pi_{M \setminus e}(t) + t \pi_{M/e}(t)$. Thus the Proposition follows from the inductive hypothesis applied to both $M \setminus e$ and M/e . $\square$

Proof of Theorems 9.2 and 12.3. We proceed by induction on the cardinality of the ground set E . If M_φ has a loop, then everything is zero and both theorems are trivial. If all elements are coloops, then $U_\varphi \cong \mathbb{C}^{\times E}$ and both theorems are trivial. So we may assume that there is an element $e \in E$ that is neither a loop nor a coloop.

Fix a positive integer i and consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{OS}^i(M_{\varphi'}) & \xrightarrow{\rho_e} & \mathrm{OS}^i(M_\varphi) & \xrightarrow{\partial_e} & \mathrm{OS}^{i-1}(M_{\varphi''}) & \longrightarrow & 0 \\ & & \downarrow \xi_{\varphi'} & & \downarrow \xi_\varphi & & \downarrow \xi_{\varphi''} & & \downarrow \\ H^{i-2}(U_{\varphi''}) & \xrightarrow{d_{i-1}} & H^i(U_{\varphi'}) & \xrightarrow{a_i} & H^i(U_\varphi) & \xrightarrow{b_i} & H^{i-1}(U_{\varphi''}) & \xrightarrow{d_i} & H^{i+1}(U_{\varphi'}) \end{array}$$

Here the top row is the exact sequence from Lemma 13.1, and the bottom row is the exact sequence from Equation 2. Commutativity of the left-middle square is clear. Commutativity of the right-middle square is the second statement for which we will not provide an argument in these notes; see [OT92, Lemma 5.86] for details.

Our inductive hypothesis tells us that both $\xi_{\varphi'}$ and $\xi_{\varphi''}$ are isomorphisms. Since $\xi_{\varphi''}$ is an isomorphism and ∂_e is surjective, b_i is surjective, and therefore $d_i = 0$. We similarly have $d_{i-1} = 0$, therefore the bottom row reduces to a short exact sequence. It then follows from the five lemma that ξ_φ is an isomorphism. This proves Theorem 12.3, and Theorem 9.2 follows from Proposition 13.3. $\square$

Exercise 13.4. Show that $\partial_e : \Lambda_{\mathbb{C}}[x_f \mid f \in E] \rightarrow \Lambda_{\mathbb{C}}[x_f \mid f \in E \setminus \{e\}]$ descends to a map $\partial_e : \mathrm{OS}^*(M) \rightarrow \mathrm{OS}^*(M/e)$. This involves showing that an arbitrary monomial times an arbitrary generator of the Orlik–Solomon ideal for M gets mapped to an element of the Orlik–Solomon ideal for M/e .

Exercise 13.5. Verify Lemma 13.1 explicitly in the case where $M = U_{3,4}$, and when $M = M_{K_4}$.

14 Log concavity (first modern theorem)

The first part of these notes has been devoted to an exposition of two classical theorems describing the cohomology of the complement of a complex hyperplane arrangement in terms of the corresponding matroid. In this section we state a much more recent theorem about matroids. Before we start, let's recall all of the reasons we have for being interested in the (ordinary and reduced) characteristic polynomial and Poincaré polynomial.

- If M is realizable over a finite field, then evaluating the (reduced) characteristic polynomial of M at the cardinality of the field gives the number of points on the (projectivized) complement of the associated hyperplane arrangement (Proposition 8.3 and Exercise 8.11).
- The characteristic polynomial of a graphical matroid M_Γ is equal to the chromatic polynomial of Γ divided by a power of t (Proposition 8.7).
- If M is realizable over the complex numbers, then the (reduced) Poincaré polynomial of M is equal to the Poincaré polynomial of the (projectivized) complement of the associated hyperplane arrangement (Theorem 9.2).
- More generally, the coefficients of the Poincaré polynomial are equal to the dimensions of the graded pieces of the Orlik–Solomon algebra (Proposition 13.3).³

Definition 14.1. Let $a_0, a_1, \dots, a_m$ be a sequence of positive numbers. This sequence is **concave** if $a_i \geq \frac{a_{i-1} + a_{i+1}}{2}$ for all i . It is **log concave** if $\log a_0, \dots, \log a_m$ is concave. Note that we have

$$\begin{aligned}
 a_0, a_1, \dots, a_m \text{ log concave} &\Leftrightarrow \log a_i \geq \frac{\log a_{i-1} + \log a_{i+1}}{2} \text{ for all } i \\
 &\Leftrightarrow 2 \log a_i \geq \log a_{i-1} + \log a_{i+1} \text{ for all } i \\
 &\Leftrightarrow \log(a_i^2) \geq \log(a_{i-1} a_{i+1}) \text{ for all } i \\
 &\Leftrightarrow a_i^2 \geq a_{i-1} a_{i+1} \text{ for all } i.
 \end{aligned}$$

If $f(t) = \sum_{i=0}^m a_i t^i$ is a polynomial with positive coefficients, we will say that $f(t)$ is log concave if and only if its sequence of coefficients is log concave.

Example 14.2. Here is a log concave polynomial that I have encountered in the wild (it is the

³It is also possible to define the reduced Orlik–Solomon algebra, which gives us a reduced version of this result.

Kazhdan–Lusztig polynomial of the matroid $M_{K_{20}}$):

$$\begin{aligned}
f(t) &= 1 \\
&+ 524097 t \\
&+ 45762931992 t^2 \\
&+ 66208557177786 t^3 \\
&+ 11100857399288280 t^4 \\
&+ 404180066561961690 t^5 \\
&+ 4042252614171772000 t^6 \\
&+ 11522756204094885750 t^7 \\
&+ 7879824460254822075 t^8 \\
&+ 585243816844111425 t^9.
\end{aligned}$$

Since the log of an integer is roughly proportional to the number of digits in its decimal expansion, you can almost “see” log concavity without any calculations!

Remark 14.3. It is clear that a concave sequence is **unimodal**, meaning that there is some j for which $a_0 \leq a_1 \leq \dots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \dots \geq a_{m-1} \geq a_m$. Since the function log is monotonic, the same is true of a log concave sequence.

Here is our first big theorem.

Theorem 14.4. *For any matroid M , the polynomials $\pi_M(t)$ and $\bar{\pi}_M(t)$ are log concave.*

Theorem 14.4 has a long history. Unimodality of $\pi_M(t)$ was conjectured for graphical matroids in 1968 by Read, and for all matroids in 1971 by Rota. Log concavity of $\pi_M(t)$ was conjectured for graphical matroids in 1974 by Hogar, and for all matroids in 1976 by Welsh. The theorem was proved for matroids that are realizable over $\mathbb{C}$ by Huh [Huh12], for all realizable matroids by Huh and Katz [HK12], and finally for all matroids by Adiprasito, Huh, and Katz [AHK18]. In this course, we will outline the proof of Adiprasito, Huh, and Katz. The following basic lemma and corollaries explain why it is sufficient to treat the case of $\bar{\pi}_M(t)$ in Theorem 14.4.

Lemma 14.5. *Suppose $f(t) = \sum_{i=0}^m a_i t^i$ is log concave. Then we have $a_{i-1}a_i \geq a_{i-2}a_{i+1}$ for all i .*

Proof. We have

$$\begin{aligned}
a_{i-1}a_i^2 - a_{i-2}a_i a_{i+1} &\geq a_{i-1}^2 a_{i+1} - a_{i-2}a_i a_{i+1} \\
&= a_{i+1}(a_{i-1}^2 - a_{i-2}a_i) \\
&\geq 0.
\end{aligned}$$

Dividing both sides by a_i gives the desired result. □

Corollary 14.6. *If $f(t)$ is log concave and $\lambda > 0$, then $(1 + \lambda t)f(t)$ is also log concave.*

Proof. Write $f(t) = \sum a_i t^i$, so that

$$(1 + \lambda t)f(t) = \sum (a_i + \lambda a_{i-1})t^i.$$

We have

$$\begin{aligned} & (a_i + \lambda a_{i-1})^2 - (a_{i-1} + \lambda a_{i-2})(a_{i+1} + \lambda a_i) \\ = & a_i^2 + 2\lambda a_{i-1}a_i + \lambda^2 a_{i-1}^2 - (a_{i-1}a_{i+1} + \lambda a_{i-2}a_{i+1} + \lambda a_{i-1}a_i + \lambda^2 a_{i-2}a_i) \\ = & a_i^2 - a_{i-1}a_{i+1} + \lambda(a_{i-1}a_i - a_{i-2}a_{i+1}) + \lambda^2(a_{i-1}^2 - a_{i-2}a_i), \end{aligned}$$

which is non-negative by Lemma 14.5. □

Remark 14.7. By Corollary 14.6, log concavity of $\pi_M(t) = (1 + t)\bar{\pi}_M(t)$ will follow from log concavity of $\bar{\pi}_M(t)$.

The following result can be traced back to Isaac Newton!

Corollary 14.8. *Any real rooted polynomial $f(t)$ with positive coefficients is log concave.*

Proof. Since the coefficients are all positive, the roots $r_1, \dots, r_d$ must all be negative. We have $f(t) = a_0(1 + \lambda_1 t) \cdots (1 + \lambda_d t)$, where $\lambda_i = -r_i^{-1}$. Log concavity then follows from Corollary 14.6 by induction on d . □

Example 14.9. By Example 8.6, we have

$$\bar{\pi}_{M_{K_n}}(t) = (1 + 2t)(1 + 3t) \cdots (1 + (n - 1)t),$$

which is log concave by Corollary 14.8.

Remark 14.10. Example 14.9 might lead us to hope that $\bar{\pi}_M(t)$ is always real rooted, but this is in fact very unusual. For example, we have $\bar{\pi}_{U_{3,4}}(t) = 1 + 3t + 3t^2$, which does not have real roots.

Remark 14.11. Suppose that a group G acts on the matroid $M = (E, \mathcal{I})$. That is, G acts on E by permutations, preserving independence. Then the Orlik–Solomon algebra $\text{OS}(M)$ is naturally a graded representation of G . The following generalization of Theorem 14.4 was conjectured in [GPY17], and is still wide open:

For all i , there exists an inclusion of G -representations $\text{OS}^{i-1}(M) \otimes \text{OS}^{i+1}(M) \subset \text{OS}^i(M)^{\otimes 2}$, (5)

and similarly for the reduced Orlik–Solomon algebra. Note that, when G is the trivial group, this is precisely equivalent to Theorem 14.4. In the case where $M = M_{K_n}$ and $G = S_n$ with its natural action, Equation (12.3) is known to hold whenever $n \leq 44$ [MMPR23] or $i \leq 9$ [Hom].

Exercise 14.12. Suppose that

$$\bar{\pi}_M(t) = a_0 + a_1t + \cdots + a_{d-1}t^{d-1}.$$

Use Proposition 7.4 to show that

$$\bar{\pi}_{\text{tr } M}(t) = a_0 + a_1t + \cdots + a_{d-2}t^{d-2}.$$

Show that this allows us to reduce Theorem 14.4 to the statement that $a_{d-2}^2 \geq a_{d-3}a_{d-1}$.

Exercise 14.13. Suppose that $a_0, \dots, a_d$ is a log concave sequence of positive numbers. Generalize Lemma 14.5 by showing that, for all $0 \leq i \leq j \leq k \leq l \leq d$ with $i + l = j + k$, we have $a_j a_k \geq a_i a_l$.
Hint: First restrict to the case where $j = i + 1$, and induct on the quantity $k - j$. Then treat the general case with a second induction, this time on the quantity $j - i$.

Exercise 14.14. Prove that Equation (5) always holds when $i = 1$.

15 Correlation (second modern theorem)

Let Γ be a connected graph and let e, f be distinct edges that are not loops. We will choose a spanning tree T uniformly at random, and write $\mathbb{P}(e \in T)$ to denote the probability that T contains e . More concretely, we have

$$\mathbb{P}(e \in T) := \frac{\text{the number of spanning trees that contain } e}{\text{the total number of spanning trees}}.$$

We write $\mathbb{P}(e \in T \mid f \in T)$ to denote the probability that T contains e given the information that T contains f . More concretely, we have

$$\mathbb{P}(e \in T \mid f \in T) := \frac{\text{the number of spanning trees that contain both } e \text{ and } f}{\text{the number of spanning trees that contain } f}.$$

Example 15.1. Suppose that $\Gamma = K_3$ is a triangle, and e and f are two of the edges. Then $\mathbb{P}(e \in T) = 2/3$ and $\mathbb{P}(e \in T \mid f \in T) = 1/2$.

Intuitively, it makes sense that having $f \in T$ can only make it less likely that we have $e \in T$, and this is in fact the case. We will prove the following classical result in the next lecture.

Theorem 15.2 (Kirchhoff's theorem). *We always have $\mathbb{P}(e \in T \mid f \in T) \leq \mathbb{P}(e \in T)$.*

Now let us formulate the analogous statement for matroids. Let $M = (E, \mathcal{I})$ be a matroid, and let e and f be distinct elements of E that are not loops. We can choose a basis B uniformly at random, and define the probabilities

$$\mathbb{P}(e \in B) \quad \text{and} \quad \mathbb{P}(e \in B \mid f \in B)$$

as above. Theorem 15.2 motivates the following question.

Question 15.3. *Is it the case that we always have $\mathbb{P}(e \in B) \leq \mathbb{P}(e \in B \mid f \in B)$?*

Let's introduce the following quantities:

$$\begin{aligned} b_{ef} &:= \text{the number of bases containing both } e \text{ and } f \\ b^{ef} &:= \text{the number of bases containing neither } e \text{ nor } f \\ b_e^f &:= \text{the number of bases containing } e \text{ but not } f \\ b_f^e &:= \text{the number of bases containing } f \text{ but not } e. \end{aligned}$$

Then we have

$$\begin{aligned}
\frac{\mathbb{P}(e \in B \mid f \in B)}{\mathbb{P}(e \in B)} &= \frac{b_{ef}}{b_{ef} + b_f^e} \div \frac{b_{ef} + b_e^f}{b_{ef} + b^{ef} + b_e^f + b_f^e} \\
&= \frac{b_{ef}^2 + b_{ef}b^{ef} + b_{ef}b_e^f + b_{ef}b_f^e}{b_{ef}^2 + b_{ef}b_e^f + b_{ef}b_f^e + b_e^fb_f^e} \\
&= \frac{b_{ef}b^{ef} + c}{b_e^fb_f^e + c},
\end{aligned}$$

where $c = b_{ef}^2 + b_{ef}b_e^f + b_{ef}b_f^e$. Hence

$$\frac{\mathbb{P}(e \in B \mid f \in B)}{\mathbb{P}(e \in B)} \leq 1 \quad \Leftrightarrow \quad \frac{b_{ef}b^{ef}}{b_e^fb_f^e} \leq 1.$$

Let's define the **correlation constant**

$$\alpha(M, e, f) := \frac{b_{ef}b^{ef}}{b_e^fb_f^e},$$

and reformulate Question 15.3 as follows.

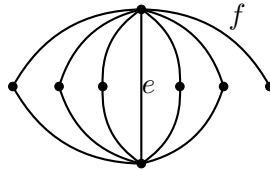
Question 15.4. *Theorem 15.2 says that*

$$\alpha(M, e, f) \leq 1$$

for graphical matroids. Does this hold for all matroids?

As you can probably guess from the exposition, the answer is no!

Example 15.5. Let $M = \text{tr}(M_\Gamma)$, where Γ is as shown:



Note that truncation does not preserve the property of being graphical, and M is indeed not a graphical matroid. Since Γ has 8 vertices, M_Γ has rank 7, and M has rank 6. That is, a basis for M is a collection of six edges that do not form any cycles. We have

$$\begin{aligned}
b_{ef} &= 5 \cdot 2^4 = 80 \\
b^{ef} &= 5 \cdot 2^4 = 80 \\
b_e^f &= 2^5 = 32 \\
b_f^e &= 2^5 + 5 \cdot 4 \cdot 2^3 = 192.
\end{aligned}$$

Thus

$$\alpha(M, e, f) = \frac{b_{ef} b^{ef}}{b_e^f b_f^e} = \frac{80 \cdot 80}{32 \cdot 192} = \frac{25}{24}.$$

Here is our second big theorem, which was proved by Huh, Schröter, and Wang [HSW22].

Theorem 15.6. *If M is a matroid of rank d and e and f are distinct elements of the ground set that are not loops, then*

$$\alpha(M, e, f) \leq 2 - \frac{2}{d}.$$

Exercise 15.7. *If M is a uniform matroid, show that $\alpha(M, e, f) < 1$. You can interpret this statement as follows: Suppose that d Muppets are chosen at random from a group of n . If Gonzo's name is called first, the probability that Fozzie's name will be called goes down.*

Exercise 15.8. *Fix a prime p , and let*

$$E := \{e, f\} \cup (\mathbb{F}_p \times \{2, 3, 4, 5\}).$$

Define $\varphi : E \rightarrow \mathbb{F}_p^5 = \mathbb{F}_p\{x_1, \dots, x_5\}$ as follows:

$$\begin{aligned} \varphi(e) &= x_1 \\ \varphi(f) &= x_2 + x_3 + x_4 + x_5 \\ \varphi(a, i) &= ax_1 + x_i. \end{aligned}$$

Show that

$$\alpha(M, e, f) = \frac{8}{7}.$$

Remark 15.9. For a long time, $\frac{8}{7}$ was the conjecturally the largest possible value of $\alpha(M, e, f)$ [BH20, Conjecture 3.12(2)]. However, a new example with $\alpha(M, e, f) = \frac{100}{81}$ was found by June Huh in 2026. There is still a wide gap between $\frac{100}{81}$ and the upper bound of 2 from Theorem 15.6.

16 Kirchhoff's theorem

In this lecture, we will prove Theorem 15.2. The proof will take the whole lecture, so there will be no exercises.

Let Γ be a connected graph with edge set E and vertex set V . An **arrow** is by definition an edge of Γ along with a choice of direction; we denote the set of arrows by A . For any arrow a , we write $\bar{a}$ for the reversed arrow; we therefore have $E \cong A/\sim$, where $a \sim \bar{a}$. For any $a \in A$, we write $[a] \in E$ to denote the underlying unoriented edge. We have **head** and **tail** maps $h, t : A \rightarrow V$.

Definition 16.1. Let r and s be distinct vertices of Γ . An **r/s flow** is a function $i : A \rightarrow \mathbb{R}$ satisfying the following properties:

- For all $a \in A$, $i_{\bar{a}} = -i_a$.
- For any vertex $u \notin \{r, s\}$, $\sum_{t(a)=u} i_a = 0$.

Lemma 16.2. *If i is an r/s flow, then*

$$-\sum_{h(a)=r} i_a = \sum_{t(a)=r} i_a = \sum_{h(a)=s} i_a = -\sum_{t(a)=s} i_a.$$

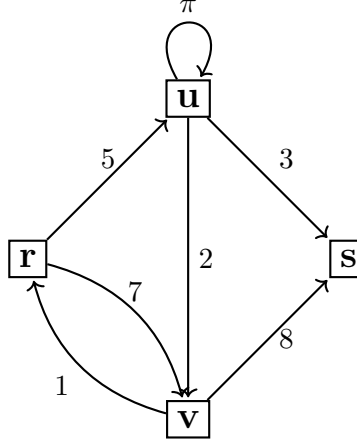
Proof. The first and last equality follow from the fact that $h(a) = u$ if and only if $t(\bar{a}) = u$. For the middle equality, we first observe that $\sum_a i_a = 0$, since the contribution of each arrow cancels that of its opposite. We therefore have

$$\sum_{t(a)=r} i_a = \sum_{t(a) \neq s} i_a = \sum_{h(a) \neq s} i_{\bar{a}} = -\sum_{h(a) \neq s} i_a = -\sum_a i_a + \sum_{h(a)=s} i_a = \sum_{h(a)=s} i_a,$$

and the lemma is proved. □

The quantity in Lemma 16.2 is denoted $|i|$. If $|i| = 1$, we say that i is a **unit flow**.

Example 16.3. Here is a picture of an r/s flow i with $|i| = 11$. There is one edge between r and u , and therefore two arrows. The flow i takes the value 5 on the arrow from r to u , and therefore it takes the value -5 on the arrow from u to r . There are two edges between r and v , and therefore four arrows. The values taken by i on these four arrows are ± 7 and ± 1 .



Definition 16.4. We define the **energy** $E(i) := \frac{1}{2} \sum_a i_a^2$.

Definition 16.5. A function $\varphi : V \rightarrow \mathbb{R}$ is called a **potential function** for i if, for all a ,

$$i_a = \varphi(h(a)) - \varphi(t(a)).$$

We observe that the flow i admits a potential function if and only if the sum of the values of i over any oriented cycle is zero, in which case the potential function is unique up to a constant.

Let N denote the total number of spanning trees in Γ , and for any arrow a , let N_a be the number of spanning trees T such that the unoriented edge $[a]$ is contained in T and the unique oriented path in T from r to s includes the arrow a .

Proposition 16.6. *The following statements hold.*

- (1) *There exists a unique unit flow i admitting a potential function φ .*
- (2) *We have $E(i) = \varphi(s) - \varphi(r)$.*
- (3) *For all unit flows j , $E(i) \leq E(j)$, with equality only if $i = j$.*
- (4) *For any arrow a , $i_a = (N_a - N_{\bar{a}})/N$.*

Proof. To prove (2) and (3), we assume that i is a unit flow and φ is a potential function for i . The existence of such an i will be established when we prove (4), and uniqueness is a consequence of (3).

We begin with the proof of (2). We have

$$\begin{aligned}
E(i) &= \frac{1}{2} \sum_a i_a^2 \\
&= \frac{1}{2} \sum_a i_a (\varphi(h(a)) - \varphi(t(a))) \\
&= \frac{1}{2} \sum_u \varphi(u) \sum_{h(a)=u} i_a - \frac{1}{2} \sum_u \varphi(u) \sum_{t(a)=u} i_a \\
&= \frac{1}{2} (\varphi(s)|i| - \varphi(r)|i|) - \frac{1}{2} (\varphi(r)|i| - \varphi(s)|i|) \\
&= \varphi(s) - \varphi(r).
\end{aligned}$$

Next, we prove (3). Let $k = j - i$, so $|k| = 0$. We have

$$\sum_a i_a k_a = \sum_a k_a (\varphi(h(a)) - \varphi(t(a))) = \sum_u \varphi(u) \left(\sum_{h(a)=u} k_a - \sum_{t(a)=u} k_a \right),$$

which vanishes because $|k| = 0$. Thus

$$E(j) = \frac{1}{2} \sum_a j_a^2 = \frac{1}{2} \sum_a (i_a + k_a)^2 = E(i) + E(k) + \sum_a i_a k_a = E(i) + E(k),$$

which is greater than or equal to $E(i)$ with equality if and only if $k = 0$.

Finally, we prove (1) and (4) by showing that $i_a := (N_a - N_{\bar{a}})/N$ defines a unit flow that sums to zero around all oriented cycles. We begin by showing that it is a flow. The fact that $i_{\bar{a}} = -i_a$ is clear. For the second condition, we fix a vertex $u \notin \{r, s\}$, and we want to show that $\sum_{h(a)=u} i_a = 0$. What does a particular spanning tree T contribute to this sum?

- If the unique oriented path in T from r to s does not pass through u , then T contributes nothing.
- If it does pass through u , let a be the arrow in the path with $h(a) = u$ and let b be the arrow with $t(b) = u$. Then T contributes $1/N$ to i_a and $-1/N$ to i_b , for a net contribution of zero.

Thus the sum vanishes, so i is a flow. To see that it is a unit flow, we perform a similar analysis to compute the sum $|i| = \sum_{h(a)=t} i_a$. In this case, every tree T contributes $1/N$, so the sum evaluates to 1.

It remains only to prove that the sum of the values of i around any oriented cycle is zero. To do this, we define a **bush** to be an ordered pair of (non-spanning) trees (T_r, T_s) with $r \in T_r$, $s \in T_s$, and $T_r \sqcup T_s$ a spanning forest. Then N_a is equal to the number of bushes (T_r, T_s) with $t(a) \in T_r$ and $h(a) \in T_s$.

Given an oriented cycle $(a_1, \dots, a_k)$ we need to show that

$$\sum_{j=1}^k i_{a_j} = 0.$$

What is the contribution of a particular bush (T_r, T_s) to this sum? It is

$$\frac{|\{j \mid t(a_j) \in T_r \text{ and } h(a_j) \in T_s\}| - |\{j \mid t(a_j) \in T_s \text{ and } h(a_j) \in T_r\}|}{N},$$

which is equal to zero. $\square$

The **effective resistance** $R_{\text{eff}}(\Gamma, r, s)$ is defined to be the energy $E(i)$ for the unique unit r/s flow i admitting a potential function. Proposition 16.6 has the following two corollaries.

Corollary 16.7. *For any arrow a , we have $R_{\text{eff}}(\Gamma, t(a), h(a)) = \mathbb{P}([a] \in T)$.*

Proof. Let i be the unique $t(a)/h(a)$ flow admitting a potential function, and let φ be such a potential function. By Proposition 16.6(2), we have

$$R_{\text{eff}}(\Gamma, t(a), h(a)) = E(i) = \varphi(h(a)) - \varphi(t(a)) = i_a.$$

By Proposition 16.6(4), this is equal to $(N_a - N_{\bar{a}})/N$. We clearly have $N_{\bar{a}} = 0$, and N_a is equal to the number of spanning trees containing the edge $[a]$, therefore we may conclude that $R_{\text{eff}}(\Gamma, t(a), h(a)) = N_a/N = \mathbb{P}([a] \in T)$. $\square$

Corollary 16.8. *Let f be an edge in Γ that does not connect r and s . Then*

$$R_{\text{eff}}(\Gamma, r, s) \geq R_{\text{eff}}(\Gamma/f, r, s).$$

Proof. Let i be the unique good flow on Γ . Note that there is a canonical injection from arrows of Γ/f to arrows of Γ , so we can define a function j on the arrows of Γ/f by putting $j_a = i_a$ for all a . It is easy to check that j is a flow. Let b be an orientation of f . Then

$$R_{\text{eff}}(\Gamma, r, s) = E(i) = E(j) + i_b^2 \geq E(j) \geq R_{\text{eff}}(\Gamma/f, r, s),$$

where the last inequality follows from Proposition 16.6(3). $\square$

Proof of Theorem 15.2. Let e and f be distinct edges of Γ that are not loops. If e and f are parallel, then $\mathbb{P}(e \in T \mid f \in T) = 0$, and the theorem is trivial. Thus we may assume that this is not the case.

Let a be an orientation of e . By the two corollaries, we have

$$\mathbb{P}_\Gamma(e \in T) = R_{\text{eff}}(\Gamma, t(a), h(a)) \geq R_{\text{eff}}(\Gamma/f, t(a), h(a)) = \mathbb{P}_{\Gamma/f}(e \in T).$$

By the definition of contraction, this last quantity is equal to $\mathbb{P}_\Gamma(e \in T \mid f \in T)$. $\square$

17 The Chow ring

The goal for the next few weeks is to define and study the Chow ring of a matroid, which we will use to prove Theorems 14.4 and 15.6.

Remark 17.1. In Lectures 9–13, we started by discussing the space U_φ , and then later introduced the ring $\text{OS}(M)$ with the property that $\text{OS}(M_\varphi) \cong H^*(U_\varphi)$. Here we will do something similar, but we will do it in the opposite order. That is, we'll start by introducing the Chow ring of any matroid, and then we'll say that, if the matroid is realizable, then the Chow ring is isomorphic to the cohomology ring of a certain space. Unlike in the case of the Orlik–Solomon algebra, we will provide very little of the proof of this fact. However, we will give complete combinatorial proofs of all of the statements that we need for Theorems 14.4 and 15.6.

Definition 17.2. Let $M = (E, \mathcal{I})$ be a loopless matroid with $E \neq \emptyset$, and let

$$R(M) := \mathbb{R}[x_F \mid \emptyset \neq F \text{ a flat}].$$

Define two ideals

$$\mathcal{J}_1(M) := \left\langle \sum_{e \in F} x_F \mid e \in E \right\rangle \subset R(M) \quad \text{and} \quad \mathcal{J}_2(M) := \left\langle x_F x_G \mid F \text{ and } G \text{ incomparable} \right\rangle \subset R(M).$$

We define the **Chow ring**

$$\underline{\text{CH}}(M) := R(M) / \mathcal{J}_1(M) + \mathcal{J}_2(M).$$

It will be useful at times to have an alternative presentation for the Chow ring. Let

$$u_F := - \sum_{F \leq G} x_G,$$

so that

$$x_F = - \sum_{F \leq G} \mu(F, G) u_G.$$

Then we have

$$R(M) = \mathbb{R}[u_F \mid \emptyset \neq F \text{ a flat}] \quad \text{and} \quad \mathcal{J}_1(M) = \langle u_F \mid \text{rk } F = 1 \rangle.$$

Lemma 17.3. *We have*

$$\mathcal{J}_2(M) = \left\langle (u_F - u_{F \vee G})(u_G - u_{F \vee G}) \mid F \text{ and } G \text{ nonempty flats} \right\rangle.$$

Remark 17.4. If F and G are comparable, then the expression $(u_F - u_{F \vee G})(u_G - u_{F \vee G})$ is equal to zero, so it is sufficient to consider incomparable pairs.

Proof. We have

$$(u_F - u_{F \vee G})(u_G - u_{F \vee G}) = \left(- \sum_{F \leq H \not\leq G} x_H \right) \left(- \sum_{G \leq J \not\leq F} x_J \right) \in \mathcal{J}_2(M),$$

which gives us one inclusion. For the opposite inclusion, we define (solely for the purpose of this proof) a grading on $R(M)$ by putting $\deg x_F = |F|$. Then, for any incomparable F and G , $(u_F - u_{F \vee G})(u_G - u_{F \vee G})$ is equal to $x_F x_G$ plus terms of higher degree, and each of those terms has the form $x_H x_J$ for some incomparable H and J . Thus we can prove that

$$x_F x_G \in \left\langle (u_F - u_{F \vee G})(u_G - u_{F \vee G}) \mid F \text{ and } G \text{ nonempty flats} \right\rangle$$

by downward induction on $|F| + |G|$. □

Now we have a second presentation

$$\underline{\text{CH}}(M) = \mathbb{R}[u_F \mid \emptyset \neq F \text{ a flat}] / \langle u_F \mid \text{rk } F = 1 \rangle + \langle (u_F - u_{F \vee G})(u_G - u_{F \vee G}) \rangle.$$

Note that we may not simply ignore the generators u_F with $\text{rk } F = 1$, as we need them in the presentation of $\mathcal{J}_2(M)$. For example, for any rank 2 flat H , we may choose rank 1 flats F and G with $F \vee G = H$, and we have

$$u_H^2 = (u_F - u_H)(u_G - u_H) = 0 \in \underline{\text{CH}}(M).$$

Example 17.5. Let's analyze the Chow ring of the Boolean matroid Boo_3 . We have $\underline{\text{CH}}^0(\text{Boo}_3) = \mathbb{R} \cdot 1$ and $\underline{\text{CH}}^1(\text{Boo}_3) = \mathbb{R}\{u_{12}, u_{13}, u_{23}, u_{123}\}$. How about $\underline{\text{CH}}^2(\text{Boo}_3)$? We have the following exhaustive list of relations in degree 2:

- $0 = (u_i - u_{ij})(u_j - u_{ij}) = u_{ij}^2$
- $0 = (u_i - u_{123})(u_{jk} - u_{123}) = u_{123}^2 - u_{jk}u_{123}$
- $0 = (u_{ij} - u_{123})(u_{jk} - u_{123}) = u_{123}^2 - u_{ij}u_{123} - u_{jk}u_{123} + u_{ij}u_{jk} = -u_{123}^2 + u_{ij}u_{jk}$, where the last equality uses the second relation.

From this, we can conclude that $\underline{\text{CH}}^2(\text{Boo}_3) = \mathbb{R} \cdot u_{123}^2$. We will express this by saying that there is a unique isomorphism

$$\deg : \underline{\text{CH}}^2(\text{Boo}_3) \rightarrow \mathbb{R}$$

with the property that $\deg(u_{123}^2) = 1$. We will later prove a similar fact about any matroid M , with 2 replaced by $\text{rk } M - 1$.

Exercise 17.6. Suppose that $G < H$ and $\operatorname{rk} H = \operatorname{rk} G + 1$. Show that $u_H^2 = u_G u_H \in \underline{\operatorname{CH}}(M)$.
Hint: There is a flat F of rank 1 with $F \vee G = H$.

Exercise 17.7. Show that $\underline{\operatorname{CH}}^3(\operatorname{Boo}_3) = 0$.

Exercise 17.8. Consider the bilinear form on $\underline{\operatorname{CH}}^1(\operatorname{Boo}_3)$ given by the formula

$$\langle \xi, \eta \rangle = \deg(\xi \eta).$$

Show that this form is nondegenerate. What is its signature? That is, how many positive and negative eigenvalues does it have? *Hint: Find an orthogonal basis for the form (one of your basis elements could be u_{123}), and show that all basis elements have nonzero self-pairing. How many are positive and how many are negative?*

18 The wonderful variety

Let V be a complex vector space, E a nonempty finite set, and $\varphi : E \rightarrow V \setminus \{0\}$ a map with $V_E = V$. As usual, let M_φ be the associated (loopless) matroid. In this lecture, we will describe a smooth, projective, complex algebraic variety that has $\underline{\mathrm{CH}}(M_\varphi)$ as its cohomology ring. This is purely for motivation—we will not provide a proof, and we will also not rely on this section for our proofs of Theorems 14.4 and 15.6.

Recall from Lecture 8 that, for each subset $S \subset E$ we have a corresponding subspace

$$H_S := \bigcap_{e \in S} H_e \subset V^*.$$

Furthermore, $H_S = H_T$ if and only if $\bar{S} = \bar{T}$, thus we have a bijection $F \mapsto H_F$ between flats and subspaces obtained as intersections of hyperplanes, taking flats of rank k to subspaces of codimension k .

Consider the map

$$V^* \rightarrow \bigoplus_{\mathrm{rk} F \geq 1} V^*/H_F.$$

Projectivizing, we obtain a rational map

$$\mathbb{P}(V^*) \dashrightarrow \prod_{\mathrm{rk} F \geq 1} \mathbb{P}(V^*/H_F).$$

This map is not everywhere defined: if $\alpha \in H_e \subset V^*$, then the image of α in V^*/H_e is zero, so it does not define an element of $\mathbb{P}(V^*/H_e)$. However, if we restrict to the locus $\bar{U}_\varphi \subset \mathbb{P}(V^*)$, we obtained a well-defined inclusion

$$\psi : \bar{U}_\varphi \rightarrow \prod_{\mathrm{rk} F \geq 1} \mathbb{P}(V^*/H_F).$$

The **wonderful variety** X_φ is defined as the closure of the image of $\bar{U}_\varphi$ in $\prod_{\mathrm{rk} F \geq 1} \mathbb{P}(V^*/H_F)$. This variety comes with a map

$$\pi : X_\varphi \rightarrow \mathbb{P}(V^*)$$

given by projection onto to the factor indexed by the flat $F = E$, which restricts to the identity isomorphism $\pi|_{\bar{U}_\varphi} : \bar{U}_\varphi \rightarrow \bar{U}_\varphi$. That is, π is a birational map, and it is an isomorphism over $\bar{U}_\varphi$.

Next, let us define certain subvarieties of X_φ . For any flat F , let

$$U_F := H_F \setminus \bigcup_{F \subsetneq G} H_G \subset V^*,$$

so that

$$V^* = \bigsqcup_F U_F.$$

Let $\bar{U}_F = U_F/\mathbb{C}^\times$, so that

$$\mathbb{P}(V^*) = \bigsqcup_{F \subsetneq E} \bar{U}_F.$$

For any flat $\emptyset \subsetneq F \subsetneq E$, we define

$$D_F := \overline{\pi^{-1}(\bar{U}_F)} \subset X_\varphi.$$

Example 18.1. Suppose that $V = \mathbb{C}^3$, $E = \{1, 2, 3\}$ and $\varphi(e)$ is the e^{th} coordinate vector. In other words, φ is the standard realization of the boolean matroid Boo_3 . Then $\bar{U}_\varphi$ is the complement of the three coordinate lines in $\mathbb{P}^2$, and the map

$$\psi : \bar{U}_\varphi \rightarrow \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^0 \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

takes $[z_1 : z_2 : z_3]$ to

$$\left([z_1 : z_2 : z_3], [z_1 : z_2], [z_1 : z_3], [z_2 : z_3]\right).$$

The first thing to notice is that this map extends from the complement of the three coordinate lines to the complement of the three coordinate points. That is, it is well-defined provided that at least two of the numbers z_i are nonzero. This implies that D_1 , D_2 , and D_3 are all isomorphic to $\mathbb{P}^1$. More explicitly, D_1 is equal to the closure of the locus

$$\left([0 : z_2 : z_3], [0 : z_2], [0 : z_3], [z_2 : z_3]\right),$$

which is isomorphic to $\mathbb{P}(H_1) \cong \mathbb{P}^1$.

The subvariety D_{12} is slightly more subtle; we can think of it as limits of points of the form

$$\left([z_1 : z_2 : z_3], [z_1 : z_2], [z_1 : z_3], [z_2 : z_3]\right)$$

as z_1 and z_2 both approach 0. The first coordinate must go to $[0 : 0 : 1]$, and the last two coordinates must both go to $[0 : 1]$. However, the second coordinate can go to any element of $\mathbb{P}^1$, so $D_{12} \cong \mathbb{P}^1$.

The slick description of X_φ is that it is the blow-up of $\mathbb{P}^2 = \mathbb{P}(V^*)$ at the three coordinate points $\bar{U}_{ij}$.

More generally, there is a nice description of the space X_φ as an iterated blow-up. Beginning with the projective space $\mathbb{P}(V^*)$, we first blow-up the points $\bar{U}_F$ for all flats F of corank 1, then we blow up the strict transforms of the lines $\bar{U}_F$ for all flats F of corank 2, and so on. If you believe this description, then you can see that D_F is a smooth divisor (subvariety of codimension 1). These divisors have normal crossings, which is what makes X_φ “wonderful”. The following result is due to De Concini and Procesi [DCP95, Theorem 4.2(4)].

Proposition 18.2. *Let F and G be nonempty proper flats. We have $D_F \cap D_G = \emptyset$ if and only if the flats F and G are incomparable.*

Exercise 18.3. *Verify Proposition 18.2 in the case of Example 18.1.*

For any nonempty flat F , let $\eta_F \in H^2(\mathbb{P}(V^*/H_F); \mathbb{R})$ be the class represented by a hyperplane. In particular, if $\text{rk } F = 1$, then $\eta_F = 0$. Let $\pi_F : X_\varphi \rightarrow \mathbb{P}(V^*/H_F)$ be the coordinate projection. The following theorem is proved in [FY04, Corollary 2].

Theorem 18.4. *There is a degree-doubling isomorphism $\underline{\text{CH}}(M_\varphi) \rightarrow H^*(X_\varphi; \mathbb{R})$. This isomorphism takes x_F to $[D_F]$ for all nonempty proper flats F , and it takes u_F to $\pi_F^*(\eta_F)$ for all nonempty F .*

We won't prove Theorem 18.4, but we will say a little bit about why you might believe it. Here is what we would need to check to see that the homomorphism described in the theorem is even well defined:

- For all F , we have the relation $u_F = -\sum_{F \leq G} x_G$, and therefore $u_E - u_F = \sum_{F \leq G < E} x_G$, in $\underline{\text{CH}}(M_\varphi)$. Thus we would need to show that

$$\pi_F^*(\eta_E) - \pi^*(\eta_F) = \sum_{F \leq G < E} [D_G] \in H^*(X_\varphi; \mathbb{R}).$$

This can be verified using the description of X_φ as an iterated blow-up; see [BES24, Remark 3.2.6] for a sketch of the argument.

- For any rank 1 flat F , we have $u_F = 0 \in \underline{\text{CH}}(M_\varphi)$, so we have to show that $\pi_F^*(\eta_F) = 0$. This is trivial, since $\eta_F = 0$ if $\text{rk } F = 1$.
- For any pair of incomparable flats F and G , we have $x_F x_G = 0 \in \underline{\text{CH}}(M_\varphi)$, so we have to show that $[D_F] \cdot [D_G] = 0 \in H^*(X_\varphi; \mathbb{R})$. This is immediate from Proposition 18.2.

These three items together show only that the map is well defined. To prove that it is an isomorphism requires the full machinery of [FY04] and [DCP95].

19 The top degree part of the Chow ring

Let M be a loopless matroid of rank $d > 0$. The takeaway of Lecture 18 was that, in the case of a matroid M that is realizable over $\mathbb{C}$, the Chow ring is isomorphic to the cohomology ring of a smooth, nonempty, connected projective variety of dimension $d-1$, with $\underline{\text{CH}}^i(M)$ corresponding to $H^{2i}(X; \mathbb{R})$. In particular, this means that $\underline{\text{CH}}^{d-1}(M)$ should be 1-dimensional, and $\underline{\text{CH}}^i(M)$ should be zero for all $i \geq d$. The purpose of this lecture is to prove this, for arbitrary (not necessarily realizable) matroids.

Recall that $\underline{\text{CH}}(M)$ is a quotient of the polynomial ring $R(M) = \mathbb{R}[u_F \mid \emptyset \neq F \text{ a flat}]$. We define a **standard monomial** in $R(M)$ to be a monomial of the form $u_{F_1}^{a_1} \cdots u_{F_k}^{a_k}$ with

$$\emptyset = F_0 < F_1 < \cdots < F_k$$

and $a_i < \text{rk}(F_i) - \text{rk}(F_{i-1})$ for $i \in \{1, \dots, k\}$.

Lemma 19.1. *The ring $\underline{\text{CH}}(M)$ is spanned by the images of standard monomials.*

Remark 19.2. The standard monomials in fact form a basis for $\underline{\text{CH}}(M)$, but we will not need this fact. This essentially goes back to [FY04], but see [Lar] for a direct proof.

Sketch of proof. By repeated application of the relations coming from the ideal $\mathcal{J}_2(M)$, we can write any monomial in $R(M)$ as a linear combination of monomials of the form $u_{F_1}^{a_1} \cdots u_{F_k}^{a_k}$ with

$$\emptyset = F_0 < F_1 < \cdots < F_k$$

(see [Lar, Lemma 2.2] for details). Thus we only need to worry about the degrees.

First suppose that $a_1 \geq r := \text{rk}(F_1)$. Choose a chain

$$\emptyset = G_0 < G_1 < \cdots < G_r = F_1.$$

By repeated applications of Exercise 17.6, we have

$$u_{F_1}^{a_1} = u_{G_r}^{a_1} = u_{G_1} u_{G_2} \cdots u_{G_{r-1}} u_{G_r}^{a_1 - r + 1},$$

which is equal to zero because $u_{G_1} = 0$.

Now suppose that there is an index $i > 1$ such that $a_i \geq r := \text{rk}(F_i) - \text{rk}(F_{i-1})$. Choose a chain

$$F_{i-1} = G_0 < G_1 < \cdots < G_r = F_i.$$

Again by repeated applications of Exercise 17.6, we have

$$u_{F_{i-1}}^{a_{i-1}} u_{F_i}^{a_i} = u_{G_0}^{a_{i-1}} u_{G_r}^{a_i} = u_{G_0}^{a_{i-1}} u_{G_1} u_{G_2} \cdots u_{G_{r-1}} u_{G_r}^{a_i - r} = u_{G_r}^{a_{i-1} + a_i} = u_{F_i}^{a_{i-1} + a_i}.$$

We have therefore decreased the value of the number k of flats in the chain for our monomial.

Iterating this process, we either end up with a standard monomial, or with zero. $\square$

Note that there are no standard monomials of degree d or higher, and there is a unique standard monomial of degree $d - 1$, namely u_E^{d-1} . However, we have not yet proved that this monomial represents a nonzero class in $\underline{\text{CH}}(M)$. This is the content of the next theorem. The result is originally due to [FY04], but we will give a modified version of the proof in [Lar].

Theorem 19.3. *There exists an isomorphism $\deg : \underline{\text{CH}}^{d-1}(M) \rightarrow \mathbb{R}$ with $\deg(u_E^{d-1}) = 1$.*

Sketch of proof. Consider the linear map $\deg : R^{d-1}(M) \rightarrow \mathbb{R}$ defined by putting

$$\deg(u_{F_1} \cdots u_{F_{d-1}}) := \begin{cases} 1 & \text{if, for all } \emptyset \neq T \subset [d-1], \text{rk}(\vee_{e \in T} F_e) \geq |T| + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we do not require that the flats $F_1, \dots, F_{d-1}$ form a chain, nor that they are distinct. We clearly have $\deg(u_E^{d-1}) = 1$. Thus, by Lemma 19.1 and the fact that u_E^{d-1} is the unique standard monomial of degree $d - 1$, it will suffice to prove that $\deg$ descends to $\underline{\text{CH}}(M)$. That is, we need to show that $\deg$ vanishes on the ideals $\mathcal{J}_1(M)$ and $\mathcal{J}_2(M)$.

The fact that $\deg$ vanishes on $\mathcal{J}_1(M)$ is clear. Indeed, if F_e has rank 1, then one can see that $\deg(u_{F_1} \cdots u_{F_{d-1}}) = 0$ by taking $T = \{e\}$. Thus we only need to worry about $\mathcal{J}_2(M)$. That is, for any tuple of flats $(F_1, \dots, F_{d-1})$, we need to show that the degree of the following class vanishes:

$$u_{F_1} \cdots u_{F_{d-3}} (u_{F_{d-2}} - u_{F_{d-2} \vee F_{d-1}}) (u_{F_{d-1}} - u_{F_{d-2} \vee F_{d-1}}).$$

Let us write

$$\begin{aligned} m_1 &:= u_{F_1} \cdots u_{F_{d-3}} u_{F_{d-2}} u_{F_{d-1}} \\ m_2 &:= u_{F_1} \cdots u_{F_{d-3}} u_{F_{d-2}} u_{F_{d-2} \vee F_{d-1}} \\ m_3 &:= u_{F_1} \cdots u_{F_{d-3}} u_{F_{d-1}} u_{F_{d-2} \vee F_{d-1}} \\ m_4 &:= u_{F_1} \cdots u_{F_{d-3}} u_{F_{d-2} \vee F_{d-1}} u_{F_{d-2} \vee F_{d-1}}, \end{aligned}$$

so that we need to prove that $\deg(m_1) - \deg(m_2) - \deg(m_3) + \deg(m_4) = 0$.

If $\deg(m_1) = 1$, then $\deg(m_2) = \deg(m_3) = \deg(m_4) = 1$, and we win. Similarly, if $\deg(m_2) = 1$ or $\deg(m_3) = 1$, then $\deg(m_4) = 1$. Thus the only two potentially problematic cases are:

- $\deg(m_1) = 0$ and $\deg(m_2) = \deg(m_3) = \deg(m_4) = 1$, or
- $\deg(m_1) = \deg(m_2) = \deg(m_3) = 0$ and $\deg(m_4) = 1$.

We need to show that neither of these things can happen.

Let's consider the first of the two cases. Suppose that $\deg(m_1) = 0$. That means that there is some $\emptyset \neq T \subset [d-1]$ for which $\text{rk}(\vee_{e \in T} F_e) \leq |T|$. If $d-1 \notin T$, then the same T shows that $\deg(m_2) = 0$. Similarly, if $d-2 \notin T$, then the same T shows that $\deg(m_3) = 0$. So we may assume

that $\{d-2, d-1\} \subset T$. But then the same T shows that $\deg(m_4) = 0$. For the second case, which is a little bit more complicated, see [Lar]. $\square$

Exercise 19.4. For any $e \in E$, show that

$$u_E = \sum_{e \in G \subsetneq E} x_G.$$

Exercise 19.5. Suppose that $e \notin F$. Show that

$$x_F u_E = \sum_{F \cup \{e\} \subset G \subsetneq E} x_F x_G.$$

Let $F_1 < F_2 < \cdots < F_k$ be a chain of nonempty proper flats.

Exercise 19.6. If there is some i for which $\text{rk } F_i \neq i$, use descending induction on k to show that

$$x_{F_1} \cdots x_{F_k} u_E^{d-1-k} = 0.$$

(Note that the $k = d-1$ case is vacuous.)

Exercise 19.7. If $\text{rk } F_i = i$ for all i , use ascending induction on k to show that

$$x_{F_1} \cdots x_{F_k} u_E^{d-1-k} = u_E^{d-1}.$$

(Note that the $k = 0$ case is trivial.) From this, we can conclude that, for any maximal chain $F_1 < \cdots < F_{d-1}$ of nonempty proper flats, we have

$$\deg(x_{F_1} \cdots x_{F_{d-1}}) = \deg(u_E^{d-1}) = 1.$$

20 The Chow ring and the reduced Poincaré polynomial

We introduced the Chow ring with the promise that it would help us prove Theorems 14.4 and 15.6, but we have not yet made the connection. In this lecture, we will make the connection to Theorem 14.4.

Let

$$\alpha := u_E = \sum_{e \in G \subsetneq E} x_G$$

for any $e \in E$, where the last equality is Exercise 19.4. Let

$$\beta := \sum_{e \notin G \neq \emptyset} x_G$$

for any $e \in E$. This definition appears at first to depend on the choice of $e \in E$, but we can see that

$$\alpha + \beta = \sum_{\emptyset \neq G \neq E} x_G$$

does not depend on e , and therefore neither does β . Let

$$\bar{\pi}_M(t) = \sum_{k=0}^{d-1} a_k t^k$$

be the reduced characteristic polynomial of M .

Theorem 20.1. *We have $a_k = \deg(\alpha^{d-1-k} \beta^k)$ for all k .*

To prove Theorem 20.1, we will break symmetry by choosing a total ordering of the set E . That is, we will identify E with $\{1, \dots, n\}$. Let $F_1 < \dots < F_k$ be a chain of nonempty proper flats. We call this chain **initial** if $\text{rk } F_i = i$ for all i , and **descending** if $\min F_1 > \min F_2 > \dots > \min F_k > 1$. Let $D_k(M)$ denote the set of initial descending chains. Our strategy for proving Theorem 20.1 will be to show that

$$\deg(\alpha^{d-1-k} \beta^k) = |D_k(M)| = a_k.$$

Lemma 20.2. *For any k , we have*

$$\beta^k = \sum_{\text{descending chains}} x_{F_1} \cdots x_{F_k}.$$

Proof. The statement is trivial when $k = 0$. We will assume that it holds for k and prove it for $k + 1$. We have

$$\beta^{k+1} = \beta \cdot \beta^k = \sum_{\substack{\text{descending chains} \\ F_1 < \dots < F_k}} \beta \cdot x_{F_1} \cdots x_{F_k}.$$

Fix a particular descending chain, and let $e = \min F_1$. Then

$$\beta \cdot x_{F_1} \cdots x_{F_k} = \sum_{e \notin F} x_F x_{F_1} \cdots x_{F_k}.$$

If $e \notin F$, then we cannot have $F_1 \subset F$, so in order for the product to be nonzero, we must have $F < F_1$. Also $e \notin F$ implies that $\min F > \min F_1$. Thus we have

$$\beta^{k+1} = \sum_{\substack{\text{descending chains} \\ F < F_1 < \cdots < F_k}} x_F x_{F_1} \cdots x_{F_k}.$$

This completes the induction. □

Corollary 20.3. *For any k , we have $\deg(\alpha^{d-1-k}\beta^k) = |D_k(M)|$.*

Proof. By Lemma 20.2, we have

$$\deg(\alpha^{d-1-k}\beta^k) = \sum_{\text{descending chains}} \deg(x_{F_1} \cdots x_{F_k} \alpha^{d-1-k}).$$

By Exercises 19.6 and 19.7, non-initial descending chains contribute 0 to this sum and initial descending chains contribute 1. □

Theorem 20.1 is now reduced to the following proposition.

Proposition 20.4. *For any k , we have $|D_k(M)| = a_k$.*

Proof. If $k < d - 1$, then we can truncate M without changing either $D_k(M)$ or a_k (see Exercise 14.12). Thus we may assume that $k = d - 1$. Recall also that $a_{d-1} = \pm\mu(\emptyset, E)$.

By Theorem 6.6 with $F = \overline{\{1\}}$, we have

$$\mu(\emptyset, E) = - \sum_{\substack{1 \notin G \\ \text{rk } G = d-1}} \mu(\emptyset, G).$$

Let M^G be the matroid obtained from M by deleting $E \setminus G$; the lattice of flats of M^G is equal to the interval $[\emptyset, G]$ in the lattice of flats of M . In particular, the Möbius function is the same. Let $e = \min G$, and apply the same analysis to M^G . We get

$$\mu(\emptyset, G) = - \sum_{\substack{e \notin H \subset G \\ \text{rk } H = d-2}} \mu(\emptyset, H),$$

and therefore

$$\mu(\emptyset, E) = \sum_{\substack{H < G \\ 1 < \min G < \min H \\ \text{rk } G = d-1 \\ \text{rk } H = d-2}} \mu(\emptyset, H).$$

Iterating this argument, we conclude that

$$\mu(\emptyset, E) = \pm \sum_{\substack{F_1 < \dots < F_{d-1} \\ \text{initial descending}}} \mu(\emptyset, F_1).$$

Since $\text{rk } F_1 = 1$, we have $\mu(\emptyset, F_1) = -1$, and therefore

$$a_{d-1} = \pm \mu(\emptyset, E) = \pm |D_{d-1}(M)|.$$

Both are positive, so we in fact have $a_{d-1} = |D_{d-1}(M)|$. □

Exercise 20.5. *Explicitly verify the equalities $a_k = |D_k(M)| = \deg(\alpha^{2-k}\beta^k)$ when $M = M_{K_4}$. Note that this will require you to pick a total order of the ground set.*

21 The Chow ring and correlation

In the previous lecture, we made a connection between the Chow ring and the coefficients of the reduced Poincaré polynomial. In this lecture, we will make a connection between the Chow ring and the quantities appearing in the statement of Theorem 15.6.

Let $M = (E, \mathcal{I})$ be a loopless matroid of rank d . Let $\bar{M} := M \oplus \text{Boo}_1$, a matroid of rank $d + 1$ on the ground set $\bar{E} := E \sqcup \{0\}$. We will always use the notation $\bar{F} \subset \bar{E}$ to denote a flat of $\bar{M}$. For all $e \in E$, let

$$y_e := \sum_{0 \in \bar{F} \not\ni e} x_{\bar{F}} = u_{\overline{\{0,e\}}} - u_{\{0\}} = u_{\overline{\{0,e\}}} \in \underline{\text{CH}}^1(\bar{M}).$$

Note that we also have

$$y_e = u_{\overline{\{0,e\}}} - u_{\{e\}} = \sum_{0 \notin \bar{F} \ni e} x_{\bar{F}}.$$

That is, we are free to change the roles of 0 and e in the sum without changing the class in the Chow ring that it represents.

Lemma 21.1. *If a flat $\bar{F}$ contains exactly one of the elements 0 and e , then $y_e x_{\bar{F}} = 0$. In particular, $y_e^2 = 0$.*

Proof. If $\bar{F}$ contains e but not 0, then it is incomparable to $\bar{G}$ for all $\bar{G}$ such that $0 \in \bar{G} \not\ni e$. If it contains 0 but not e , then it is incomparable to $\bar{G}$ for all $\bar{G}$ such that $0 \notin \bar{G} \ni e$. $\square$

Lemma 21.2. *If $J \subset E$ is dependent, then $\prod_{e \in J} y_e = 0$.*

Proof. We may assume that J is a circuit. Choose $f \in J$, and let $I = J \setminus \{f\}$. Then

$$\prod_{e \in J} y_e = y_f \prod_{e \in I} y_e = \sum_{0 \in \bar{F} \not\ni f} x_{\bar{F}} \prod_{e \in I} y_e.$$

By Lemma 21.1, if $0 \in \bar{F}$ but $e \notin \bar{F}$, then $x_{\bar{F}} y_e = 0$. Thus the only surviving terms in this sum are the ones for which $e \in \bar{F}$ for all $e \in I$, and therefore

$$\prod_{e \in J} y_e = \sum_{I \cup \{0\} \subset \bar{F} \not\ni f} x_{\bar{F}} \prod_{e \in I} y_e.$$

Any flat that contains I also contains f , so this sum is empty. $\square$

Lemma 21.3. *If $B \subset E$ is a subset of cardinality d , then*

$$\deg \left(\prod_{e \in B} y_e \right) = \begin{cases} 1 & \text{if } B \text{ is a basis} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If B is not a basis, then it is dependent, and therefore the degree vanishes by Lemma 21.2.

Now assume that B is a basis, and write $B = \{1, \dots, d\}$. For all $k \leq d+1$, let

$$\bar{F}_k := \overline{\{0, 1, \dots, k-1\}},$$

which is a flat of rank k . Then $\bar{F}_k$ is the unique flat with the following three properties:

- $\{0, 1, \dots, k-1\} \subset \bar{F}_k$
- $k \notin \bar{F}_k$
- $\bar{F}_k$ is comparable to $\bar{F}_{k+1}$.

This means that we have

$$\begin{aligned} \prod_{e \in B} y_e &= y_1 \cdots y_{d-2} y_{d-1} y_d \\ &= y_1 \cdots y_{d-2} y_{d-1} \sum_{0 \in \bar{F} \not\supset d} x_{\bar{F}} \\ &= y_1 \cdots y_{d-2} y_{d-1} \sum_{\{0, 1, \dots, d-1\} \subset \bar{F} \not\supset d} x_{\bar{F}} \\ &= y_1 \cdots y_{d-2} y_{d-1} x_{\bar{F}_d} \\ &= \dots \\ &= y_1 \cdots y_{d-2} x_{\bar{F}_{d-1}} x_{\bar{F}_d} \\ &= \dots \\ &= x_{\bar{F}_1} \cdots x_{\bar{F}_{d-2}} x_{\bar{F}_{d-1}} x_{\bar{F}_d}. \end{aligned}$$

This has degree 1 by Exercise 19.7. □

Let $\ell_{ef} := \sum_{g \notin \{e, f\}} y_g \in \underline{\text{CH}}^1(\bar{M})$.

Proposition 21.4. *We have the following equalities:*

$$\begin{aligned} \deg(\ell_{ef}^d) &= d! b^{ef} \\ \deg(y_e \ell_{ef}^{d-1}) &= (d-1)! b_e^f \\ \deg(y_f \ell_{ef}^{d-1}) &= (d-1)! b_f^e \\ \deg(y_e y_f \ell_{ef}^{d-2}) &= (d-2)! b_{ef}. \end{aligned}$$

Proof. We have

$$\ell_{ef}^d = \sum_{\substack{B \text{ a basis} \\ e, f \notin B}} d! \prod_{g \in B} y_g,$$

which has degree $d! b^{ef}$ by Lemma 21.3. The other three statements are similar. □

Exercise 21.5. Let S and T be independent subsets of E with $\bar{S} = \bar{T}$. Show that

$$\prod_{e \in S} y_e = \prod_{e \in T} y_e.$$

Hint: Reduce to the case where $S = R \cup \{s\}$ and $T = R \cup \{t\}$, where $R = S \cap T$.

Exercise 21.6. Let F be a flat of M . By Exercise 21.5, it makes sense to define

$$y_F := \prod_{e \in S} y_e$$

for any maximal independent subset $S \subset F$. Given two flats F and G , compute $y_F \cdot y_G$. How does the subalgebra of $\underline{\text{CH}}(\bar{M})$ generated by the classes $\{y_e \mid e \in E\}$ compare to the Möbius ring $B(\mathcal{L}) \otimes \mathbb{R}$ from Exercise 5.11? Warning: They are not isomorphic.

22 The Kähler package

Let $A = \bigoplus_{i=0}^m A^i$ be a finite dimensional $\mathbb{R}$ -algebra, equipped with an isomorphism $\deg : A^m \rightarrow \mathbb{R}$. Fix an element $\ell \in A^1$. We say that ℓ is **ample** and A satisfies the **Kähler package** with respect to ℓ if the following properties hold:

- **Poincaré duality:** For all $0 \neq \eta \in A^k$, there exists $\xi \in A^{m-k}$ with $\deg(\eta\xi) \neq 0$.
- **Hard Lefschetz:** For all $k \leq m/2$, the multiplication map $\ell^{m-2k} : A^k \rightarrow A^{m-k}$ is an isomorphism.
- **Hodge–Riemann bilinear relations:** For all $k \leq m/2$, let

$$P^k := \ker \left(\ell^{m-2k+1} : A^k \rightarrow A^{m-k+1} \right) \subset A^k.$$

(The letter P stands for **primitive**.) Consider the bilinear form on A^k given by the formula

$$\langle \eta, \xi \rangle := (-1)^k \deg(\ell^{m-2k}\eta\xi).$$

This form is positive definite on the subspace $P^k \subset A^k$.

Remark 22.1. Poincaré duality implies that A^k is dual to A^{m-k} , and therefore they have the same dimension. Hard Lefschetz implies that A^k is isomorphic to A^{m-k} , and therefore they have the same dimension. However, there is no formal relationship between these two statements. That is, one can easily define an algebra that satisfies one of these properties but not the other. Once we have *both* of these statements, we know that the pairing in the third statement is nondegenerate.

Let’s take some time to unpack the third statement. Assume that the first two statements hold.

- When $k = 0$, we have $P^0 = A^0 = \mathbb{R} \cdot 1$, and the assertion is that $\langle 1, 1 \rangle = \deg(\ell^m)$ should be positive.
- When $k = 1$, we have

$$P^1 = \{\eta \in A^1 \mid \ell^{m-1}\eta = 0\} = \{\eta \in A^1 \mid \langle \ell, \eta \rangle = 0\} = (\ell P^0)^\perp.$$

Hodge–Riemann in degree 0 tells us that our pairing is negative definite on ℓP^0 , and Hodge–Riemann in degree 1 tells us that it is positive definite on P^1 .

- In general, we have an orthogonal decomposition

$$A^k = (\ell^k P^0) \oplus (\ell^{k-1} P^1) \oplus \cdots \oplus (\ell P^{k-1}) \oplus P^k.$$

The space P^k is “primitive” in the sense that it is the orthogonal complement of $\ell \cdot A^{k-1}$. Hodge–Riemann in degree $k - i$ says that our pairing is $(-1)^i$ -definite on $\ell^{k-i} P^i$.

Example 22.2. Let

$$A = \mathbb{R}[x, y, z] / \langle x^2, y^2, z^3 \rangle$$

and consider the isomorphism $\deg : A^4 \rightarrow \mathbb{R}$ with $\deg(xyz^2) = 1$. Let $\ell = x + y + z$.

Poincaré duality is easy. Hard Lefschetz in degree 0 says that $\ell^4 \neq 0$, which is clear. In degree 2, it is trivial. The one nontrivial case is degree 1. Here, it says that if

$$(x + y + z)^2(ax + by + cz) = 0,$$

then $a = b = c = 0$. This can be verified explicitly.

Now let's think about Hodge–Riemann. In degree 0, it says that $\deg(\ell^4) > 0$, which is clear. In degree 1, we have

$$P^1 = \{ax + by + cz \mid (x + y + z)^3(ax + by + cz) = 0\} = \mathbb{R}\{z - 2x, z - 2y\}.$$

We can then compute

$$\langle z - 2x, z - 2y \rangle = -\deg((x + y + z)^2(z - 2x)(z - 2y)) = 2,$$

$$\langle z - 2x, z - 2x \rangle = -\deg((x + y + z)^2(z - 2x)^2) = 6,$$

and similarly

$$\langle z - 2y, z - 2y \rangle = 6.$$

That means that our intersection form on P^1 is given by the matrix $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$. The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 8 and the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue 4, so the matrix is positive definite.

Finally, let's check Hodge–Riemann in degree 2. We have

$$P^2 = \{axy + bxz + cyz + dz^2 \mid (x + y + z)(axy + bxz + cyz + dz^2) = 0\} = \mathbb{R}\{2xy - xz - yz + z^2\},$$

and

$$\langle 2xy - xz - yz + z^2, 2xy - xz - yz + z^2 \rangle = \deg((2xy - xz - yz + z^2)^2) = 6 > 0.$$

If X is a smooth, nonempty, connected, projective variety of dimension m over the complex numbers and L is an ample line bundle on X , then the algebra $\bigoplus_{i=0}^m H^{2i}(X; \mathbb{R})$ satisfies the Kähler package with respect to the ample class $\ell := c_1(L) \in H^2(X; \mathbb{R})$. Example 22.2 illustrates this for $X = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^2$.

Let M be a loopless matroid of degree d on the ground set $E \neq \emptyset$. Let $c : P(E) \rightarrow \mathbb{R}$ be a function with the properties that $c_\emptyset = c_E = 0$. We say that c is **strictly submodular** if

$$c_S + c_T > c_{S \cap T} + c_{S \cup T}$$

for all incomparable subsets $S, T \subset E$. If a weak inequality holds, we say that c is **submodular**. For a submodular function $c : P(E) \rightarrow \mathbb{R}$, let

$$\ell_c := \sum_{F \in \mathcal{L}(M)} c_F x_F \in \underline{\text{CH}}^1(M).$$

The following theorem is due to Adiprasito, Huh, and Katz [AHK18].

Theorem 22.3. *If c is strictly submodular, then the Chow ring $\underline{\text{CH}}(M)$ satisfies the Kähler package with respect to the ample class ℓ_c .*

Remark 22.4. If M is realizable over $\mathbb{C}$, the proof of Theorem 22.3 is “easy” in that it follows from Theorem 18.4 along with well-known results in algebraic geometry. However, those results themselves are quite deep, and the proof of Theorem 18.4 is also not so simple (in particular, it is not given in these notes). The proof that we will outline of Theorem 22.3 for arbitrary matroids is difficult but elementary.

We will not prove the following statement, but we will use it in an essential way in Lecture 28. See [BES24, Remark 2.1.8] for an explanation in terms of toric geometry. (It is stated there for submodular functions and weak inequalities, but the argument can be modified.)

Lemma 22.5. *Let c be a strictly submodular function. There is another strictly submodular function c' such that $c'_F > 0$ for every nonempty proper flat F and $\ell_c = \ell_{c'} \in \underline{\text{CH}}^1(M)$.*

Exercise 22.6. *Consider the case of the matroid Boo_3 . First show that we can define a strictly submodular function by putting $c_i = 3$ and $c_{ij} = 2$ for all i, j . Now check the Kähler package for $\underline{\text{CH}}(\text{Boo}_3)$ with respect to the class ℓ_c . You should have already have done most of the necessary work for Exercise 17.8.*

Exercise 22.7. *Verify Lemma 22.5 in the very simple case of Boo_2 . That is, first find a strictly submodular function c that fails to satisfy the condition $c_F > 0$ for every nonempty proper F , and then find another strictly submodular function c' that satisfies the condition and has $\ell_c = \ell_{c'}$.*

Exercise 22.8. *Work out the analogue of Example 22.2 with*

$$A = \mathbb{R}[x, y, z, w] / \langle x^2, y^2, z^2, w^2 \rangle,$$

the cohomology ring of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

23 Proving the big theorems

The purpose of this lecture is to show that Theorems 14.4 and 15.6 follow from Theorem 22.3. Note that we will only need to use the Hodge–Riemann bilinear relations for $\underline{\text{CH}}(M)$ in degrees 0 and 1.

Lemma 23.1. *Let $A = \bigoplus_{i=0}^m$ be an algebra satisfying the Kähler package with respect to an ample class ℓ . For any $\eta \in A^1$ that is not a multiple of ℓ , we have*

$$\deg(\ell^{m-2}\eta^2) \deg(\ell^m) < \deg(\ell^{m-1}\eta)^2.$$

Proof. Consider the Hodge–Riemann bilinear form on $A^1 = \mathbb{R} \cdot \ell \oplus P^1$. Let $\xi \in P^1$ be the orthogonal projection of η onto P^1 , and consider the restriction of our form to the 2-dimensional subspace

$$V := \mathbb{R}\{\ell, \eta\} = \mathbb{R}\{\ell, \xi\}.$$

By the Hodge–Riemann bilinear relations in degrees 0 and 1, we know that

$$\langle \ell, \ell \rangle < 0 \quad \text{and} \quad \langle \xi, \xi \rangle > 0,$$

so the restriction of the form to V is indefinite. Using the basis $\{\ell, \eta\}$ for V , we may represent our form by the matrix

$$\begin{pmatrix} \langle \ell, \ell \rangle & \langle \ell, \eta \rangle \\ \langle \eta, \ell \rangle & \langle \eta, \eta \rangle \end{pmatrix} = - \begin{pmatrix} \deg(\ell^m) & \deg(\ell^{m-1}\eta) \\ \deg(\ell^{m-1}\eta) & \deg(\ell^{m-2}\eta^2) \end{pmatrix}.$$

Since the form is indefinite, this matrix has negative determinant, which proves the lemma. $\square$

Corollary 23.2. *Let $M = (E, \mathcal{I})$ be a loopless matroid of rank $d \geq 3$, and let $c : P(E) \rightarrow \mathbb{R}$ be a submodular function. Then for any $\eta \in \underline{\text{CH}}^1(M)$,*

$$\deg(\ell_c^{d-3}\eta^2) \deg(\ell_c^{d-1}) \leq \deg(\ell_c^{d-2}\eta)^2.$$

Proof. If η is a multiple of ℓ_c , then we have equality. If c is strictly submodular, then the inequality follows from Lemma 23.1. If c is only weakly submodular, then we may write it as the limit of a strictly submodular sequence c_n , and we have

$$\deg(\ell_c^{d-2}\eta)^2 - \deg(\ell_c^{d-3}\eta^2) \deg(\ell_c^{d-1}) = \lim_{n \rightarrow \infty} \left(\deg(\ell_{c_n}^{d-2}\eta)^2 - \deg(\ell_{c_n}^{d-3}\eta^2) \deg(\ell_{c_n}^{d-1}) \right).$$

Since the limit of a non-negative sequence is non-negative, the desired inequality holds. $\square$

We are now ready to prove our first big theorem.

Proof of Theorem 14.4, assuming Theorem 22.3. Let

$$\bar{\pi}_M(t) = a_0 + a_1 t + \cdots + a_{d-1} t^{d-1}$$

be the reduced Poincaré polynomial of M . By Exercise 14.12, we only need to prove the inequality $a_{d-2}^2 \geq a_{d-3}a_{d-1}$.

Fix an element $e \in E$ and let

$$c_S = \begin{cases} 0 & \text{if } e \in S \text{ or } S = \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\ell_c = \sum_{e \notin F \neq \emptyset} x_F = \beta.$$

The function c is (weakly) submodular, so Corollary 23.2 with $\eta = \alpha$ tells us that

$$\deg(\beta^{d-3}\alpha^2) \deg(\beta^{d-1}) \leq \deg(\beta^{d-2}\alpha)^2.$$

By Theorem 20.1, this translates to the statement that $a_{d-3}a_{d-1} \leq a_{d-2}^2$. $\square$

The proof of Theorem 15.6 is similar, though this time we need to restrict to a subspace of dimension 3 rather than 2.

Lemma 23.3. *Let $A = \bigoplus_{i=0}^m A^i$ be an algebra satisfying the Kähler package with respect to an ample class ℓ . For any $\eta, \xi \in A^1$, we have*

$$\det \begin{pmatrix} \deg(\ell^m) & \deg(\ell^{m-1}\eta) & \deg(\ell^{m-1}\xi) \\ \deg(\ell^{m-1}\eta) & \deg(\ell^{m-2}\eta^2) & \deg(\ell^{m-2}\eta\xi) \\ \deg(\ell^{m-1}\xi) & \deg(\ell^{m-2}\eta\xi) & \deg(\ell^{m-2}\xi^2) \end{pmatrix} \geq 0.$$

Proof. If ℓ , η , and ξ are linearly dependent, then the matrix will be singular and therefore have determinant zero, so we may assume that this is not the case. Consider the restriction of the Hodge–Riemann bilinear form to the subspace

$$V := \mathbb{R}\{\ell, \eta, \xi\} \subset A^1.$$

We know that the form has only one negative eigenvalue on all of A^1 , therefore it has at most one negative eigenvalue on V . On the other hand, we know that $\langle \ell, \ell \rangle < 0$, so it has at least one negative eigenvalue on V . Thus one eigenvalue is negative and the other two are non-negative (possibly zero). Thus the determinant of any matrix representing the form is non-positive. The matrix in the statement of the lemma represents -1 times the Hodge–Riemann bilinear form with respect to the basis $\{\ell, \eta, \xi\}$, so its determinant is non-negative. $\square$

Proof of Theorem 15.6, assuming Theorem 22.3. For each element $g \in E$, consider the submodular function $c(g) : P(\bar{E}) \rightarrow \mathbb{R}$ given by the formula

$$c(g)_{\bar{S}} = \begin{cases} 1 & \text{if } 0 \in \bar{S} \not\ni g \\ 0 & \text{otherwise.} \end{cases}$$

Then $\ell_{c(g)} = y_g \in \underline{\text{CH}}^1(\bar{M})$. Let $c = \sum_{g \notin \{e, f\}} c(g)$, so that

$$\ell_c = \sum_{g \notin \{e, f\}} y_g = \ell_{ef},$$

the element that we considered in Lecture 21. Since each $c(g)$ is submodular, so is c . Using the same limiting argument that we employed in the proof of Theorem 14.4, we can apply Lemma 23.3 to the triple $\{\ell_{ef}, y_e, y_f\}$ even though c is not strictly submodular, thus we have

$$\det \begin{pmatrix} \deg(\ell_{ef}^d) & \deg(\ell_{ef}^{d-1} y_e) & \deg(\ell_{ef}^{d-1} y_f) \\ \deg(\ell_{ef}^{d-1} y_e) & \deg(\ell_{ef}^{d-2} y_e^2) & \deg(\ell_{ef}^{d-2} y_e y_f) \\ \deg(\ell_{ef}^{d-1} y_f) & \deg(\ell_{ef}^{d-2} y_e y_f) & \deg(\ell_{ef}^{d-2} y_f^2) \end{pmatrix} \geq 0.$$

By Proposition 21.4, we may rewrite this as

$$\det \begin{pmatrix} d!b^{ef} & (d-1)!b_e^e & (d-1)!b_e^f \\ (d-1)!b_e^e & 0 & (d-2)!b_{ef} \\ (d-1)!b_e^f & (d-2)!b_{ef} & 0 \end{pmatrix} \geq 0.$$

Dividing the matrix by $(d-2)!$ and computing the determinant, we find that

$$0 \leq -d(d-1)b_{ef}^2 b^{ef} + 2(d-1)^2 b_{ef} b_e^f b_e^e.$$

Dividing by $(d-1)b_{ef}$, this implies that

$$db_{ef} b^{ef} \leq 2(d-1)b_e^f b_e^e,$$

or equivalently that

$$\alpha(M, e, f) = \frac{b_{ef} b^{ef}}{b_e^f b_e^e} \leq 2 \cdot \frac{d-1}{d}.$$

This completes the proof. □

Exercise 23.4. Verify that the function $c(g)$ defined above is submodular.

Exercise 23.5. Prove that every submodular function may be written as a limit of strictly submodular functions (which we implicitly assumed in the proof of Corollary 23.2).

24 Pullback and pushforward maps

Let M be a matroid and F a flat. We define two new matroids, called the **localization** M^F and the **contraction** M_F . The localization already appeared in Section 20: it is the matroid on the ground set F defined by deleting all of the elements of $E \setminus F$. The key properties of this matroid are that $\text{rk}(M^F) = \text{rk}(F)$ and Exercise 2.9 implies that

$$\mathcal{L}(M^F) = \{G \mid G \leq F \in \mathcal{L}(M)\}.$$

The contraction M_F is the matroid on the ground set $E \setminus F$ defined by contracting all of the elements of F . The key properties of this matroid are that $\text{rk}(M_F) = \text{crk}(F)$ and Exercise 2.9 implies that

$$\mathcal{L}(M_F) = \{G \setminus F \mid F \leq G \in \mathcal{L}(M)\}.$$

In other words, if M is loopless, then the lattice of flats of M^F is equal to $[\emptyset, F]$ and the lattice of flats of M_F is isomorphic to $[F, E]$.

Example 24.1. Given $\sigma : E \rightarrow V$,⁴ define $\sigma^F : F \rightarrow V_F$ by restricting σ and $\sigma_F : E \setminus F \rightarrow V/V_F$ by composing $\sigma_{E \setminus F}$ with the projection from V to V/V_F . Then $(M_\sigma)^F = M_{\sigma^F}$ and $(M_\sigma)_F = M_{\sigma_F}$.

Our main goal in this lecture is to define two maps relating $\underline{\text{CH}}(M)$ to $\underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F)$. These maps have geometric interpretations in the realizable case (Remark 24.4), but our main focus will be on purely algebraic definitions and properties.

Lemma 24.2. *Suppose that M is a loopless matroid and F is a nonempty proper flat. There is a surjective graded algebra homomorphism*

$$\varphi^F : \underline{\text{CH}}(M) \rightarrow \underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F)$$

that is defined on the x -generators as follows:

$$\varphi^F(x_G) = \begin{cases} 0 & \text{if } F \text{ and } G \text{ are incomparable} \\ 1 \otimes x_G & \text{if } G < F \\ x_{G \setminus F} \otimes 1 & \text{if } F < G \\ -(1 \otimes \alpha_{M^F} + \beta_{M_F} \otimes 1) & \text{if } F = G. \end{cases}$$

This homomorphism has the property that

$$\varphi^F(\alpha_M) = \alpha_{M_F} \otimes 1 \quad \text{and} \quad \varphi^F(\beta_M) = 1 \otimes \beta_{M^F}.$$

Proof. We may use the formulas in the statement of the lemma to define a surjective graded algebra

⁴We use σ rather than our usual φ because φ is the standard letter for the homomorphism in Lemma 24.2.

homomorphism

$$\varphi^F : R(M) \rightarrow \underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F).$$

To see that φ^F descends to $\underline{\text{CH}}(M)$, we need to show that it kills the ideals

$$\mathcal{J}_1(M) = \left\langle \sum_{e \in G} x_G \mid e \in E \right\rangle \quad \text{and} \quad \mathcal{J}_2(M) = \left\langle x_G x_H \mid G \text{ and } H \text{ incomparable} \right\rangle.$$

If $e \notin F$, then we have

$$\sum_{e \in G} \varphi^F(x_G) = \sum_{F < G \ni e} x_{G \setminus F} \otimes 1 = \left(\sum_{e \in H \in \mathcal{L}(M_F)} x_H \right) \otimes 1 = 0.$$

To analyze the case where $e \in F$, let's recall a few useful equalities:

- $\sum_{e \in G < E} x_G = -x_E = u_E = \alpha_M$ (Exercise 19.4)
- $\sum_{\emptyset < G < E} x_G = \alpha_M + \beta_M$ (definition of β from Lecture 20).

If $e \in F$, then we have

$$\begin{aligned} \sum_{e \in G} \varphi^F(x_G) &= \sum_{e \in G < F} 1 \otimes x_G - (1 \otimes \alpha_{M^F} + \beta_{M^F} \otimes 1) + \sum_{F < G} x_{G \setminus F} \otimes 1 \\ &= 1 \otimes \alpha_{M^F} - (1 \otimes \alpha_{M^F} + \beta_{M^F} \otimes 1) + (\alpha_{M^F} + \beta_{M^F}) \otimes 1 + x_{E \setminus F} \otimes 1 \\ &= 1 \otimes \alpha_{M^F} - (1 \otimes \alpha_{M^F} + \beta_{M^F} \otimes 1) + (\alpha_{M^F} + \beta_{M^F}) \otimes 1 - \alpha_{M^F} \otimes 1 \\ &= 0. \end{aligned}$$

The fact that φ^F kills $x_G x_H$ for G and H incomparable is a straightforward to check case-by-case. This completes the proof that $\varphi^F : \underline{\text{CH}}(M) \rightarrow \underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F)$ is a well-defined surjective graded algebra homomorphism.

We have

$$\varphi^F(\alpha_M) = \varphi^F(-x_E) = -x_{E \setminus F} \otimes 1 = \alpha_{M_F} \otimes 1.$$

For the last statement, choose $e \in F$, so that

$$\varphi^F(\beta_M) = \sum_{e \notin G} \varphi^F(x_G) = \sum_{e \notin G < F} \varphi(x_G) = 1 \otimes \beta_{M^F}.$$

This completes the proof. □

Lemma 24.3. *Suppose that M is a loopless matroid and F is a nonempty proper flat. There is a unique $\underline{\text{CH}}(M)$ -module homomorphism*

$$\psi^F : \underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F) \rightarrow x_F \underline{\text{CH}}(M)$$

with $\psi^F(1) = x_F$, where the module structure on $\underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F)$ is given by φ^F . This map satisfies the identity

$$\deg_{M_F} \otimes \deg_{M^F} = \deg_M \circ \psi^F : \underline{\text{CH}}^{\text{rk } F-1}(M_F) \otimes \underline{\text{CH}}^{\text{rk } F-1}(M^F) \rightarrow \mathbb{R}.$$

Proof. Existence will be saved for Exercise 24.6, and uniqueness follows from surjectivity of φ . The most interesting part of the statement is the degree formula. Let $F_1 < \cdots < F_{d-1}$ be a maximal chain of nonempty proper flats of M with $F_k = F$. Then

$$\begin{aligned} \psi^F \left((x_{F_{k+1} \setminus F} \cdots x_{F_{d-1} \setminus F}) \otimes (x_{F_1} \cdots x_{F_{k-1}}) \right) &= \psi^F \circ \varphi^F(x_{F_1} \cdots x_{F_{k-1}} x_{F_{k+1}} \cdots x_{F_{d-1}}) \\ &= \psi^F(1) x_{F_1} \cdots x_{F_{k-1}} x_{F_{k+1}} \cdots x_{F_{d-1}} \\ &= x_{F_1} \cdots x_{F_{d-1}}. \end{aligned}$$

By Exercise 19.7, we have

$$\deg_{M_F} \otimes \deg_{M^F} \left((x_{F_{k+1} \setminus F} \cdots x_{F_{d-1} \setminus F}) \otimes (x_{F_1} \cdots x_{F_{k-1}}) \right) = 1 \cdot 1 = 1$$

and

$$\deg_M \circ \psi^F \left((x_{F_{k+1} \setminus F} \cdots x_{F_{d-1} \setminus F}) \otimes (x_{F_1} \cdots x_{F_{k-1}}) \right) = \deg_M(x_{F_1} \cdots x_{F_{d-1}}) = 1,$$

thus the two degree maps coincide. □

Remark 24.4. Given a complex realization $\sigma : E \rightarrow V$, one can construct an isomorphism

$$X_{\sigma_F} \times X_{\sigma^F} \cong D_F \subset X_\sigma.$$

(See if you can convince yourself that this is plausible when F has rank 1 or corank 1.) The ring homomorphism φ^F may be interpreted as the pullback from the cohomology of X_σ to the cohomology of D_F , and the module homomorphism ψ^F may be interpreted as the pushforward in cohomology from D_F to X_σ . For this reason, φ^F is called a **pullback map** and ψ^F a **pushforward map**, regardless of whether the matroid is realizable.

Exercise 24.5. Show that

$$\varphi^F(u_G) = \begin{cases} 1 \otimes u_G & \text{if } G \leq F \\ u_{(F \vee G) \setminus F} \otimes 1 & \text{otherwise.} \end{cases}$$

Use this to show that the kernel of φ^F is equal to

$$\left\langle u_G \mid F < G \text{ and } \text{rk } F + 1 = \text{rk } G \right\rangle + \left\langle u_G - u_{F \vee G} \mid G \not\leq F \right\rangle.$$

If you need guidance, see [Lar, Lemma 2.5].

Exercise 24.6. Using Exercise 24.5, show that $x_F \cdot \ker(\varphi^F) = 0$. Use this to prove that the pushforward homomorphism ψ^F exists.

Exercise 24.7. Suppose that $c : P(E) \rightarrow \mathbb{R}$ is strictly submodular. Show that there exist strictly submodular functions $c' : P(E \setminus F) \rightarrow \mathbb{R}$ and $c'' : P(F) \rightarrow \mathbb{R}$ such that

$$\varphi^F(\ell_c) = \ell_{c'} \otimes 1 + 1 \otimes \ell_{c''}.$$

Hint: This is easy if $c_F = 0$. In general it is more subtle!

25 Deletion maps

Let $M = (E, \mathcal{I})$ be a loopless matroid of rank d , and let $e \in E$ be any element. By Exercise 2.9, the flats of $M \setminus e$ are precisely those subsets $F \subset E \setminus \{e\}$ such that either F or $F \cup \{e\}$ (or possibly both) is a flat of M . We will write $F \cup e$ rather than $F \cup \{e\}$ for convenience.

Lemma 25.1. *There is a graded ring homomorphism $\theta_e : \underline{\text{CH}}(M \setminus e) \rightarrow \underline{\text{CH}}(M)$ given by the formula*

$$\theta_e(x_F) := x_F + x_{F \cup e},$$

with the convention that $x_S = 0$ if S is not a flat. If e is not a coloop, we have

$$\deg_{M \setminus e} = \deg_M \circ \theta_e : \underline{\text{CH}}^{d-1}(M) \rightarrow \mathbb{R}.$$

Proof. We can define $\theta_e : R(M \setminus e) \rightarrow \underline{\text{CH}}(M)$ by the given formula, and we need to show that it kills $\mathcal{J}_1(M \setminus e)$ and $\mathcal{J}_2(M \setminus e)$. Fix an element $f \in E \setminus \{e\}$, and consider the corresponding generator of $\mathcal{J}_1(M \setminus e)$. We have

$$\sum_{f \in F \in \mathcal{L}(M \setminus e)} \theta_e(x_F) = \sum_{f \in F \in \mathcal{L}(M \setminus e)} (x_F + x_{F \cup e}) = \sum_{f \in G \in \mathcal{L}(M)} x_G = 0.$$

Now consider a pair F and G of incomparable flats of $M \setminus e$. We have

$$\theta_e(x_F x_G) = (x_F + x_{F \cup e})(x_G + x_{G \cup e}) = 0,$$

since all four pairs of subsets appearing in the expansion are incomparable. Thus θ_e indeed descends to $\underline{\text{CH}}(M \setminus e)$.

We have

$$\theta_e(\alpha_{M \setminus e}^{d-1}) = (-1)^{d-1} \theta_e(x_{E \setminus \{e\}}^{d-1}) = (-1)^{d-1} x_E^{d-1} = \alpha_M^{d-1}.$$

If e is not a coloop, then $\text{rk}(M \setminus e) = \text{rk}(M) = d$, and both $\alpha_{M \setminus e}^{d-1}$ and α_M^{d-1} have degree 1. $\square$

Remark 25.2. Suppose that M admits a realization $\sigma : E \rightarrow V$ over $\mathbb{C}$, and let $\sigma' := \sigma|_{E \setminus \{e\}}$ be the corresponding realization of $M \setminus e$. If e is not a coloop, then $\{H_F \mid F \in \mathcal{L}(M \setminus e)\}$ is a subset of $\{H_F \mid F \in \mathcal{L}(M)\}$, and the corresponding projection

$$\prod_{\substack{F \in \mathcal{L}(M) \\ \text{rk } F \geq 1}} \mathbb{P}(V^*/H_F) \rightarrow \prod_{\substack{F \in \mathcal{L}(M \setminus e) \\ \text{rk } F \geq 1}} \mathbb{P}(V^*/H_F)$$

takes X_σ to $X_{\sigma'}$. The homomorphism θ_e is isomorphic to the pullback on cohomology along this map. The statement about degrees in Lemma 25.1 reflects the fact that the map from X_σ to $X_{\sigma'}$ is birational. (In fact, it is given by a sequence of blowups.)

Let $\underline{\text{CH}}_{(e)} \subset \underline{\text{CH}}(M)$ denote the image of the homomorphism θ_e . At the moment, we do not yet know that θ_e is injective. We can, however, prove the following lemma.

Lemma 25.3. *Suppose that e is not a coloop. If Poincaré duality holds for $\underline{\text{CH}}(M \setminus e)$, then the homomorphism θ_e is injective, and therefore $\underline{\text{CH}}_{(e)} \cong \underline{\text{CH}}(M \setminus e)$.*

Proof. Suppose that $\eta \in \underline{\text{CH}}^k(M \setminus e)$ is in the kernel of θ_e . If $\eta \neq 0$, then Poincaré Duality implies that there is an element $\xi \in \underline{\text{CH}}^{d-1-k}(M \setminus e)$ such that $\deg_{M \setminus e}(\eta\xi) \neq 0$. Then Lemma 25.1 implies that $\deg_M \circ \theta_e(\eta\xi) \neq 0$, which contradicts the fact that $\theta_e(\eta) = 0$. $\square$

Let $\mathcal{S}_e \subset \mathcal{L}(M \setminus e)$ be the collection of nonempty proper flats F for which both F and $F \cup e$ are flats of M . The following theorem was first proved in [BHM⁺22], but we will follow the exposition in [Wan]. We will defer the proof until the next lecture.

Theorem 25.4. *Suppose that e is not a coloop. There is a direct sum decomposition*

$$\underline{\text{CH}}(M) = \underline{\text{CH}}_{(e)} \oplus \bigoplus_{F \in \mathcal{S}_e} x_{F \cup e} \underline{\text{CH}}_{(e)}$$

of $\underline{\text{CH}}(M)$ into indecomposable graded $\underline{\text{CH}}(M \setminus e)$ -modules, orthogonal with respect to the Poincaré pairing on $\underline{\text{CH}}(M)$.

Remark 25.5. The decomposition in Theorem 25.4 is called the **semi-small decomposition** of $\underline{\text{CH}}(M)$. The reason for this terminology is that the map $\pi : X_\sigma \rightarrow X_{\sigma'}$ in Remark 25.2 is a semi-small map of algebraic varieties, meaning that there is no irreducible subvariety $Y \subset X_\sigma$ such that $2 \dim Y - \dim \pi(Y) > \dim X_{\sigma'}$. The key property of a semi-small map is that the cohomology of the source satisfies the Hard Lefschetz property and the Hodge–Riemann bilinear relations with respect to the pullback of an ample class on the target (see [BHM⁺22, Section 1.5] and references therein for further discussion). We will eventually use the decomposition in Theorem 25.4 to see that the analogous properties hold for $\underline{\text{CH}}(M)$ and $\underline{\text{CH}}(M \setminus e)$.

We now formulate a technical lemma that relates the deletion maps to the pushforward and pullback maps that we defined in the previous lecture, and in so doing gives us some understanding of the summand $x_{F \cup e} \underline{\text{CH}}_{(e)}$ appearing in Theorem 25.4. This part is not an easy read.

Suppose that $F \in \mathcal{S}_e$. We have $M^F = (M \setminus e)^F$, since both are obtained from M by deleting every element of $E \setminus F$. Similarly, we have $M^{F \cup e} \setminus e = M^F$. Now let's compare the matroids $(M \setminus e)_F$ and $M_{F \cup e}$. Flats of the latter have the form $G \setminus (F \cup e)$ where G is a flat of M containing $F \cup e$, while flats of the former have the form $G \setminus (F \cup e)$ where G is a flat of M containing F but not necessarily e . Hence every flat of $M_{F \cup e}$ is a flat of $(M \setminus e)_F$, but not vice versa.

Consider the homomorphism $\omega_{e,F} : \underline{\text{CH}}((M \setminus e)_F) \rightarrow \underline{\text{CH}}(M_{F \cup e})$ defined by putting

$$\omega_{e,F}(x_G) = \begin{cases} x_G & \text{if } G \text{ is a flat of } M_{F \cup e} \\ 0 & \text{otherwise} \end{cases}$$

for every nonempty flat G of $(M \setminus e)_F$. It is easy to check that this is well defined. Next, consider

the following composition:

$$\underline{\text{CH}}(M \setminus e) \xrightarrow{\varphi_{M \setminus e}^F} \underline{\text{CH}}((M \setminus e)_F) \otimes \underline{\text{CH}}((M \setminus e)^F) = \underline{\text{CH}}((M \setminus e)_F) \otimes \underline{\text{CH}}(M^F) \xrightarrow{\omega_{e,F} \otimes \text{id}} \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F).$$

Call this composition $\gamma_{e,F}$. Finally, consider the following composition:

$$\underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F) \xrightarrow{\text{id} \otimes \theta_e} \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^{F \cup e}) \xrightarrow{\psi^{F \cup e}} x_{F \cup e} \underline{\text{CH}}_{(e)}.$$

Call this composition Ψ_e^F . We refer to [Wan, Lemma 4.4 and Proposition 4.5] for the proof of the following result, which is not conceptually difficult but is a lot to keep track of in a lecture.

Lemma 25.6. *The homomorphism Ψ_e^F is a surjective homomorphism of $\underline{\text{CH}}(M \setminus e)$ -modules, where the module structure on the source is given by $\gamma_{e,F}$. Furthermore, for any pair of elements $\mu, \nu \in \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$ whose product lies in cohomological degree $d - 3$, we have*

$$\deg_M (\Psi_e^F(\mu) \Psi_e^F(\nu)) = -(\deg_{M_{F \cup e}} \otimes \deg_{M^F})(\mu \nu).$$

Exercise 25.7. *Work out the set $\mathcal{S}_e$ and the direct sum decomposition in Theorem 25.4 in the case where $M = U_{3,4}$ is the uniform matroid of rank 3 on the set $\{1, 2, 3, 4\}$ and $e = 1$. Verify that the summands are orthogonal with respect to the Poincaré pairing.*

Exercise 25.8. *Assuming that Poincaré duality holds for Chow rings of matroids, use the degree formula in Lemma 25.6 to show that Ψ_e^F is injective, and therefore an isomorphism.*

26 Poincaré duality and the semi-small decomposition

In this lecture we simultaneously prove Poincaré duality for $\underline{\text{CH}}(M)$ along with the semi-small decomposition (Theorem 25.4), following the exposition in [Wan, Section 4.2]. The proof will be by induction on the cardinality of the ground set. The base cases for the induction will be all Boolean matroids. (The induction will proceed by deleting an element that is not a coloop, and a matroid in which every element is a coloop is Boolean.)

We will take the base case as given. There are two reasonable approaches to proving Poincaré duality (and in fact the full Kähler package) for the Chow ring of a Boolean matroid. The first is to note that Boolean matroids are realizable, and therefore it is true for geometric reasons. In fact, the wonderful variety for a Boolean matroid is a toric variety (associated with a permutahedron), which makes it easier to identify the Chow ring with the cohomology ring than it is for arbitrary realizable matroids. Alternatively, one can formulate and prove a version of the semi-small decomposition when e is a coloop, and thus reduce everything to the case of a Boolean matroid of rank 1 (for which the Chow ring is equal to $\mathbb{R}$). This is the approach taken in both [BHM⁺22] and [Wan]. The only reason that we eschew it here is to avoid having to consider the coloop case in addition to the non-coloop case in every step of every proof.

Without further ado, assume that $M = (E, \mathcal{I})$ is a loopless matroid of rank d , that $e \in E$ is not a coloop, and that Poincaré duality and the semi-small decomposition hold for all matroids on smaller ground sets.

Lemma 26.1. *For all $F \in \mathcal{S}_e$, the degree $d - 1$ part of the subspace $x_{F \cup e} \underline{\text{CH}}_{(e)} \subset \underline{\text{CH}}(M)$ is equal to zero.*

Remark 26.2. This had better be true, or else it would intersect the subspace $\underline{\text{CH}}_{(e)}$, which we know is nonzero in degree $d - 1$.

Proof. By Lemma 25.6, we have a surjective map $\Psi_e^F : \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F) \rightarrow x_{F \cup e} \underline{\text{CH}}_{(e)}$. This map raises degree by 1 (due to the presence of the pushforward $\psi^{F \cup e}$ in the definition), so it is sufficient to show that the source vanishes in degree $d - 2$.

If $\text{rk } F = k$, then $\text{rk}(M_{F \cup e}) = d - k - 1$ and $\text{rk}(M^F) = k$, so the top nonvanishing degree of $\underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$ is $(d - k - 2) + (k - 1) = d - 3$. $\square$

Lemma 26.3. *The restriction of the Poincaré pairing to $\underline{\text{CH}}_{(e)} \subset \underline{\text{CH}}(M)$ is nondegenerate.*

Remark 26.4. This had also better be true if the decomposition is orthogonal and Poincaré duality holds for all of $\underline{\text{CH}}(M)$.

Proof. Suppose that $0 \neq \eta \in \underline{\text{CH}}_{(e)}$. Write $\eta = \theta_e(\mu)$ for some $\mu \in \underline{\text{CH}}(M \setminus e)$. By Poincaré duality for $\underline{\text{CH}}(M \setminus e)$, there exists an element $\nu \in \underline{\text{CH}}(M \setminus e)$ such that $\deg_{M \setminus e}(\mu\nu) \neq 0$. Let $\xi = \theta_e(\nu)$. Then Lemma 25.1 tells us that $\deg_M(\eta\xi) = \deg_{M \setminus e}(\mu\nu) \neq 0$. $\square$

Lemma 26.5. *For all $F \in \mathcal{S}_e$, the restriction of the Poincaré pairing on $\underline{\text{CH}}(M)$ to the subspace $x_{F \cup e} \underline{\text{CH}}_{(e)}$ is nondegenerate.*

Proof. This follows from Lemma 25.6 in a manner similar to the way in which Lemma 26.5 followed from Lemma 25.1. Suppose that $0 \neq \eta \in x_{F \cup e} \underline{\text{CH}}_{(e)}$. We need to find $\xi \in x_{F \cup e} \underline{\text{CH}}_{(e)}$ such that $\deg_M(\eta\xi) \neq 0$. By Lemma 25.6, there exists $\mu \in \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$ with $\eta = \Psi_e^F(\mu)$. By our inductive hypothesis, $\underline{\text{CH}}(M_{F \cup e})$ and $\underline{\text{CH}}(M^F)$ both satisfy Poincaré duality, so there exists an element $\nu \in \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$ such that $\deg_{M_{F \cup e}} \otimes \deg_{M^F}(\mu\nu) \neq 0$. Let $\xi = \Psi_e^F(\nu) \in x_{F \cup e} \underline{\text{CH}}_{(e)}$. The last part of Lemma 25.6 says that $\deg_M(\eta\xi) = -(\deg_{M_{F \cup e}} \otimes \deg_{M^F})(\mu\nu) \neq 0$. $\square$

Lemma 26.6. *The subspaces appearing in Theorem 25.4 are orthogonal under the Poincaré pairing.*

Proof. Suppose that $F \in \mathcal{S}_e$. We have

$$\underline{\text{CH}}_{(e)} \cdot x_{F \cup e} \underline{\text{CH}}_{(e)} \subset x_{F \cup e} \underline{\text{CH}}_{(e)}$$

which is zero in degree $d - 1$ by Lemma 26.1, thus any element of $\underline{\text{CH}}_{(e)}$ pairs trivially with $x_{F \cup e} \underline{\text{CH}}_{(e)}$.

Next assume that F and G are distinct elements of $\mathcal{S}_e$. If they are incomparable, then $x_{F \cup e} \underline{\text{CH}}_{(e)}$ and $x_{G \cup e} \underline{\text{CH}}_{(e)}$ are clearly orthogonal, so we may assume that $F < G$. In that case, $F \cup e$ and G are incomparable, so

$$x_{F \cup e} x_{G \cup e} = x_{F \cup e} (x_G + x_{G \cup e}) = x_{F \cup e} \theta_e(x_G). \quad (6)$$

This implies that

$$x_{F \cup e} \underline{\text{CH}}_{(e)} \cdot x_{G \cup e} \underline{\text{CH}}_{(e)} \subset x_{F \cup e} \underline{\text{CH}}_{(e)}.$$

Once again, Lemma 26.1 implies that $x_{F \cup e} \underline{\text{CH}}_{(e)}$ and $x_{G \cup e} \underline{\text{CH}}_{(e)}$ are orthogonal. $\square$

Lemma 26.7. *We have*

$$\underline{\text{CH}}(M) = \underline{\text{CH}}_{(e)} + \sum_{F \in \mathcal{S}_e} x_{F \cup e} \underline{\text{CH}}_{(e)}.$$

That is, the subspaces span.

Proof. The statement is obvious in degree 0. Now let's prove it in degree 1. Let G be a nonempty proper flat of M .

- If $e \notin G$ and $G \cup e$ is not a flat of M , then $x_G = \theta_e(x_G) \in \underline{\text{CH}}_{(e)}$.
- If $e \notin G$ and $G \cup e$ is a flat of M , then $G \in \mathcal{S}_e$, and $x_G = \theta_e(x_G) - x_{G \cup e} \in \underline{\text{CH}}_{(e)} + x_{G \cup e} \underline{\text{CH}}_{(e)}$.
- If $e \in G$ and $F := G \setminus \{e\}$ is not a flat of M , then $x_G = \theta_e(x_F) \in \underline{\text{CH}}_{(e)}$.
- If $F := G \setminus \{e\}$ is a nonempty flat of M , then $F \in \mathcal{S}_e$, so $x_G = x_F \cdot 1 \in x_{F \cup e} \underline{\text{CH}}_{(e)}$.
- Finally, if $G = \{e\}$, then x_G can be written as a linear combination of elements x_H for H of rank strictly greater than 1, all of which are covered by the previous four cases.

From the degree 1 case, we can conclude that

$$\underline{\text{CH}}_{(e)} + \sum_{F \in \mathcal{S}_e} x_{F \cup e} \underline{\text{CH}}_{(e)} = \underline{\text{CH}}_{(e)} \cdot \underline{\text{CH}}^{\leq 1}(M).$$

Thus we need to prove that $\underline{\text{CH}}_{(e)} \cdot \underline{\text{CH}}^{\leq 1}(M) = \underline{\text{CH}}(M)$. We will start by proving that

$$\underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M) = \underline{\text{CH}}^2(M). \quad (7)$$

Once we can show this, then for every $k > 0$, it will follow that

$$\underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^k(M) = \underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M) \cdot \underline{\text{CH}}^{k-1}(M) = \underline{\text{CH}}^2(M) \cdot \underline{\text{CH}}^{k-1}(M) = \underline{\text{CH}}^{k+1}(M).$$

Thus we will generate all of $\underline{\text{CH}}(M)$ by induction on k .

Okay, now it remains only to prove Equation (7). We know from the degree 1 case that

$$\begin{aligned} \underline{\text{CH}}^2(M) &= \underline{\text{CH}}^1(M) \cdot \underline{\text{CH}}^1(M) \\ &= \left(\underline{\text{CH}}_{(e)}^1 + \mathbb{R}\{x_{F \cup e} \mid F \in \mathcal{S}_e\} \right) \cdot \underline{\text{CH}}^1(M) \\ &= \underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M) + \sum_{F \in \mathcal{S}_e} x_{F \cup e} \underline{\text{CH}}^1(M), \end{aligned}$$

thus we can reduce the lemma to showing that $x_{F \cup e} \underline{\text{CH}}^1(M) \subset \underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M)$ for all $F \in \mathcal{S}_e$. Applying the degree 1 case yet again, we can reduce to showing that $x_{F \cup e} x_{G \cup e} \in \underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M)$ for all $F, G \in \mathcal{S}_e$.

If F and G are incomparable, then $x_{F \cup e} x_{G \cup e} = 0$. If $F < G$, then Equation (6) tells us that $x_{F \cup e} x_{G \cup e} = \theta_e(x_G) x_{F \cup e} \in \underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M)$. Thus the only remaining case is where $F = G$. To tackle this case, we begin by observing that

$$x_{F \cup e} \alpha_M = x_{F \cup e} \sum_{e \in G < E} x_G = \sum_{e \in G < F \cup e} x_{F \cup e} x_G + x_{F \cup e}^2 + \sum_{F \cup e < G < E} x_{F \cup e} x_G. \quad (8)$$

On the other hand, if we fix some $f \notin F \cup e$, we can also write

$$x_{F \cup e} \alpha_M = x_{F \cup e} \sum_{f \in G < E} x_G = \sum_{F \cup \{e, f\} \leq G < E} x_{F \cup e} x_G. \quad (9)$$

(The fact that e is not a coloop implies that $F \cup e \neq E$, so such an f necessarily exists.) Comparing Equations (8) and (9), we have

$$-x_{F \cup e}^2 = \sum_{e \in G < F \cup e} x_{F \cup e} x_G + \sum_{\substack{F \cup e < G < E \\ f \notin G}} x_{F \cup e} x_G.$$

If $e \in G < F \cup e$, then

$$x_{F \cup e} x_G = (x_F + x_{F \cup e}) x_G = \theta_e(x_F) x_G \in \underline{\text{CH}}_{(e)}^1 \cdot \underline{\text{CH}}^1(M).$$

If $F \cup e < G < E$, then

$$x_{F \cup e} x_G = x_{F \cup e} \theta_e(x_{G \setminus e}) \in \underline{\mathrm{CH}}_{(e)}^1 \cdot \underline{\mathrm{CH}}^1(M).$$

This completes the proof. □

Corollary 26.8. *Poincaré duality and the semi-small decomposition both hold for $\underline{\mathrm{CH}}(M)$.*

Proof. We begin with the semi-small decomposition. We have proved that the subspaces span (Lemma 26.7), so we just need to prove that any two of them intersect trivially. This follows from nondegeneracy (Lemmas 26.3 and 26.5) and orthogonality (Lemma 26.6) of the Poincaré pairing. More concretely, if η lies in one subspace, then nondegeneracy implies that it pairs nontrivially with some ξ in the same subspace, thus η cannot lie in any of the other subspaces.

Now that we know that $\underline{\mathrm{CH}}(M)$ decomposes as an orthogonal direct sum of subspaces on which the Poincaré pairing is nondegenerate, we can conclude that the Poincaré pairing is nondegenerate on all of $\underline{\mathrm{CH}}(M)$. □

Exercise 26.9. *In Exercise 17.8, you showed by hand that the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{Boo}_3)$ is nondegenerate. Can you do the same for $\underline{\mathrm{CH}}(\mathrm{Boo}_4)$?*

27 Generalities about Hard Lefschetz and Hodge–Riemann

In the previous lecture, we established the semi-small decomposition of $\underline{\mathrm{CH}}(M)$ and proved that $\underline{\mathrm{CH}}(M)$ satisfies Poincaré duality. In the last two lectures, we turn our attention to the Hard Lefschetz theorem (HL) and the Hodge–Riemann bilinear relations (HR). This lecture will be focused on general lemmas that are not specific to Chow rings of matroids, and the proof for Chow rings of matroids will happen in the next lecture.

We begin by showing that HL and HR are well behaved under the operation of tensor product.

Lemma 27.1. *Suppose that $A = \bigoplus_{i=0}^m A^i$ satisfies HL and HR with respect to the ample class $a \in A^1$ and $B = \bigoplus_{j=0}^n B^j$ satisfies HL and HR with respect to the ample class $b \in B^1$. Then $A \otimes B$ satisfies HL and HR with respect to the ample class $a \otimes 1 + 1 \otimes b$.*

Proof. Let $P^i \subset A^i$ and $Q^j \subset B^j$ denote the spaces of primitive elements for $i \leq m/2$ and $j \leq n/2$. As discussed in Lecture 22, we have

$$A^i = (a^i P^0) \oplus (a^{i-1} P^1) \oplus \cdots \oplus (a P^{i-1}) \oplus P^i$$

and

$$B^j = (b^j Q^0) \oplus (b^{j-1} Q^1) \oplus \cdots \oplus (b Q^{j-1}) \oplus Q^j.$$

Let

$$R_{ij}^k := \bigoplus_{s+t+i+j=k} (a^s P^i) \otimes (b^t Q^j),$$

so that

$$(A \otimes B)^k = \bigoplus_{i,j} R_{ij}^k.$$

You will show in Exercise 27.4 that this is an orthogonal decomposition of $(A \otimes B)^k$. Let

$$R_{ij} := \bigoplus_{k=0}^{m+n} R_{ij}^k \subset A \otimes B.$$

One can immediately see that multiplication by $a \otimes 1 + 1 \otimes b$ takes R_{ij} to itself, so it is sufficient to prove HL and HR for each R_{ij} . We have

$$R_{ij} = P^i \otimes Q^j \otimes \mathbb{R}[a, b] / \langle a^{m-2i+1}, b^{n-2j+1} \rangle,$$

so this reduces to proving HR and HL for the ring $\mathbb{R}[a, b] / \langle a^{m-2i+1}, b^{n-2j+1} \rangle$ with respect to the ample class $a + b$. This is simply the cohomology ring of $\mathbb{CP}^{m-2i} \times \mathbb{CP}^{n-2j}$, so it is true for geometric reasons. For an outline of a purely algebraic proof, see [Wan, Proposition 4.6]. $\square$

The **signature** of a nondegenerate bilinear form is equal to the dimension of a maximal positive definite subspace minus the dimension of a maximal negative definite subspace. More concretely,

one can always choose a basis with respect to which the form is represented by a diagonal matrix whose entries consist of r positive numbers and s negative numbers, and the signature is $r - s$.

Lemma 27.2. *Suppose that an algebra $A = \bigoplus_{i=0}^m A^i$ satisfies HL with respect to an ample class ℓ . Then HR is equivalent to the statement that the signature of the Hodge–Riemann bilinear form on A^k is equal to*

$$\sum_{i=0}^k (-1)^{k-i} (\dim A^i - \dim A^{i-1}).$$

Proof. As noted in Lecture 22, HL implies that we have an orthogonal decomposition

$$A^k = (\ell^k P^0) \oplus (\ell^{k-1} P^1) \oplus \cdots \oplus (\ell P^{k-1}) \oplus P^k.$$

HR in degree i is the statement that the restriction of our form to $\ell^{k-i} P^i$ is $(-1)^{k-i}$ -definite. If we already know HR in degree $k-1$, HR in degree k is equivalent to the statement that the signature of the form on A^k is equal to

$$\sum_{i=0}^k (-1)^{k-i} \dim P^i.$$

Now recall that P^i is defined as the kernel of a surjective map $A^i \rightarrow A^{m-i+1}$. This implies that

$$\dim P^i = \dim A^i - \dim A^{m-i+1} = \dim A^i - \dim A^{i-1},$$

where the last equality follows from HL. □

The following lemma will be extremely important in the next lecture, and in a sense explains why the semi-small decomposition is at all useful to us!

Lemma 27.3. *Suppose that $\ell_0, \ell_1 \in A^1$, and let $\ell_t = t\ell_1 + (1-t)\ell_0$ be the straight line family connecting the two. Suppose that HL holds for ℓ_t for all $t \in [0, 1]$, and that HR holds for ℓ_0 . Then HR also holds for ℓ_1 .*

Proof. By Lemma 27.2, deducing HR from HL reduces to a signature calculation. The signature is a locally constant function on the space of nondegenerate bilinear forms, so the signature of the Hodge–Riemann pairing with respect to ℓ_1 is equal to that of the Hodge–Riemann pairing with respect to ℓ_0 . □

Exercise 27.4. *Show that the decomposition $(A \otimes B)^k = \bigoplus_{i,j} R_{ij}^k$ is orthogonal with respect to the Hodge–Riemann bilinear form on $(A \otimes B)^k$. Hint: By definition, P^i is killed by a^{m-2i+1} , so $a^s P^i$ is killed by $a^{m-2i+1-s}$. This and some careful bookkeeping will get you to the desired statement.*

28 Hard Lefschetz and Hodge–Riemann for matroids

We are now ready to show that $\underline{\text{CH}}(M)$ satisfies the Hard Lefschetz theorem and the Hodge–Riemann bilinear relations, thus completing the proof of Theorem 22.3. As in Lecture 26, we will proceed by induction on the ground set, taking all Boolean matroids as our base case. Once again we will not prove the base case; everything written at the beginning of Lecture 26 applies equally well here.

Assume that $M = (E, \mathcal{I})$ is a loopless matroid of rank d and that HL and HR hold for all matroids on smaller ground sets. Our first lemma shows that we can use our inductive hypothesis of HR for smaller matroids to prove HL for M . This is the most satisfying step in the proof of Theorem 22.3.

Lemma 28.1. *The Hard Lefschetz theorem holds for M . That is, for any strictly submodular function $c : P(E) \rightarrow \mathbb{R}$, $\underline{\text{CH}}(M)$ satisfies HL with respect to the ample class ℓ_c .*

Proof. Suppose that $k \leq (d-1)/2$, $\eta \in \underline{\text{CH}}^k(M)$ and $\ell_c^{d-1-2k}\eta = 0$. We need to prove that $\eta = 0$. Our strategy will be to prove that $\eta x_F = 0$ for every nonempty proper flat F . If this is the case, then η is killed by $\underline{\text{CH}}^1(M)$, and therefore by every element of $\underline{\text{CH}}(M)$ of positive degree. This implies that η pairs trivially with every element of $\underline{\text{CH}}^{d-1-k}(M)$ under the Poincaré pairing, and therefore that $\eta = 0$ by Poincaré duality.

Let F be a nonempty proper flat. We will write

$$\eta^F := \varphi^F(\eta) \in \underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F).$$

We know that

$$x_F \eta = \psi^F(1)\eta = \psi^F(\varphi^F(\eta)) = \psi^F(\eta^F),$$

so it will be sufficient to show that $\eta^F = 0$.

By Exercise 24.7, there exist strictly submodular functions $c' : P(E \setminus F) \rightarrow \mathbb{R}$ and $c'' : P(F) \rightarrow \mathbb{R}$ such that

$$\varphi^F(\ell_c) = \ell_{c'} \otimes 1 + 1 \otimes \ell_{c''}.$$

Let

$$\ell_c^F := \varphi^F(\ell_c) \in \underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F),$$

and note that $\underline{\text{CH}}(M_F) \times \underline{\text{CH}}(M^F)$ satisfies HR with respect to ℓ_c^F by our inductive hypothesis and Lemma 27.1. Observe that

$$(\ell_c^F)^{d-1-2k} \eta^F = \varphi^F(\ell_c^{d-1-2k} \eta) = 0.$$

This means that η^F is a primitive element of degree k in $\underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F)$, and therefore HR for $\underline{\text{CH}}(M_F) \otimes \underline{\text{CH}}(M^F)$ implies that

$$(-1)^k (\deg_{M_F} \otimes \deg_{M^F}) \left((\ell_c^F)^{d-2-2k} \eta^F \eta^F \right) = \langle \eta^F, \eta^F \rangle \geq 0,$$

with equality if and only if $\eta^F = 0$.

We next use the degree formula in Lemma 24.3 to reinterpret this degree. We have

$$\begin{aligned}
(\deg_{M_F} \otimes \deg_{M^F}) \left((\ell_c^F)^{d-2-2k} \eta^F \eta^F \right) &= \deg_M \circ \psi^F \left((\ell_c^F)^{d-2-2k} \eta^F \eta^F \right) \\
&= \deg_M \circ \psi^F \circ \varphi^F \left(\ell_c^{d-2-2k} \eta \eta \right) \\
&= \deg_M \left(\psi^F(1) \ell_c^{d-2-2k} \eta \eta \right) \\
&= \deg_M \left(x_F \ell_c^{d-2-2k} \eta \eta \right).
\end{aligned}$$

For the last part of the argument, we cite Lemma 22.5 in order to assume that $c_F > 0$ for every nonempty proper flat F . Since $\ell_c^{d-1-2k} \eta = 0$ and $\ell_c = \sum_F c_F x_F$, we have

$$0 = \deg_M(\ell_c^{d-1-2k} \eta \eta) = \sum_F c_F \deg_M(x_F \ell_c^{d-2-2k} \eta \eta) = (-1)^k \sum_F c_F \langle \eta^F, \eta^F \rangle.$$

The right-hand side can only be zero if $\eta^F = 0$ for all F . This completes the proof. $\square$

Suppose that $c : P(E) \rightarrow \mathbb{R}$ is strictly submodular. Choose an arbitrary strictly submodular function $c' : P(E \setminus \{e\}) \rightarrow \mathbb{R}$, and extend it to $P(E)$ by putting $c'_S := c'_{S \setminus e}$ for all S . Then $c' : P(E) \rightarrow \mathbb{R}$ is submodular, but it is *not* strictly submodular: if S and T are incomparable but $S \setminus e \subset T$, then $c'_S + c'_T = c'_{S \cap T} + c'_{S \cup T}$, which demonstrates that the submodularity inequality can be an equality. For any $t \in [0, 1]$, let $c(t) = tc + (1-t)c'$. Then $c(0) = c'$, $c(1) = c$, and $c(t)$ is strictly submodular for all $t \in (0, 1]$.

We are going to study the classes

$$\ell_c, \ell_{c'}, \ell_{c(t)} = t\ell_c + (1-t)\ell_{c'} \in \underline{\text{CH}}^1(M).$$

Lemma 28.1 tells us that $\underline{\text{CH}}(M)$ satisfies HL with respect to $\ell_{c(t)}$ for all $t \in (0, 1]$. Lemma 27.3 tells us that, if we can show that $\underline{\text{CH}}(M)$ satisfies HR with respect to $\ell_{c'}$, this will imply HR with respect to ℓ_c . (Note that HR for $\ell_{c'}$ implies HL for $\ell_{c'}$.) Thus we have reduced our task to proving HR with respect to $\ell_{c'}$. The key point is that $\ell_{c'} = \theta_e(\ell_{c'})$, and now we see the relevance of Remark 25.5. The crucial property that we need to prove is that $\underline{\text{CH}}(M)$ satisfies HR with respect to an ample class for $M \setminus e$, which is exactly the meaning of semi-smallness in the geometric setting.

We will use the semi-small decomposition of Theorem 25.4:

$$\underline{\text{CH}}(M) = \underline{\text{CH}}_{(e)} \oplus \bigoplus_{F \in \mathcal{S}_e} x_{F \cup e} \underline{\text{CH}}_{(e)}.$$

We know that this is a decomposition of $\underline{\text{CH}}(M \setminus e)$ -modules, and therefore that it is preserved by the action of $\ell_{c'}$. We also know that it is an orthogonal decomposition, so it will suffice to prove HR on each of the summands. The fact that HR holds for $\underline{\text{CH}}_{(e)}$ follows from our inductive hypothesis and the fact $\deg_{M \setminus e} = \deg_M \circ \theta_e$ (Lemma 25.1). So all that remains is to prove HR for the action

of $\ell_{c'}$ on $x_{F \cup e} \underline{\text{CH}}_{(e)}$.

Recall the homomorphism

$$\gamma_{e,F} := (\omega_{e,F} \otimes \text{id}) \circ \varphi_{M \setminus e}^F : \underline{\text{CH}}(M \setminus e) \rightarrow \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$$

defined in Lecture 25. We know from our inductive hypothesis and Lemma 27.1 that HR holds for $\underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$.

Lemma 28.2. *The element $\ell_F := \gamma_{e,F}(\ell_{c'})$ is an ample class for $\underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$.*

Proof. Exercise 24.7 and Lemma 27.1 together tell us that $\varphi_{M \setminus e}^F$ takes ample classes to ample classes, so we only need to show that $\omega_{e,F}$ takes ample classes to ample classes. But there is nothing to check here: if $c'' : P(E \setminus (F \cup e)) \rightarrow \mathbb{R}$ is any function, then $\omega_{e,F}$ takes $\ell_{c''}$ to $\ell_{c''}$. $\square$

Lemma 28.3. *The summand $x_{F \cup e} \underline{\text{CH}}_{(e)}$ satisfies HR with respect to $\ell_{c'}$. That is, if*

$$Q^k := \ker \left(\ell_{c'}^{d-2k} : x_F \underline{\text{CH}}_{(e)}^{k-1} \rightarrow x_F \underline{\text{CH}}_{(e)}^{d-k-1} \right),$$

then the restriction of the Hodge–Riemann bilinear form from $\underline{\text{CH}}^k(M)$ to Q^k is positive definite.

Proof. Suppose $0 \neq \eta \in Q^k$. By Lemma 25.6, there is a nonzero element $\mu \in \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$ of degree $k-1$ such that $\eta = \Psi_e^F(\mu)$. The first thing we'll show is that μ is primitive with respect to the ample class $\ell_F \in \underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$. That is, we will show that $\ell_F^{d-2k} \mu = 0$. Since $\eta \in Q^k$, we have

$$0 = \ell_{c'}^{d-2k} \eta = \ell_{c'}^{d-2k} \Psi_e^F(\mu) = \Psi_e^F \left(\ell_F^{d-2k} \mu \right).$$

Exercise 25.8 tells us that Ψ_e^F is injective, so this implies that $\ell_F^{d-2k} \mu = 0$. Now we can apply HR for $\underline{\text{CH}}(M_{F \cup e}) \otimes \underline{\text{CH}}(M^F)$ to deduce that

$$\begin{aligned} 0 < \langle \mu, \mu \rangle &= (-1)^{k-1} (\deg_{M_{F \cup e}} \otimes \deg_{M^F}) (\ell_F^{d-2k-1} \mu \mu) \\ &= (-1)^k \deg_M \left(\Psi_e^F(\ell_F^{d-2k-1} \mu) \Psi_e^F(\mu) \right) \\ &= (-1)^k \deg_M \left(\ell_{c'}^{d-2k-1} \Psi_e^F(\mu) \Psi_e^F(\mu) \right) \\ &= (-1)^k \deg_M \left(\ell_{c'}^{d-2k-1} \eta \eta \right) \\ &= \langle \eta, \eta \rangle, \end{aligned}$$

where the second and third equalities come from Lemma 25.6. This completes the proof. $\square$

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