

Intersection cohomology of Popov–Vinberg varieties

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Abstract. The Popov–Vinberg variety of a simply connected, split, semisimple algebraic group G is a singular affine variety that contains the basic affine space G/U as a Zariski open subset. It is defined as the spectrum of the ring of functions on G/U , and can also be identified with the universal symplectic implosion for the maximal compact subgroup of G . We provide a recursive procedure for computing the intersection cohomology of this variety, with an emphasis on the case where $G = \mathrm{SL}_n$.

1 Introduction

Background. Let G be a simply connected, split, semisimple algebraic group over a field $\mathbb{F}$, and let $U \subset G$ be a maximal unipotent subgroup. The quotient G/U is a quasi-affine variety, and if $\mathbb{F}$ has characteristic zero, its coordinate ring includes exactly one copy of every finite dimensional irreducible representation of G . Despite the fact that G/U is not affine, it is known as the **basic affine space**; see [BGG70, BGG75, BK99, Pol01, GR15, GK25] for a small sample of the literature on this space.² Even though U is not reductive, the coordinate ring $\mathcal{O}(G/U) \cong \mathcal{O}(G)^U$ is finitely generated, and we may therefore consider the GIT quotient $X_G := G//U = \mathrm{Spec} \mathcal{O}(G/U)$. This is an affine variety that contains G/U as a Zariski open subset. It was first studied by Popov and Vinberg [VP72], so we call it the **Popov–Vinberg variety** of G . It is singular unless G is a power of SL_2 , and the orbits of G in X_G are in bijection with parabolic subgroups that contain U .

When $\mathbb{F} = \mathbb{C}$, the Popov–Vinberg variety is also called the **universal symplectic implosion** for the maximal compact subgroup $K \subset G$. Symplectic implosion may be viewed as an abelianisation construction in symplectic geometry: given a symplectic manifold M with a Hamiltonian action of a compact group K , one produces a new space M_{impl} with an action of the maximal torus of K , with the property that the symplectic reductions of M by K and M_{impl} by the torus agree. The space M_{impl} is called the **implosion** of M . General implosions can be recovered from the implosion of T^*K in a simple manner. (As discussed in [DKM24], this can be viewed as a composition in the real version of the Moore–Tachikawa category.) For this reason, the implosion of T^*K is called “universal”, and it is isomorphic to X_G as a stratified Kähler variety [GJS02].

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²Sometimes the word “basic” is replaced by “base” or “fundamental”.

Results. Our main result (Theorem 3.4) is a recursive procedure for computing the intersection cohomology Poincaré polynomial of the Popov–Vinberg variety of a complex group. We focus particularly on the case where $G = \mathrm{SL}_n$, in which case our recursion only involves groups of the form SL_m for $m \leq n$ (Corollary 4.1). The only non-trivial calculation of this type that has appeared in the literature before is that of $\mathrm{IH}^*(X_{\mathrm{SL}_3})$ [HJ14]; we provide an appendix that lists the intersection cohomology Poincaré polynomials of X_{SL_n} for $n \leq 13$. We also prove that the $(2i)^{\mathrm{th}}$ intersection cohomology Betti number of X_{SL_n} is a polynomial in n of degree at most $i/2$ (Proposition 5.2), and we conjecture that this polynomial is a non-negative linear combination of binomial coefficients (Conjecture 5.3).

We then consider the generating series $\Psi(t, u)$, where the coefficient of u^n is equal to the Hilbert series of the SL_n -equivariant intersection cohomology of X_{SL_n} . Our recursion translates into a surprisingly simple functional equation³ (Proposition 4.5):

$$\Psi(t^{-1}, u)\Psi(t, -u) = 1.$$

One reason for studying this generating series is that the direct sum over all n of the equivariant intersection homology⁴ groups of X_{SL_n} naturally forms an algebra with $\Psi(t, u)$ as its Hilbert series; we define this algebra and initiate its study in Section 6.

Methods. Our proof of Theorem 3.4 proceeds by lifting the complex group to a group scheme over $\mathbb{Z}$, base changing to a field of positive characteristic, and employing the Grothendieck–Lefschetz trace formula to compute the l -adic étale intersection cohomology of the intersection cohomology (IC) sheaf. By a standard comparison result (e.g. [KW06, Proposition 10.4.1(i)]), the resulting Poincaré polynomial is the same as that of the topological intersection cohomology of the Popov–Vinberg variety over the complex numbers. These methods are very close to those used to compute Poincaré polynomials of stalks of the IC sheaves of Schubert varieties [KL80], toric varieties [DL91, Fie91], hypertoric varieties [PW07], and arrangement Schubert varieties [EPW16]. In each of those cases, the polynomials computed are examples of Kazhdan–Lusztig–Stanley (KLS) polynomials [Sta92, Pro18], and this case is no different. We are in effect computing KLS polynomials of the Boolean poset of subsets of simple roots of G , with respect to an exotic rank function and kernel. This is not a perspective that we pursue, but it is implicit when we invoke the main result of [Pro18] to prove Theorem 3.4.

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³See Remark 4.6 for a precise statement about the extent to which this equation determines $\Psi(t, u)$.

⁴Replacing equivariant cohomology with equivariant homology does not change the dimensions.

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2 The Popov–Vinberg variety

We fix a field $\mathbb{F}$. Let G be a simply connected, split, semisimple algebraic group over $\mathbb{F}$, let $T \subset B \subset G$ be a maximal torus and a Borel subgroup, and let $U := [B, B]$ be the corresponding maximal unipotent subgroup. Let

$$X_G := \operatorname{Spec} \mathcal{O}(G/U)$$

be the associated Popov–Vinberg variety. The action of G on X_G has finitely many orbits, indexed by parabolic subgroups $B \subset P \subset G$, and the stabilizer of the orbit indexed by P is equal to the commutator $[P, P]$. Let us make this more concrete, following [GJS02, Section 6].

Let $\Lambda^* := \operatorname{Hom}(T, \mathbb{G}_m)$ be the weight lattice, and let $\Phi = \Phi^+ \cup \Phi^- \subset \Lambda^*$ be the roots. Consider the **simple roots** $\{\alpha_1, \dots, \alpha_{\operatorname{rk} G}\} \subset \Phi$ and the **fundamental weights** $\{\varpi_1, \dots, \varpi_{\operatorname{rk} G}\} \subset \Lambda^*$. For any subset $S \subset \Delta := \{1, \dots, \operatorname{rk} G\}$, we denote by $\Phi_S \subset \Phi$ the set $\{\alpha \in \Phi \mid (\varpi_i, \alpha) = 0 \text{ for all } i \in S\}$. Consider the decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

of the Lie algebra of G into root spaces. We have the following subgroups of G :

- A parabolic subgroup P_S , with Lie algebra $\mathfrak{p}_S = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_S \cup \Phi^+} \mathfrak{g}_\alpha$.
- The unipotent radical $U_S \subset P_S$, with Lie algebra $\bigoplus_{\alpha \in \Phi^+ \setminus \Phi_S^+} \mathfrak{g}_\alpha$.
- The Levi subgroup $L_S \subset P_S$ with root system Φ_S , which has the property that P_S is the semidirect product of U_S and L_S .
- The commutator subgroup $G_S := [L_S, L_S]$.
- The commutator subgroup $H_S := [P_S, P_S]$, which is the semidirect product of U_S and G_S .

Note that, with our conventions, as the set $S \subset \Delta$ becomes larger, Φ_S and P_S both become smaller. In particular, when $S = \Delta$, we have $P_\Delta = B$, $H_\Delta = U_\Delta = U$, $L_\Delta = T$, and G_Δ is the trivial group. At the other extreme, when $S = \emptyset$, we have $G_\emptyset = H_\emptyset = L_\emptyset = P_\emptyset = G$, and $U_\emptyset$ is trivial. The orbit of G in X_G corresponding to the subset S is isomorphic to G/H_S . The open orbit corresponding to $S = \Delta$ is equal to G/U , and the closed orbit corresponding to $S = \emptyset$ is a single point.

Example 2.1. Suppose that $G = \operatorname{SL}_n$ and $B \subset G$ is the subgroup of upper triangular matrices. We have a canonical isomorphism $\Lambda^* \cong \mathbb{Z}^n/\mathbb{Z}$. Let e_i denote the image in Λ^* of the i^{th} standard basis vector for $\mathbb{Z}^n$, and let $\alpha_i = e_i - e_{i+1}$ be the corresponding simple root.

There is a bijection between compositions (ordered partitions) of n and subsets of Δ given by sending an r -tuple $\sigma = (\sigma_1, \dots, \sigma_r)$ of positive integers with $\sigma_1 + \dots + \sigma_r = n$ to the subset

$$S_\sigma := \{\sigma_1, \sigma_1 + \sigma_2, \dots, \sigma_1 + \dots + \sigma_{r-1}\} \subset \Delta.$$

Given a composition σ , we will write $P_\sigma := P_{S_\sigma}$, and similarly for L_σ , G_σ , and H_σ . We then have the following explicit descriptions:

- The parabolic subgroup P_σ consists of block upper triangular matrices of determinant 1, and has dimension $\frac{1}{2}(n^2 - 2 + \sum_{i=1}^r \sigma_i^2)$.
- The unipotent radical $U_\sigma \subset P_\sigma$ consists of block upper triangular matrices whose diagonal blocks are identity matrices, and has dimension $\frac{1}{2}(n^2 - \sum_{i=1}^r \sigma_i^2)$.
- The Levi subgroup $L_\sigma \subset P_\sigma$ consists of block diagonal matrices of determinant 1.
- The commutator subgroup $G_\sigma := [L_\sigma, L_\sigma] \cong \mathrm{SL}_{\sigma_1} \times \dots \times \mathrm{SL}_{\sigma_r}$ consists of block diagonal matrices whose individual blocks each have determinant 1.
- The commutator subgroup $H_\sigma := [P_\sigma, P_\sigma]$ consists of block upper triangular matrices whose diagonal blocks each have determinant 1.

The open orbit corresponds to the composition $\sigma = (1, \dots, 1)$, and has dimension $\frac{1}{2}n(n+1) - 1$, while the unique fixed point corresponds to the composition $\sigma = (n)$.

Let V_ϖ be the irreducible representation of G with highest weight ϖ , and let $v_\varpi \in V_\varpi$ be a nonzero highest weight vector. Let

$$E_G := V_{\varpi_1} \oplus \dots \oplus V_{\varpi_{\mathrm{rk} G}}.$$

For any subset $S \subset \Delta$, let

$$v_S := \sum_{i \in S} v_{\varpi_i} \in E_G,$$

and note that the stabilizer of v_S is equal to H_S . Let $O_S := G \cdot v_S \cong G/H_S$. We have $H_\Delta = U$, thus there is a unique G -invariant map from G/U to E_G taking the identity coset $e \cdot U \in G/U$ to the vector v_Δ . Since E_G is affine, this map factors through the Popov–Vinberg variety X_G .

Example 2.2. When $G = \mathrm{SL}_n(\mathbb{F})$, we have $V_{\varpi_i} = \bigwedge^i(\mathbb{F}^n)$ and $v_{\varpi_i} = e_1 \wedge \dots \wedge e_i$. The map from SL_n/U to E_{SL_n} sends the coset of the identity to the vector

$$v_\Delta = e_1 + e_1 \wedge e_2 + \dots + e_1 \wedge e_2 \wedge \dots \wedge e_{n-1}.$$

Remark 2.3. If we wanted to work with groups that are not simply connected, we would need to replace Δ with a choice of a collection of dominant weights that generates the weight lattice. To simplify the exposition, we restrict our attention to simply connected groups, for which the collection of fundamental weights is a canonical such choice.

The following theorem is proved in [GJS02, Theorem 6.11].

Theorem 2.4. *Suppose that $\mathbb{F} = \mathbb{C}$. The induced map from X_G to E_G is a closed embedding, thus we may identify X_G with a closed subvariety of E_G . For any subset $S \subset \Delta$, we have $v_S \in X_G \subset E$, and we have a G -equivariant stratification*

$$X_G = \bigsqcup_{S \subset \Delta} O_S. \quad (1)$$

For any subset $S \subset \Delta$, consider the Popov–Vinberg variety X_{G_S} for the group G_S . Since $G_S \cap U$ is a maximal unipotent subgroup of G_S , there is a unique G_S -equivariant map $\varphi_S : X_{G_S} \rightarrow X_G$ taking the identity coset $e \cdot (G_S \cap U)$ to v_Δ . Equivalently, if we identify X_{G_S} with the GIT quotient $H_S // U := \text{Spec}(\mathbb{F}[H_S]^U)$, the map $\varphi_S : H_S // U \rightarrow G // U$ is induced by the inclusion of H_S into G . Furthermore, we have a map $G \times X_{G_S} \rightarrow X_G$ taking (g, x) to $g \cdot \varphi_S(x)$, and this descends to a map

$$\psi_S : (G \times X_{G_S}) / H_S \rightarrow X_G.$$

The following lemma is proved in [GJS02, Lemma 6.13].

Lemma 2.5. *Suppose that $\mathbb{F} = \mathbb{C}$. The map ψ_S is an open embedding with image*

$$\bigsqcup_{S \subset S' \subset \Delta} O_{S'}.$$

The map φ_S is a closed embedding that exhibits X_{G_S} as a G_S -equivariant normal slice to the orbit $O_S \subset X_G$, taking the point $0 \in X_{G_S}$ to the point $v_S \in X_G$.

Example 2.6. Continuing with Example 2.1, write $X_n := X_{\text{SL}_n}$ and

$$X_\sigma := X_{(\text{SL}_n)_{S_\sigma}} \cong X_{\text{SL}_{\sigma_1} \times \cdots \times \text{SL}_{\sigma_n}} \cong X_{\sigma_1} \times \cdots \times X_{\sigma_r}.$$

The closed inclusion $\varphi_\sigma : X_\sigma \rightarrow X_n$ is induced by the block diagonal inclusion of the subgroup $G_\sigma = \text{SL}_{\sigma_1} \times \cdots \times \text{SL}_{\sigma_r}$ into SL_n . If we take

$$\begin{aligned} E_{\sigma_1} &= \bigoplus_{i=1}^{\sigma_1-1} \bigwedge^i \mathbb{F}\{e_1, \dots, e_{\sigma_1}\} \\ E_{\sigma_2} &= \bigoplus_{i=1}^{\sigma_2-1} \bigwedge^i \mathbb{F}\{e_{\sigma_1+1}, \dots, e_{\sigma_1+\sigma_2}\} \\ &\vdots \\ E_{\sigma_r} &= \bigoplus_{i=1}^{\sigma_r-1} \bigwedge^i \mathbb{F}\{e_{\sigma_1+\cdots+\sigma_{r-1}+1}, \dots, e_n\}, \end{aligned}$$

then φ_σ is the restriction of the map $E_{\text{SL}_{\sigma_1}} \times \cdots \times E_{\text{SL}_{\sigma_r}} \rightarrow E_{\text{SL}_n}$ that sends $(a_1, \dots, a_r)$ to

$$a_1 + v_{\varpi_{\sigma_1}} \wedge (1 + a_2) + v_{\varpi_{\sigma_1+\sigma_2}} \wedge (1 + a_3) + \cdots + v_{\varpi_{\sigma_1+\cdots+\sigma_{n-1}}} \wedge (1 + a_r).$$

In particular the element $0 \in X_\sigma$ is sent to the element

$$v_{\varpi_{\sigma_1}} + v_{\varpi_{\sigma_1+\sigma_2}} + \cdots + v_{\varpi_{\sigma_1+\cdots+\sigma_{n-1}}} = v_{S_\sigma} \in X_{\mathrm{SL}_n}.$$

The following technical lemma will be needed for the proof of Theorem 3.4.

Lemma 2.7. *For any G and any subset $S \subset \Delta$, there exists an action of $\mathbb{G}_m$ on X_G such that the normal slice map $\varphi_S : X_{G_S} \rightarrow X_G$ is $\mathbb{G}_m$ -equivariant with respect to the scalar action of $\mathbb{G}_m$ on X_{G_S} and the chosen action of $\mathbb{G}_m$ on X_G . In particular, this action fixes the point $v_S = \varphi_S(0) \in X_G$.*

Proof. For each cocharacter $\rho : \mathbb{G}_m \rightarrow T$, consider the homomorphism $\lambda_\rho : \mathbb{G}_m \rightarrow \mathrm{Aut}(E_G)$ given by letting $\mathbb{G}_m$ act on V_{ϖ_i} with weight $\langle \rho, \varpi_i \rangle$. Note that this is not the same as the action of $\mathbb{G}_m$ on E_G given by composing ρ with the action of G on E_G , however these two actions do agree on the highest weight vectors $v_{\varpi_i} \in E_G$, and therefore on the element $v_{S'} \in E_G$ for any $S' \subset \Delta$. In addition, the action of $\mathbb{G}_m$ via λ_ρ commutes with the action of G . In particular, for any $t \in \mathbb{G}_m$ and $g \in G$, we have

$$\lambda_\rho(t) \cdot (g \cdot v_\Delta) = g \cdot (\lambda_\rho(t) \cdot v_\Delta) = g \cdot (\rho(t) \cdot v_\Delta) = (g \rho(t)) \cdot v_\Delta \in G/U \subset E_G,$$

therefore our action of $\mathbb{G}_m$ on E_G restricts to an action on X_G .

We will take ρ to be the unique cocharacter with the property that $\langle \rho, \varpi_i \rangle = 0$ if $i \in S$ and 1 if $i \notin S$; such a cocharacter exists because G is simply connected, and the fundamental weights therefore form a basis for the weight lattice. The map $\varphi_S : X_{G_S} \rightarrow X_G$ is equivariant with respect to the action of $\mathbb{G}_m$ on X_G given by λ_ρ . $\square$

3 Recursive formula

We begin this section with a review of the statement from [Pro18] that will be our main technical tool, providing a link between point counts over finite fields and intersection cohomology. Let

$$X = \bigsqcup_{x \in Q} V_x$$

be a stratified algebraic variety over $\mathbb{F}_q$, where Q is a finite poset and $x \leq y$ if and only if the stratum V_x is contained in the closure of V_y .

Next, suppose that, for each $x \in P$, we have an affine subvariety $C_x \subset X$ that is a **conical normal slice** to the stratum V_x . The phrase “normal slice” means that C_x intersects V_x in a single point, and in a neighborhood of that point, X looks like the product of V_x with C_x . In particular, we require that, for all $x \leq y \in Q$, the restriction of the IC sheaf of $\overline{V}_y(\mathbb{F}_q)$ to $C_x(\mathbb{F}_q)$ is isomorphic to the IC sheaf of $X_{xy}(\mathbb{F}_q)$. The word “conical” means that we have a stratification preserving action of $\mathbb{G}_m$ on X (depending on the element $x \in Q$) that takes C_x to itself and contracts it to the point of intersection with V_x . For all $x \leq y \in Q$, let $V_{xy} := C_x \cap V_y$, and let $X_{xy} := \overline{V}_{xy}$. Then, for

all $x \leq z \in Q$, we have [Pro18, Lemma 3.1]

$$X_{xz} = \bigsqcup_{x \leq y \leq z} V_{xy}.$$

Finally, suppose that we have a collection of polynomials $\{\kappa_{xy}(t) \mid x \leq y \in Q\}$ such that, for all $x \leq y \in Q$ and all $s \geq 1$, we have

$$\kappa_{xy}(q^s) = |V_{xy}(\mathbb{F}_{q^s})|.$$

Choose a prime l that does not divide q , and for all $x \leq y \in Q$, let

$$f_{xy}(t) := \sum_{i \geq 0} t^i \dim \mathrm{IH}_{\mathrm{\acute{e}t}}^{2i}(X_{xy}(\bar{\mathbb{F}}_q); \mathbb{Q}_l)$$

be the even degree l -adic étale intersection cohomology Poincaré polynomial of $X_{xy}(\bar{\mathbb{F}}_q)$, and let $d_{xy} = \dim X_{xy}$.

Theorem 3.1. [Pro18, Theorem 3.3] *For all $x < z \in Q$, the l -adic étale intersection cohomology of X_{xz} vanishes in odd degree, $\deg f_{xz}(t) < d_{xz}/2$, and*

$$t^{d_{xz}} f_{xz}(t^{-1}) = \sum_{x \leq y \leq z} \kappa_{xy}(t) f_{yz}(t).$$

Remark 3.2. We give a brief outline of the idea behind the proof of Theorem 3.1. The Grothendieck–Lefschetz trace formula (an l -adic étale version of the Lefschetz fixed point theorem in topology) tells us that

$$\sum_{j \geq 0} (-1)^j \mathrm{tr} \left(\mathrm{Fr} \circ \mathrm{IH}_{\mathrm{\acute{e}t},c}^j(X_{xz}(\bar{\mathbb{F}}_q); \mathbb{Q}_l) \right) = \sum_{p \in X_{xz}(\mathbb{F}_q)} \sum_{j \geq 0} (-1)^j \mathrm{tr} \left(\mathrm{Fr}^j \circ \mathrm{IH}_{\mathrm{\acute{e}t}}^j(\mathrm{IC}_{X_{xz}(\bar{\mathbb{F}}_q),p}(\mathbb{Q}_l)) \right),$$

where $\mathrm{IH}_{\mathrm{\acute{e}t},c}$ denotes compactly supported intersection cohomology, $\mathrm{IC}_{X_{xy}(\bar{\mathbb{F}}_q),p}(\mathbb{Q}_l)$ denotes the stalk of the IC sheaf at the point p with coefficients in $\mathbb{Q}_l$, and Fr denotes the Frobenius automorphism. By Poincaré duality, $\mathrm{IH}_{\mathrm{\acute{e}t},c}^j(X_{xz}(\bar{\mathbb{F}}_q); \mathbb{Q}_l)$ is isomorphic to $\mathrm{IH}_{\mathrm{\acute{e}t}}^{2d_{xz}-j}(X_{xz}(\bar{\mathbb{F}}_q); \mathbb{Q}_l)$. For each point $p \in V_{xy} \subset X_{xz}$, we have

$$\mathrm{IH}_{\mathrm{\acute{e}t}}^*(\mathrm{IC}_{X_{xz}(\bar{\mathbb{F}}_q),p}(\mathbb{Q}_l)) \cong \mathrm{IH}_{\mathrm{\acute{e}t}}^*(X_{yz}(\bar{\mathbb{F}}_q); \mathbb{Q}_l),$$

so the right-hand side becomes

$$\sum_{x \leq y \leq z} \kappa_{xy}(q) \sum_{j \geq 0} (-1)^j \mathrm{tr} \left(\mathrm{Fr}^j \circ \mathrm{IH}_{\mathrm{\acute{e}t}}^j(X_{yz}(\bar{\mathbb{F}}_q); \mathbb{Q}_l) \right).$$

Suppose that we knew that all of these intersection cohomology groups vanished in odd degree and that the Frobenius automorphism always acts on IH^{2i} as multiplication by q^i . This would allow us to express our trace formula succinctly as

$$q^{d_{xz}} f_{xz}(q^{-1}) = \sum_{x \leq y \leq z} \kappa_{xy}(q) f_{yz}(q).$$

Moreover, if this holds with q replaced by q^s for any $s \geq 1$, then it holds with q replaced by a formal variable t .

The difficult part is showing that the intersection cohomology vanishes in odd degree and that the Frobenius acts in the prescribed way, which is proved using a delicate induction. This approach was employed for classical Schubert varieties in [KL80], for toric varieties in [DL91, Fie91], for hypertoric varieties in [PW07], and arrangement Schubert varieties in [EPW16]. The main result of [Pro18] is a unification of these arguments into a machine that can be used off the shelf.

We now apply Theorem 3.1 in our setting. Let G be a simply connected, split, semisimple group scheme over the integers, and let X_G be its associated base affine space, which is again a scheme over $\mathbb{Z}$. Lemma 3.3 says that the main technical results of the previous section hold over a finite field of sufficiently large characteristic. The lemma follows, for example, from repeated applications of [Poo17, Theorem 3.2.1(i)].

Lemma 3.3. *If q is a power of a sufficiently large prime, then Theorem 2.4 and Lemma 2.5 both hold with $\mathbb{C}$ replaced by the finite field $\mathbb{F}_q$.*

Let $m_1, \dots, m_{\text{rk } G}$ be the exponents of G , and let

$$f_G(t) := (1 - t^{m_1+1}) \cdots (1 - t^{m_{\text{rk } G}+1}).$$

Let $d_G := \deg f_G(t) = \dim X_G$. This polynomial will be relevant to us in two different ways. First, we have (see e.g. [Car89, Theorem 9.4.10] or [MT11, §24.1])

$$|G(\mathbb{F}_q)| = (-1)^{\text{rk } G} q^{d_G - \text{rk } G} f_G(q) \tag{2}$$

for any prime power q . Second, we have

$$\frac{1}{f_G(t)} = \sum_{i=0}^{\infty} t^i \dim H_{G(\mathbb{C})}^{2i}(\bullet; \mathbb{Q}),$$

where we write $\bullet$ to denote a point. Let

$$P_G(t) := \sum_{i=0}^{\infty} t^i \dim \text{IH}^{2i}(X_G(\mathbb{C}); \mathbb{Q}).$$

If we choose a prime l that does not divide q , then we may replace the topological intersection cohomology of the space $X_G(\mathbb{C})$ with the l -adic étale intersection cohomology of the variety $X_G(\overline{\mathbb{F}}_q)$ without changing the Poincaré polynomial [KW06, Proposition 10.4.1(i)].

Theorem 3.4. *The intersection cohomology of $X_G(\mathbb{C})$ vanishes in odd degree. If G is not the*

trivial group, then⁵ $\deg P_G(t) < d_G/2$, and

$$\frac{P_G(t)}{f_G(t)} = \sum_{S \subset \Delta} \frac{t^{d_{G_S}} P_{G_S}(t^{-1})}{f_{G_S}(t)}. \quad (3)$$

Remark 3.5. Before proving Theorem 3.4, we explain why it allows us to compute $P_G(t)$ recursively. There is a slight subtlety here, since $G_\emptyset = G$ and therefore $P_G(t)$ appears on both sides of the equation. Let us rewrite this equation as

$$P_G(t) - t^{d_G} P_G(t^{-1}) = \sum_{\emptyset \neq S \subset \Delta} \frac{f_G(t) t^{d_{G_S}} P_{G_S}(t^{-1})}{f_{G_S}(t)}. \quad (4)$$

Since $\deg P_G(t) < d_G/2$, the polynomial $t^{d_G} P_G(t^{-1})$ vanishes in degree less than or equal to $d_G/2$, and therefore $P_G(t)$ can be obtained from the right hand side of the equation by truncation.

Proof. Choose a prime power q as in Lemma 3.3. Theorem 2.4 and Lemma 3.3 tell us that the G -orbits in X_G are indexed by subsets $S \subset \Delta$, and the number of $\mathbb{F}_q$ -points on the orbit O_S is equal to

$$|G(\mathbb{F}_q)/H_S(\mathbb{F}_q)| = \frac{|G(\mathbb{F}_q)|}{|G_S(\mathbb{F}_q)| \cdot |U_S(\mathbb{F}_q)|}.$$

Using Equation (2) and the fact that $\dim U_S = (d_G - \operatorname{rk} G) - (d_{G_S} - \operatorname{rk} G_S)$, this is equal to

$$\frac{(-1)^{\operatorname{rk} G} f_G(q)}{(-1)^{\operatorname{rk} G_S} f_{G_S}(q)}.$$

Lemmas 2.5 and 3.3 tell us that O_S has a normal slice isomorphic to X_{G_S} , and Lemma 2.7 says that this slice is conical.

We will now apply Theorem 3.1 to the variety X_G , which is stratified by the Boolean poset of subsets of Δ . We assume that G is nontrivial, so that $\emptyset$ is strictly contained in Δ . We have $(X_G)_{\emptyset\Delta} = X_G$, and for every subset $S \subset \Delta$, we have $V_{\emptyset S} = V_S = O_S$ and $X_{S\Delta} = C_S \cong X_{G_S}$. More generally, for any $S \subset T \subset \Delta$, V_{ST} is isomorphic to an orbit of G_S on X_{G_S} , and therefore has polynomial point count. Theorem 3.1 then tells us that the intersection cohomology of X_G vanishes in odd degree, $\dim P_G(t) < d_G/2$, and

$$t^{d_G} P_G(t^{-1}) = \sum_{S \subset \Delta} \frac{(-1)^{\operatorname{rk} G} f_G(t)}{(-1)^{\operatorname{rk} G_S} f_{G_S}(t)} P_{G_S}(t). \quad (5)$$

Equation (3) follows from this formula by replacing t with t^{-1} and making use of the identity $(-1)^{\operatorname{rk} G} f_G(t^{-1}) = t^{-\operatorname{rk} G} f_G(t)$. $\square$

Remark 3.6. We now explain how to categorify Equation (3). Let IC_{X_G} be the intersection cohomology sheaf on $X_G(\mathbb{C})$. For each subset $S \subset \Delta$, let $\iota_S : O_S(\mathbb{C}) \hookrightarrow X_G(\mathbb{C})$ be the inclusion of the corresponding orbit. The stratification of $X_G(\mathbb{C})$ by orbits induces a filtration by supports on

⁵When G is the trivial group, $P_G(t) = 1$ and $d_G = 0 = \deg P_G(t)$, so this inequality fails.

the complex of global sections of an injective resolution of IC_{X_G} . This filtered complex gives rise to a spectral sequence with

$$E_1^{p,q} = \bigoplus_{\mathrm{codim} O_S = p} \mathbb{H}_{G(\mathbb{C})}^{p+q}(\iota_S^! \mathrm{IC}_{X_G}),$$

converging to $\mathrm{IH}_{G(\mathbb{C})}^*(X_G(\mathbb{C}); \mathbb{Q})$ [BGS96, Section 3.4]. The hypercohomology of the complex $\iota_S^! \mathrm{IC}_{X_G}$ is a $G(\mathbb{C})$ -equivariant local system on $O_S(\mathbb{C})$ whose fiber at the point v_S is equal to $\mathrm{IH}_c^*(X_{G_S}(\mathbb{C}); \mathbb{Q})$, the compactly supported intersection cohomology of the normal slice. Since the stabilizer $H_S(\mathbb{C})$ is connected, this local system is canonically trivial, and we therefore have

$$\begin{aligned} \mathbb{H}_{G(\mathbb{C})}^*(\iota_S^! \mathrm{IC}_{X_G}) &\cong \mathrm{H}_{G(\mathbb{C})}^*(O_S(\mathbb{C}); \mathbb{Q}) \otimes \mathrm{IH}_c^*(X_{G_S}(\mathbb{C}); \mathbb{Q}) \\ &\cong \mathrm{H}_{H_S(\mathbb{C})}^*(\bullet; \mathbb{Q}) \otimes \mathrm{IH}_c^*(X_{G_S}(\mathbb{C}); \mathbb{Q}) \\ &\cong \mathrm{H}_{G_S(\mathbb{C})}^*(\bullet; \mathbb{Q}) \otimes \mathrm{IH}_c^*(X_{G_S}(\mathbb{C}); \mathbb{Q}), \end{aligned}$$

where the last isomorphism follows from the fact that the inclusion of $G_S(\mathbb{C})$ into $H_S(\mathbb{C})$ is a homotopy equivalence. Returning to our spectral sequence, this means that

$$E_1^{p,q} \cong \bigoplus_{\substack{\mathrm{codim} O_S = p \\ j+k=p+q}} H_{G_S(\mathbb{C})}^j(\bullet; \mathbb{Q}) \otimes \mathrm{IH}_c^k(X_{G_S}(\mathbb{C}); \mathbb{Q}).$$

The left-hand side of Equation (3) is the Hilbert series of the E_∞ -page (with respect to the total grading) and the right-hand side is the Hilbert series of the E_1 -page. Thus Equation (3) implies that the spectral sequence degenerates at the E_1 -page. In other words, taking the associated graded with respect to the filtration of $\mathrm{IH}_{G(\mathbb{C})}^*(X_G(\mathbb{C}); \mathbb{Q})$ induced by the orbit stratification, we have

$$\mathrm{gr} \mathrm{IH}_{G(\mathbb{C})}^*(X_G(\mathbb{C}); \mathbb{Q}) \cong E_\infty = E_1 = \bigoplus_S \mathrm{H}_{G_S(\mathbb{C})}^*(\bullet; \mathbb{Q}) \otimes \mathrm{IH}_c^*(X_{G_S}(\mathbb{C}); \mathbb{Q}),$$

which categorifies Equation (3).

Remark 3.7. Turning Remark 3.6 into an alternate proof of Equation (3) would require showing independently that the intersection cohomology vanishes in odd degree and that the spectral sequence degenerates, which would require a calculation of mixed Hodge weights. This could be achieved via an inductive argument similar in flavor to the one referenced in Remark 3.2.

Section 4 will be devoted to understanding Equation 3 in the case where $G = \mathrm{SL}_n$. However, we will conclude this section by considering the case where $G = G_2$.

Example 3.8. Consider the exceptional group G_2 , and let

$$P_{G_2}(t) := \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}(X_{G_2}(\mathbb{C}); \mathbb{Q}).$$

We have $\Delta = \{1, 2\}$, where ϖ_1 is short and ϖ_2 is long. The space X_{G_2} has four orbits: the dense orbit G_2/U , the two intermediate orbits $O_{\{1\}}$ and $O_{\{2\}}$, and the fixed point $\{v_\emptyset\}$. We have

$G_{\{1\}} \cong G_{\{2\}} \cong \mathrm{SL}_2$, hence the two intermediate orbits have normal slices isomorphic to $X_{\mathrm{SL}_2} \cong \mathbb{A}^2$. We have $f_{G_2}(t) = (1 - t^2)(1 - t^6)$ [Dic01] and $f_{\mathrm{SL}_2}(t) = 1 - t^2$, so Equation (4) says that

$$P_{G_2}(t) - t^8 P_{G_2}(t^{-1}) = (1 - t^2)(1 - t^6) \left(2 \frac{t^2}{1 - t^2} + 1 \right) = 1 + t^2 - t^6 - t^8.$$

Since the degree of $P_{G_2}(t)$ is strictly less than $8/2 = 4$, this implies that $P_{G_2}(t) = 1 + t^2$.

4 Type A

In this section, we interpret Theorem 3.4 in the special case where $G = \mathrm{SL}_n$. As in Example 2.6, we write $X_n := X_{\mathrm{SL}_n}$. Similarly, we write $d_n := d_{\mathrm{SL}_n} = \binom{n+1}{2} - 1$, $P_n(t) := P_{\mathrm{SL}_n}(t)$, and

$$f_n(t) := f_{\mathrm{SL}_n}(t) = (1 - t^2) \cdots (1 - t^n).$$

For any composition $\sigma = (\sigma_1, \dots, \sigma_r)$ of n , we have $G_\sigma \cong \mathrm{SL}_{\sigma_1} \times \cdots \times \mathrm{SL}_{\sigma_r}$ and $X_\sigma \cong X_{\sigma_1} \times \cdots \times X_{\sigma_r}$, so Equation (3) may be translated as follows:

$$\frac{P_n(t)}{f_n(t)} = \sum_r \sum_{\sigma_1 + \cdots + \sigma_r = n} \prod_{i=1}^r \frac{t^{d_{\sigma_i}} P_{\sigma_i}(t^{-1})}{f_{\sigma_i}(t)}. \quad (6)$$

This recursion can be reformulated in terms of a sum with a simpler index set.

Corollary 4.1. *For all positive integers n , we have*

$$P_n(t) - t^{d_n} P_n(t^{-1}) = \sum_{s=1}^{n-1} \frac{f_n(t)}{f_s(t) f_{n-s}(t)} t^{d_s} P_s(t^{-1}) P_{n-s}(t).$$

Proof. By Equation (6), we have

$$\begin{aligned} P_n(t) &= f_n(t) \sum_{r=1}^n \sum_{\sigma_1 + \cdots + \sigma_r = n} \prod_{i=1}^r \frac{t^{d_{\sigma_i}} P_{\sigma_i}(t^{-1})}{f_{\sigma_i}(t)} \\ &= P_n(t) + f_n(t) \sum_{r=2}^n \sum_{\sigma_1 + \cdots + \sigma_r = n} \prod_{i=1}^r \frac{t^{\sigma_i} P_{\sigma_i}(t^{-1})}{f_{\sigma_i}(t)} \\ &= P_n(t) + f_n(t) \sum_{\sigma_r=1}^{n-1} \frac{t^{\sigma_r} P_{\sigma_r}(t^{-1})}{f_{\sigma_r}(t)} \sum_{r=2}^n \sum_{\sigma_1 + \cdots + \sigma_{r-1} = n - \sigma_r} \prod_{i=1}^{r-1} \frac{t^{d_{\sigma_i}} P_{\sigma_i}(t^{-1})}{f_{\sigma_i}(t)} \\ &= P_n(t) + f_n(t) \sum_{\sigma_r=1}^{n-1} \frac{t^{\sigma_r} P_{\sigma_r}(t^{-1})}{f_{\sigma_r}(t)} \cdot \frac{P_{n-\sigma_r}(t)}{f_{n-\sigma_r}(t)}. \end{aligned}$$

Letting $s = \sigma_r$ gives the desired formula. □

Example 4.2. When $n = 2$, Corollary 4.1 tells us that

$$P_2(t) - t^2 P_2(t^{-1}) = 1 - t^2.$$

Since we know that the degree of $P_2(t)$ is strictly less than $d_2/2 = 1$, this implies that $P_2(t) = 1$. This is consistent with the fact that $X_2 \cong \mathbb{A}^2$.

Example 4.3. When $n = 3$, Corollary 4.1 tells us that

$$P_3(t) - t^5 P_3(t^{-1}) = \frac{f_3(t)}{f_1(t)f_2(t)} P_1(t^{-1})P_2(t) + \frac{t^2 f_3(t)}{f_2(t)f_1(t)} P_2(t^{-1})P_1(t) = 1 + t^2 - t^3 - t^5.$$

As we know that the degree of $P_3(t)$ is strictly less than $d_3/2 = 5/2$, this implies that we have $P_3(t) = 1 + t^2$. This agrees with the calculation of $P_3(t)$ in [HJ14].

Example 4.4. When $n = 4$, Corollary 4.1 tells us that

$$\begin{aligned} P_4(t) - t^9 P_4(t^{-1}) &= \frac{f_4(t)}{f_1(t)f_3(t)} P_1(t^{-1})P_3(t) + \frac{t^2 f_4(t)}{f_2(t)f_2(t)} P_2(t^{-1})P_2(t) + \frac{t^5 f_4(t)}{f_3(t)f_1(t)} P_3(t^{-1})P_1(t) \\ &= (1 - t^4)(1 + t^2) + t^2(1 - t^3)(1 + t^2) + t^5(1 - t^4)(1 + t^{-2}) \\ &= 1 + 2t^2 + t^3 - t^6 - 2t^7 - t^9. \end{aligned}$$

Since we know that the degree of $P_4(t)$ is strictly less than $d_4/2 = 9/2$, this implies that we have $P_4(t) = 1 + 2t^2 + t^3$. See the appendix for calculations of $P_n(t)$ up to $n = 13$.

Consider the generating series

$$\Psi(t, u) := 1 + \sum_{n=1}^{\infty} u^n \frac{P_n(t)}{f_n(t)}.$$

The recursion for $P_n(t)$ in Corollary 4.1 can be translated into a function equation for $\Psi(t, u)$ as follows.

Proposition 4.5. *We have $\Psi(t^{-1}, u)\Psi(t, -u) = 1$.*

Proof. Using the fact that $(-1)^{n-1} f_n(t^{-1}) = t^{-d_n} f_n(t)$, we have

$$\begin{aligned} \left(\Psi(t^{-1}, u) - 1 \right) \left(\Psi(t, -u) - 1 \right) &= \sum_{m=1}^{\infty} u^m \frac{P_m(t^{-1})}{f_m(t^{-1})} \cdot \sum_{n=1}^{\infty} (-u)^n \frac{P_n(t)}{f_n(t)} \\ &= - \sum_{m=1}^{\infty} (-u)^m \frac{t^{d_m} P_m(t^{-1})}{f_m(t)} \cdot \sum_{n=1}^{\infty} (-u)^n \frac{P_n(t)}{f_n(t)} \\ &= - \sum_{k=2}^{\infty} \frac{(-u)^k}{f_k(t)} \sum_{m=1}^{k-1} \frac{f_k(t)}{f_m(t)f_{k-m}(t)} t^{d_m} P_m(t^{-1}) P_{k-m}(t), \end{aligned}$$

where the last line is obtained by putting $k = m + n$. Corollary 4.1 says that the internal sum is

equal to $P_k(t) - t^{d_k} P_k(t^{-1})$, thus we have

$$\begin{aligned}
\left(\Psi(t^{-1}, u) - 1\right) \left(\Psi(t, -u) - 1\right) &= \sum_{k=2}^{\infty} \frac{(-u)^k}{f_k(t)} t^{d_k} P_k(t^{-1}) - \sum_{k=2}^{\infty} \frac{(-u)^k}{f_k(t)} P_k(t) \\
&= - \sum_{k=2}^{\infty} \frac{u^k}{f_k(t^{-1})} P_k(t^{-1}) - \sum_{k=2}^{\infty} \frac{(-u)^k}{f_k(t)} P_k(t) \\
&= \left(1 + u - \Psi(t^{-1}, u)\right) + \left(1 - u - \Psi(t, -u)\right) \\
&= 2 - \Psi(t^{-1}, u) - \Psi(t, -u).
\end{aligned}$$

Adding $\Psi(t^{-1}, u) + \Psi(t, -u) - 1$ to both sides gives the desired equation. $\square$

Remark 4.6. The power series $\Psi(t, u)$ is uniquely characterized by Proposition 4.5 along with the fact that the coefficient of u is equal to 1 and, for all $n \geq 2$, $f_n(t)$ times the coefficient of u^n is equal to a polynomial in t of degree strictly less than $d_n/2$.

Remark 4.7. The **plethystic logarithm**

$$\text{PLog } \Psi(t, u) = \sum_{i, n \geq 0} e(i, n) t^i u^n$$

is the power series with integer coefficients uniquely determined by the equation

$$\Psi(t, u) = \prod_{n, i \geq 0} \frac{1}{(1 - t^i u^n)^{e(i, n)}}.$$

Numerical evidence strongly suggests that these coefficients are non-negative, and furthermore that there exist polynomials $Q_n(t)$ with non-negative integer coefficients such that

$$\text{PLog } \Psi(t, u) = u + t^2 \sum_{n=2}^{\infty} \frac{u^n Q_n(t)}{f_n(t)}.$$

We thank Vladimir Dotsenko and Balázs Szendrői for conjecturing the non-negativity of the coefficients $e(i, n)$, Max Alekseyev for checking this conjecture for all $i, n \leq 50$, and Anton Mellit and Hjalmar Rosengren for making the stronger conjecture, which was verified by Szendrői up to $n = 7$. We also thank MathOverflow [Pro] for providing a forum for this discussion.

5 The Betti numbers

In this section we study the coefficient of t^i in $P_n(t)$ as a function of n . Let

$$P_n(t) = \sum_{i \geq 0} c_i(n) t^i.$$

In order to make use of the recursion in Corollary 4.1, it will also be useful to write

$$\frac{f_n(t)}{f_s(t)f_{n-s}(t)} = \sum_{i \geq 0} b_{i,s}(n)t^i.$$

Lemma 5.1. *For all $i \geq 0$ and $s \geq 1$, the function $b_{i,s}(n)$ is constant for $n \geq \max(i, 1) + s$.*

Proof. We treat the $i = 0$ and $i = 1$ cases separately. We have $\frac{f_n(0)}{f_s(0)f_{n-s}(0)} = 1$, so $b_{0,s}(n) = 1$ for all $n \geq 1 + s$. Using the Taylor series expansion of $\frac{1}{1-t^r}$, we can see that the coefficient of t in $\frac{f_n(t)}{f_s(t)f_{n-s}(t)}$ is 0, so $b_{1,s}(n) = 0$ for all $0 < s < n$.

We now proceed by induction on the quantity $i + s$. We have

$$\frac{f_n(q)}{f_s(q)f_{n-s}(q)} = (1-q) \binom{n}{s}_q,$$

and the well-known Gaussian binomial coefficient identity

$$\binom{n}{s}_q = q^s \binom{n-1}{s}_q + \binom{n-1}{s-1}_q$$

translates to the identity

$$\frac{f_n(t)}{f_s(t)f_{n-s}(t)} = t^s \frac{f_{n-1}(t)}{f_s(t)f_{n-1-s}(t)} + \frac{f_{n-1}(t)}{f_{s-1}(t)f_{n-s}(t)}.$$

Taking coefficients of t^i , this means that

$$b_{i,s}(n) = b_{i-s,s}(n-1) + b_{i,s-1}(n-1).$$

Since we have already treated the cases when $i = 0$ and $i = 1$, we may assume that $i \geq 2$. This means that our inductive hypothesis implies that the right-hand side is constant for $n-1 \geq i+s-1$, or equivalently for $n \geq i+s$. $\square$

Proposition 5.2. *Fix a non-negative integer i . The function $c_i(n)$ is given by a polynomial in n of degree at most $i/2$ for all $n \geq i$.*

Proof. We proceed by induction on i . The base case is $i = 0$, which holds because $c_0(n) = 1$ for all n . Now fix $i > 0$ and assume that the statement holds for all $j < i$. Our strategy will be to prove that the quantity $c_i(n) - c_i(n-1)$ is given by a polynomial in n for all $n > i$, which implies that $c_i(n)$ is given by a polynomial in n for all $n \geq i$.

Let $n > i$ be given. By Corollary 4.1, we have

$$c_i(n) - c_{d_n-i}(n) = \sum_{s=1}^{n-1} \sum_{\substack{p+q+r=i \\ p,q,r \geq 0}} b_{p,s}(n) c_{d_s-q}(s) c_r(n-s).$$

Since $n > i > 0$, we have $c_{d_n-i}(n) = 0$. If $s > i \geq q$, then $c_{d_s-q}(s) = 0$, so we may restrict the upper bound of the outer sum to i , which is independent of n . Thus we have

$$c_i(n) = \sum_{s=1}^i \sum_{\substack{p+q+r=i \\ p,q,r \geq 0}} b_{p,s}(n) c_{d_s-q}(s) c_r(n-s). \quad (7)$$

By Lemma 5.1, $b_{p,s}(n)$ is constant for $n \geq \max(p, 1) + s$. The condition that $n \geq 1 + s$ is automatic from the fact that $s \leq i < n$. If $n < p + s$, then we have $p + q + r = i < n < p + s$ with both inequalities strict. This means that $s - q \geq 2$, which in turn implies that $c_{d_s-q}(s) = 0$. Thus every term in our sum is equal to a constant times $c_r(n-s)$.

When $r = i$, we necessarily have $p = q = 0$. The only nonzero term of this form occurs when $s = 1$, and we obtain $c_i(n-1)$. When $r = i-1$, we either have $p = 1$ and $q = 0$, in which case $b_{1,s}(n) = 0$ for all s , or $p = 0$ and $q = 1$, in which case $c_{d_s-1}(s) = 0$ for all s . When $r \leq i-2$, our inductive hypothesis implies that $c_r(n-s)$ is given by a polynomial in n of degree at most $r/2 \leq i/2 - 1$ whenever $n-s \geq r$. We know that $n > i = p + q + r$, and therefore $n-s > p + q + r - s$. We also know that $c_{d_s-q}(s) = 0$ unless $s \leq q + 1$. Thus, for every nonzero term, $n-s > p + q + r - s \geq p + r - 1 \geq r - 1$, and therefore $n-s \geq r$.

We have now shown that $c_i(n)$ is equal to $c_i(n-1)$ plus a function of n that agrees with a polynomial of degree at most $i/2 - 1$ whenever $n > i$. This implies that $c_i(n)$ is given by a polynomial in n of degree at most $i/2$ whenever $n \geq i$. $\square$

Since the function $c_i(n)$ takes integer values, Proposition 5.2 implies that there exist integers $a_{i,k}$ for $0 \leq k \leq i/2$ with the property that, for all $n \geq i$,

$$c_i(n) = \sum_{k=0}^{\lfloor i/2 \rfloor} a_{i,k} \binom{n-i}{k}.$$

Conjecture 5.3. *The integers $a_{i,k}$ are all non-negative.*

Conjecture 5.3 is motivated by the following corollary of Proposition 5.2, in which we compute the coefficients $a_{i,k}$ for all $i \leq 9$.

Corollary 5.4. *We have the following identities:*

$$\text{for all } n \geq 0, \quad c_0(n) = 1$$

$$\text{for all } n \geq 1, \quad c_1(n) = 0$$

$$\text{for all } n \geq 2, \quad c_2(n) = n - 2$$

$$\text{for all } n \geq 3, \quad c_3(n) = n - 3$$

$$\text{for all } n \geq 4, \quad c_4(n) = \binom{n-4}{2} + 2\binom{n-4}{1}$$

$$\text{for all } n \geq 5, \quad c_5(n) = 2\binom{n-5}{2} + 4\binom{n-5}{1} + \binom{n-5}{0}$$

$$\text{for all } n \geq 6, \quad c_6(n) = \binom{n-6}{3} + 5\binom{n-6}{2} + 9\binom{n-6}{1} + 6\binom{n-6}{0}$$

$$\text{for all } n \geq 7, \quad c_7(n) = 3\binom{n-7}{3} + 12\binom{n-7}{2} + 20\binom{n-7}{1} + 15\binom{n-7}{0}$$

$$\text{for all } n \geq 8, \quad c_8(n) = \binom{n-8}{4} + 9\binom{n-8}{3} + 30\binom{n-8}{2} + 53\binom{n-8}{1} + 50\binom{n-8}{0}$$

$$\text{for all } n \geq 9, \quad c_9(n) = 4\binom{n-9}{4} + 25\binom{n-9}{3} + 73\binom{n-9}{2} + 125\binom{n-9}{1} + 123\binom{n-9}{0}.$$

Proof. By Proposition 5.2, we can compute $c_i(n)$ for $n \geq i$ by polynomial interpolation as n ranges from i to $\lfloor 3i/2 \rfloor$. We do this using the formulas in the appendix. $\square$

Example 5.5. Corollary 5.4 says that, for all $n \geq 2$, $\dim \mathrm{IH}^4(X_n(\mathbb{C}); \mathbb{Q}) = c_2(n) = n - 2$, or equivalently $\dim \mathrm{IH}_{\mathrm{SL}_n(\mathbb{C})}^4(X_n(\mathbb{C}); \mathbb{Q}) = n - 1$. Indeed, $\dim \mathrm{IH}_{\mathrm{SL}_n(\mathbb{C})}^4(X_n(\mathbb{C}); \mathbb{Q})$ is equal to the coefficient of t^2 in the power series $P_n(t)/f_n(t)$; Equation (6) shows us that this coefficient is $n - 1$, corresponding to the $n - 1$ different compositions consisting of a two and a bunch of ones. At the level of vector spaces, Remark 3.6 tells us that $\mathrm{IH}_{\mathrm{SL}_n}^4(X_n(\mathbb{C}); \mathbb{Q})$ decomposes as a direct sum of $n - 1$ copies of the 1-dimensional vector space $H_{\mathrm{SL}_2}^0(\bullet; \mathbb{Q}) \otimes \mathrm{IH}_c^4(X_{\mathrm{SL}_2}(\mathbb{C}); \mathbb{Q})$, one for each such composition.⁶ Similarly, $\dim \mathrm{IH}_{\mathrm{SL}_n(\mathbb{C})}^6(X_n(\mathbb{C}); \mathbb{Q}) = n - 2$, and $\mathrm{IH}_{\mathrm{SL}_n}^6(X_n(\mathbb{C}); \mathbb{Q})$ decomposes as a direct sum of $n - 2$ copies of the 1-dimensional vector space $H_{\mathrm{SL}_3}^0(\bullet; \mathbb{Q}) \otimes \mathrm{IH}_c^6(X_{\mathrm{SL}_3}(\mathbb{C}); \mathbb{Q})$, one for each composition consisting of a three and a bunch of ones.

Proposition 5.6. *If i is even, then $a_{i,i/2} = 1$. If i is odd, then $a_{i,(i-1)/2} = (i - 1)/2$.*

Proof. If i is even, then Equation (7) gives us the difference equation

$$c_i(n) - c_i(n - 1) = c_{i-2}(n - 2) + O((i - 4)/2).$$

If i is odd, we get

$$c_i(n) - c_i(n - 1) = c_{i-2}(n - 2) + c_{i-3}(n - 3) + O((i - 5)/2).$$

⁶Since all of these terms appear in the same entry of the E_1 -page, the filtration with respect to which we need to take the associated graded is trivial.

The proposition now follows from induction on i . □

Remark 5.7. The formulas for $c_2(n)$ and $c_3(n)$ are related to identities involving Stirling numbers. More precisely, to prove directly that $c_2(n) = n - 2$, assume inductively that

$$P_m(t) = 1 + (m - 2)t^2 + \dots$$

for all $m < n$. We observe that

$$\frac{f_n(t)}{f_{\sigma_1}(t) \dots f_{\sigma_r}(t)} = 1 + (k_\sigma - 1)t^2 + \dots,$$

where k_σ is the number of i such that $\sigma_i > 1$. The inductive step is equivalent to the identity

$$n - 2 = \sum_{r=2}^n (-1)^r (n - r - 1) \binom{n-1}{r-1},$$

which in turn is equivalent to the statement

$$\sum_{k=0}^{n-1} (-1)^k k \binom{n-1}{k} = 0.$$

The left-hand side is equal to $(-1)^{n-1}(n-1)!$ times the Stirling number of the second kind $S(1, n-1)$ [Sta12, Equation 1.94a], which counts the number of ways of partitioning a 1-element set into $n-1$ nonempty subsets, and therefore vanishes for $n \geq 3$.

To prove directly that $c_3(n) = n - 3$, a similar induction, along with our calculation of $c_2(n)$, allow us to reduce to the identity

$$\sum_{r=1}^n (-1)^r \sum_{\sigma_1 + \dots + \sigma_r = n} (r - k_\sigma) = 0. \tag{8}$$

We are grateful to Paul Balister for explaining the following proof of Equation (8). The internal sum can be rewritten as a sum over i of the number of compositions σ into r parts with $\sigma_i = 1$. This in turn is equal to the sum over k of the number of compositions of k into $i-1$ parts times the number of compositions of $n-k-1$ into $r-i$ parts. Letting

$$A_k := \sum_{r=1}^k (-1)^r \sum_{\sigma_1 + \dots + \sigma_r = n} 1,$$

we can therefore express the left-hand side of Equation (8) as $-\sum_i A_{i-1} A_{n-i}$. But $A_k = 0$ for $k \geq 2$, so this sum vanishes for $n \geq 4$.

6 Two bigraded rings

In this section, X_n will always mean $X_n(\mathbb{C})$, and intersection homology and cohomology will always be taken with rational coefficients. Let m and n be positive integers, and consider the normal slice map

$$\varphi_{m,n} : X_m \times X_n \rightarrow X_{m+n}$$

from Example 2.6 with $r = 2$. This is a normally nonsingular inclusion, and therefore induces a graded map

$$(\varphi_{m,n})_* : \mathrm{IH}_*(X_m) \otimes \mathrm{IH}_*(X_n) \rightarrow \mathrm{IH}_*(X_{m+n})$$

on intersection homology. Given a third positive integer l , Example 2.6 shows that

$$\varphi_{l,m+n} \circ (\mathrm{id}_{X_l} \times \varphi_{m,n}) = \varphi_{l,m,n} = \varphi_{l+m,n} \circ (\varphi_{l,m} \times \mathrm{id}_{X_n}),$$

therefore our two natural maps

$$\mathrm{IH}_*(X_l) \otimes \mathrm{IH}_*(X_m) \otimes \mathrm{IH}_*(X_n) \rightarrow \mathrm{IH}_*(X_{l+m+n})$$

agree. This in turn means that our maps define an associative, bigraded ring structure on the vector space

$$R := \mathbb{Q} \oplus \bigoplus_{n=1}^{\infty} \mathrm{IH}_*(X_n).$$

We can also build a version of this ring using equivariant intersection homology groups

$$\mathrm{IH}_{2i}^{\mathrm{SL}_n}(X_n) := \mathrm{IH}_{\mathrm{SL}_n}^{2i}(X_n)^*.$$

For any m and n , we have maps

$$\mathrm{IH}_*^{\mathrm{SL}_m}(X_m) \otimes \mathrm{IH}_*^{\mathrm{SL}_n}(X_n) \cong \mathrm{IH}_*^{\mathrm{SL}_m \times \mathrm{SL}_n}(X_m \times X_n) \rightarrow \mathrm{IH}_*^{\mathrm{SL}_m \times \mathrm{SL}_n}(X_{m+n}) \rightarrow \mathrm{IH}_*^{\mathrm{SL}_{m+n}}(X_{m+n}),$$

where the first map is induced by the $(\mathrm{SL}_m \times \mathrm{SL}_n)$ -equivariant normally nonsingular inclusion $\varphi_{m,n}$ and the second is induced by the inclusion of $\mathrm{SL}_m \times \mathrm{SL}_n$ into SL_{m+n} . We thus obtain an associative bigraded ring structure on

$$\hat{R} := \mathbb{Q} \oplus \bigoplus_{n=1}^{\infty} \mathrm{IH}_*^{\mathrm{SL}_n}(X_n).$$

Remark 6.1. One motivation for studying the ring $\hat{R}$ is that its Hilbert series is equal to the power series $\Psi(t, u)$ from Proposition 4.5. If the ring $\hat{R}$ were commutative, then the non-negativity of the plethystic logarithm discussed in Remark 4.7 would be equivalent to the statement that $\hat{R}$ is an infinite polynomial ring with $e(i, n)$ generators in bidegree (i, n) . However, this ring is in fact very far from being commutative, as we can see below.

While we are unable to give a complete description of either R or $\hat{R}$, it is possible to describe these rings in low homological degree. Let $x \in \mathrm{IH}_0^{\mathrm{SL}_1}(X_1) = \mathrm{IH}_0(\bullet)$, $y \in \mathrm{IH}_4^{\mathrm{SL}_2}(X_2) \cong \mathrm{H}_4^{\mathrm{SL}_2}(\bullet)$,

and $z \in \mathrm{IH}_6^{\mathrm{SL}_3}(X_3) \cong \mathrm{H}_6^{\mathrm{SL}_3}(\bullet)$ be generators of their respective 1-dimensional vector spaces. These classes freely generate the ring $\hat{R}$ in homological degree at most 6:

- For all $n \geq 1$, $\mathrm{IH}_0^{\mathrm{SL}_n}(X_n)$ is 1-dimensional with generator x^n .
- For all $n \geq 1$, $\mathrm{IH}_2^{\mathrm{SL}_n}(X_n) = 0$.
- For all $n \geq 1$, $\mathrm{IH}_4^{\mathrm{SL}_n}(X_n)$ is $(n-1)$ -dimensional with basis $\{x^{i-1}yx^{n-1-i} \mid 1 \leq i \leq n-1\}$, corresponding to the decomposition described in Example 5.5.
- For all $n \geq 2$, $\mathrm{IH}_6^{\mathrm{SL}_n}(X_n)$ is $(n-2)$ -dimensional with basis $\{x^{i-1}zx^{n-2-i} \mid 1 \leq i \leq n-2\}$, also corresponding to the decomposition described in Example 5.5.

The subring $R \subset \hat{R}$ contains the classes x , $xy - yx$, and $xz - zx$, and is freely generated by these classes in homological degree at most 6.

A Appendix

Here we list the polynomials $P_n(t)$ for $n \leq 13$.

$$\begin{aligned}
P_1(t) &= 1 \\
P_2(t) &= 1 \\
P_3(t) &= 1 + t^2 \\
P_4(t) &= 1 + 2t^2 + t^3 \\
P_5(t) &= 1 + 3t^2 + 2t^3 + 2t^4 + t^5 + 2t^6 \\
P_6(t) &= 1 + 4t^2 + 3t^3 + 5t^4 + 5t^5 + 6t^6 + 5t^7 + 4t^8 + t^9 \\
P_7(t) &= 1 + 5t^2 + 4t^3 + 9t^4 + 11t^5 + 15t^6 + 15t^7 + 20t^8 + 13t^9 + 12t^{10} + 9t^{11} + 9t^{12} + t^{13} \\
P_8(t) &= 1 + 6t^2 + 5t^3 + 14t^4 + 19t^5 + 29t^6 + 35t^7 + 50t^8 + 51t^9 + 55t^{10} + 55t^{11} + 58t^{12} + 43t^{13} \\
&\quad + 38t^{14} + 30t^{15} + 16t^{16} + 5t^{17} \\
P_9(t) &= 1 + 7t^2 + 6t^3 + 20t^4 + 29t^5 + 49t^6 + 67t^7 + 103t^8 + 123t^9 + 160t^{10} + 178t^{11} + 213t^{12} \\
&\quad + 212t^{13} + 229t^{14} + 215t^{15} + 202t^{16} + 162t^{17} + 137t^{18} + 109t^{19} + 83t^{20} + 35t^{21} \\
P_{10}(t) &= 1 + 8t^2 + 7t^3 + 27t^4 + 41t^5 + 76t^6 + 114t^7 + 186t^8 + 248t^9 + 354t^{10} + 445t^{11} + 569t^{12} \\
&\quad + 666t^{13} + 797t^{14} + 867t^{15} + 944t^{16} + 968t^{17} + 972t^{18} + 938t^{19} + 888t^{20} + 767t^{21} \\
&\quad + 624t^{22} + 539t^{23} + 420t^{24} + 277t^{25} + 138t^{26} \\
P_{11}(t) &= 1 + 9t^2 + 8t^3 + 35t^4 + 55t^5 + 111t^6 + 179t^7 + 308t^8 + 446t^9 + 683t^{10} + 931t^{11} \\
&\quad + 1284t^{12} + 1639t^{13} + 2131t^{14} + 2554t^{15} + 3068t^{16} + 3516t^{17} + 3978t^{18} + 4299t^{19} \\
&\quad + 4620t^{20} + 4722t^{21} + 4738t^{22} + 4655t^{23} + 4443t^{24} + 4047t^{25} + 3552t^{26} + 2937t^{27} \\
&\quad + 2514t^{28} + 2029t^{29} + 1484t^{30} + 873t^{31} + 265t^{32} \\
P_{12}(t) &= 1 + 10t^2 + 9t^3 + 44t^4 + 71t^5 + 155t^6 + 265t^7 + 479t^8 + 742t^9 + 1202t^{10} + 1749t^{11} \\
&\quad + 2561t^{12} + 3511t^{13} + 4828t^{14} + 6255t^{15} + 8049t^{16} + 9969t^{17} + 12172t^{18} + 14362t^{19} \\
&\quad + 16721t^{20} + 18888t^{21} + 20965t^{22} + 22755t^{23} + 24178t^{24} + 25133t^{25} + 25498t^{26} \\
&\quad + 25195t^{27} + 24670t^{28} + 23456t^{29} + 21772t^{30} + 19414t^{31} + 16711t^{32} + 14123t^{33} \\
&\quad + 12023t^{34} + 9482t^{35} + 6833t^{36} + 4006t^{37} + 1317t^{38} \\
P_{13}(t) &= 1 + 11t^2 + 10t^3 + 54t^4 + 89t^5 + 209t^6 + 375t^7 + 710t^8 + 1165t^9 + 1980t^{10} + 3043t^{11} \\
&\quad + 4692t^{12} + 6807t^{13} + 9838t^{14} + 13505t^{15} + 18404t^{16} + 24159t^{17} + 31296t^{18} + 39361t^{19} \\
&\quad + 48823t^{20} + 58981t^{21} + 70278t^{22} + 81886t^{23} + 93869t^{24} + 105612t^{25} + 116901t^{26} \\
&\quad + 126688t^{27} + 135618t^{28} + 142267t^{29} + 147027t^{30} + 148755t^{31} + 147909t^{32} + 144539t^{33} \\
&\quad + 139430t^{34} + 131305t^{35} + 120931t^{36} + 108095t^{37} + 93604t^{38} + 80199t^{39} + 68481t^{40} \\
&\quad + 55663t^{41} + 42067t^{42} + 27881t^{43} + 13597t^{44}
\end{aligned}$$

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