Locally Compact Quantum Metric Spaces

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Object of the talk

Problem addressed in this talk

How might we define a notion of quantum locally compact metric space suitable for a generalization of Gromov-Hausdorff convergence?

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We will present the core ideas from my two papers on this topic:

1. *Bounded-Lipschitz distances on the state space of a C*-algebra*, F. Latrémolière, Taiwanese Journal of Mathematics **11** (2007) 2 , 447-469 , ArXiv: math.OA/0510340
2. *Locally Compact Quantum Metric Spaces*, F. Latrémolière, submitted (2012), Arxiv: 1208.2398

Structure of this Presentation

1. *Noncommutative Metric Geometry*
   - Classical Framework
   - The Compact Case

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2. *Bounded-Lipschitz Metrics*
   - The problem of Noncompact Metric Spaces
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   - Examples

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   - The problem of Noncompact Metric Spaces
   - Bounded-Lipschitz Distances
   - Examples
3. *Quantum Locally Compact Metric Spaces*
   - The Monge-Kantorovich distance
   - Topography and Tame subsets of States
   - Quantum Locally Compact Metric Spaces
   - Examples

GPS

Gel'fand duality and Noncommutative Geometry

Theorem (Gel'fand-Naimark duality)

The category of C-algebras, with *-morphisms as arrows, is a concrete realization of the dual category of locally compact spaces, with proper continuous maps as arrows.*

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Founding Allegory of Noncommutative Geometry

Noncommutative geometry is the study of noncommutative generalizations of algebras of functions on spaces.

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Founding Allegory of Noncommutative **Metric** *Geometry*

Noncommutative metric geometry is the study of noncommutative generalizations of algebras of Lipschitz functions on metric spaces.

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Motivation

Noncommutative geometry provides a replacement for some ill-behaved spaces (e.g. orbit spaces, foliation spaces) and a natural framework for quantum geometry, i.e. the study of geometric properties of systems governed by quantum physics (e.g. Moyal planes).

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First Problem of Noncommutative Metric Geometry

What should a noncommutative analogue of a Lipschitz algebra be?

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First Problem of Noncommutative Metric Geometry

What should a noncommutative analogue of a Lipschitz algebra be? For a locally compact metric space, Gel'fand duality suggests that a noncommutative Lipschitz algebra be based on a C*-algebra. What extra structure does the metric provide?

Lipschitz Seminorms

A natural dual object for the metric is the Lipschitz seminorm:

Definition

Let (X, m) be a metric space. For any function $f : X \to \mathbb{R}$, we define:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y \right\}.$$

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Questions

1. Can we recover the metric from its Lipschitz seminorm?
2. What makes a Lipschitz seminorm special among all seminorms?

The Extended Monge-Kantorovich metric

For a C*-algebra $\mathfrak{A}$, let $\mathfrak{sa}(\mathfrak{A})$ be its self-adjoint part and $\mathfrak{u}\mathfrak{A}$ be $\mathfrak{A}$ if $\mathfrak{A}$ is unital or the minimal unitization of $\mathfrak{A}$ otherwise.

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Definition

A Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ is a C*-algebra $\mathfrak{A}$ and a seminorm L densely defined on $\mathfrak{sa}(\mathfrak{u}\mathfrak{A})$ such that $\{a \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$.

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Definition (Monge (1871), Kantorovich (1940), Kantorovich-Rubinstein (1958))

The extended Monge-Kantorovich metric associated with a Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ is defined for any two $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}) \text{ and } \mathsf{L}(a) \leq 1\}.$$

First Topological property of the extended Monge-Kantorovich metric

From our generic setup, we already observe:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair.

1. mk_{L} *is an extended metric,*
2. *The topology induced by* mk_{L} *on* $\mathscr{S}(\mathfrak{A})$ *is finer than the weak* topology,*
3. *The closed balls of* mk_{L} *of finite diameter are closed in the relative weak* topology on* $\mathscr{S}(\mathfrak{A})$.

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3. *The closed balls of mk_{L} of finite diameter are closed in the relative weak* topology on $\mathscr{S}(\mathfrak{A})$.*

Proof.

Use the density of the domain of L and the fact that $\mathfrak{sa}(\mathfrak{A})$ spans $\mathfrak{A}$. $\square$

The classical extended Monge-Kantorovich metric

Theorem

Let (X, m) be a locally compact metric space and identify X with the space of Dirac probability measures over X (i.e. the Gel'fand spectrum of $C_0(X)$). Then:

$$\forall x, y \in X \quad \mathsf{m}(x, y) = \mathsf{mk}_{\mathsf{L}}(x, y).$$

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The extended Monge-Kantorovich metric is well-behaved when working over *compact* metric spaces:

Theorem (Wasserstein, Dobrushin (1970))

Let (X, m) be a compact metric space. The extended Monge-Kantorovich metric mk_{L} associated with m is a metric which metrizes the weak topology on the state space $\mathscr{S}(C(X))$ of $C(X)$.*

Compact Quantum Metric Spaces

Based on this observation, Rieffel introduced:

Definition (Rieffel, 1998)

A compact quantum metric space $(\mathfrak{A}, \mathsf{L})$ consists of an order-unit space $\mathfrak{A}$ and a seminorm L densely defined on $\mathfrak{A}$, satisfying:

$$\{a \in \mathfrak{A} : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}},$$

and such that the distance:

$$\mathsf{mk}_{\mathsf{L}} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}) \mapsto \sup\{|\varphi(a) - \psi(a)| : a \in \mathfrak{A}, \mathsf{L}(a) \leq 1\}$$

metrizes the weak* topology on the state space $\mathscr{S}(\mathfrak{A})$.

Characterization of Compact Quantum Metric Spaces

The key observation of Rieffel is that one may characterize compact quantum metric spaces in C*-algebraic terms:

Theorem (Rieffel, 1998)

A Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ with $\mathfrak{A}$ unital is a compact quantum metric space when:

1. *There exists $r \geq 0$ such that for all $a \in \mathfrak{sa}(\mathfrak{A})$ we have* $\inf\{\|a + \lambda 1_{\mathfrak{A}}\|_{\mathfrak{A}} : \lambda \in \mathbb{R}\} \leq r\mathsf{L}(a)$,
2. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leq 1, \|a\|_{\mathfrak{A}} \leq r\}$ *is precompact in norm.*

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2. *$\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leq 1, \|a\|_{\mathfrak{A}} \leq r\}$ is precompact in norm.*

Proof.

Use Kadison functional representation and Arzéla-Ascoli theorems. □

Examples: Ergodic Actions of Compact Groups with continuous Lengths

For any C*-algebra $\mathfrak{A}$, let $\mathfrak{sa}(\mathfrak{A})$ be its self-adjoint part and $\|\cdot\|_{\mathfrak{A}}$ be its norm.

Theorem (Rieffel, 1998)

Let α be a strongly continuous action of a compact group G on a unital C-algebra $\mathfrak{A}$ and ℓ be a continuous length function on G. Let $e \in G$ be the unit of G. For all $a \in \mathfrak{A}$, define:*

$$\mathsf{L}(a) = \sup\left\{ \frac{\|\alpha_g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{e\} \right\}.$$

If $\{a \in \mathfrak{A} : \forall g \in G \quad \alpha_g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$, then $(\mathfrak{sa}(\mathfrak{A}), \mathsf{L})$ is a compact quantum metric space.

Rieffel introduces the Rieffel-Gromov-Hausdorff distance dist_q on the class of compact, quantum metric space.

Finite Dimensional Approximations of Quantum Tori

Rieffel introduces the Rieffel-Gromov-Hausdorff distance dist_q on the class of compact, quantum metric space.This allows to prove such results as:

Theorem (Latrémolière, 2005)

Let G be an Abelian compact group endowed with a continuous length function ℓ. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of closed subgroups of G converging to G for the Hausdorff distance induced by ℓ. Let σ be a skew bicharacter of the Pontryagin dual $\widehat{G}$ of G, and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of such bicharacters converging pointwise to σ on $\widehat{G}$ and such that for all $n \in \mathbb{N}$, the bicharacter σ_n is lifted from a bicharater of $\widehat{H_n}$ — which we denote σ_n again. Then $(C^*(\widehat{H_n},\sigma_n))_{n\in\mathbb{N}}$ converges to $C^*(\widehat{G},\sigma)$ in the Rieffel-Gromov-Hausdorff distance.

GPS

Topological properties of the extended Monge-Kantorovich metric — Non-Compact Case

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Problem 1

The extended Monge-Kantorovich metric is not a metric in general.

Proof.

Let δ_x denote the Dirac measure at $x \in \mathbb{R}$. Let L be the Lipschitz seminorm associated with the usual metric on $\mathbb{R}$. $\square$

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Then:

$$\mathsf{mk}_{\mathsf{L}}\left(\delta_0, \sum_{n\in\mathbb{N}} 2^{-n-1}\delta_{2^{2n}}\right) = \infty.$$

$\square$

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For a non-compact locally compact metric space (X, m), the extended Monge-Kantorovich metric is less well-behaved.

Problem 2

The extended Monge-Kantorovich metric does not metrize the weak* topology, even on its closed balls.

Proof.

Working in $\mathbb{R}$ again, we have:

$$\forall n \in \mathbb{N} \quad \mathsf{mk}_{\mathsf{L}}\left(\delta_0, \frac{n}{n+1}\delta_0 + \frac{1}{n+1}\delta_{n+1}\right) = 1$$

yet $\left(\delta_0, \frac{n}{n+1}\delta_0 + \frac{1}{n+1}\delta_{n+1}\right)_{n\in\mathbb{N}}$ weak* converges to δ_0. $\square$

Topological properties of the extended Monge-Kantorovich metric — Non-Compact Case

For a non-compact locally compact metric space (X, m), the extended Monge-Kantorovich metric is less well-behaved.

Problem 3

The state space of a non-unital C*-algebra is not weak* locally compact.

Topological properties of the extended Monge-Kantorovich metric — Non-Compact Case

For a non-compact locally compact metric space (X, m), the extended Monge-Kantorovich metric is less well-behaved.

Core Problem

All these matters are attributable to one main feature of the non-compact case: points, and more generally probability measures, can escape at infinity.

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Approach

In the noncompact situation, we have to somehow deal with the matter of escape at the boundary.

Bounded-Lipschitz Distances

A first possible approach to the noncompact case is:

Definition (Latrémolière, 2007)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair and $r > 0$. The Bounded-Lipschitz distance $\mathsf{bl}_{\mathsf{L},r}$ on the state space $\mathscr{S}(\mathfrak{A})$ is defined, for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, by:

$$\sup\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \|a\|_{\mathfrak{A}} \leq r, \mathsf{L}(a) \leq 1\}.$$

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Bounded-Lipschitz distances have two advantages:

1. They are indeed metrics on the state space,
2. If $\mathfrak{A} = C_0(X)$ where (X, m) is a locally compact metric space, and L is the associated Lipschitz seminorm, then the Bounded-Lipschitz distances metrize the weak* topology on $\mathscr{S}(C_0(X))$.

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Definition (Latrémolière, 2007)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair. Then L is called a quasi-Lip-norm on $\mathfrak{A}$ when $\mathsf{bl}_{\mathsf{L},1}$ metrizes the weak* topology on $\mathscr{S}(\mathfrak{A})$.

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What is a good characterization of quasi-Lip-norms?

The weakly uniform topology

General Principle

We characterize (quasi-)Lip-norms in terms of the topology of a “Lipschitz ball” — showing that the ball is totally bounded in the right topology. The insight is to introduce the right topology!

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Example

Rieffel's characterization of compact quantum metric space is based on a topological condition on some Lipschitz ball: namely $\{a \in \mathfrak{A} : \mathsf{L}(a) \leq 1, \|a\|_{\mathfrak{A}} \leq 1\}$ is norm precompact.

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Challenge

The matter is to define the proper topology on $\mathfrak{A}$ appropriate for the Bounded-Lipschitz situation.

Weakly Uniform Topology

Definition (Latrémolière, 2007)

Let $\mathfrak{A}$ be a C*-algebra. Let $\mathfrak{S}$ be the set of all weak* compact subsets of the state space $\mathscr{S}(\mathfrak{A})$. For $\mathscr{K} \in \mathfrak{S}$, we define:

$$\mathsf{P}_{\mathscr{K}} : a \in \mathfrak{A} \longmapsto \sup\{|\varphi(a)| : \varphi \in \mathscr{K}\}.$$

The weakly uniform topology on $\mathfrak{A}$ is the locally convex topology defined by the set $\{\mathsf{P}_{\mathscr{K}} : \mathscr{K} \in \mathfrak{S}\}$.

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Definition (Latrémolière, 2007)

With the same notations, let $q_{\mathscr{K}}$ be the seminorm defined on $\mathfrak{A}$ by $a \in \mathfrak{A} \longmapsto \sup\{|\varphi(aa^*)|, |\varphi(a^*a)| : \varphi \in \mathscr{K}\}$. The strongly uniform topology on $\mathfrak{A}$ is the locally convex topology defined by the set $\{q_{\mathscr{K}} : \mathscr{K} \in \mathfrak{S}\}$.

Metrizing the weakly uniform topology on bounded subsets

Theorem (Latrémolière, 2007)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair, with $\mathfrak{A}$ separable.

1. *If $\mathfrak{A}$ is unital, then the weakly uniform topology is the norm topology,*
2. *The weakly uniform topology is always stronger than the weak topology and weaker than the strict topology (with examples for strict comparison when $\mathfrak{A}$ is not unital),*
3. *If $\mathfrak{B} \subseteq \mathfrak{A}$ is a norm bounded subset of $\mathfrak{A}$, then for any strictly positive element $h \in \mathfrak{A}$, the metric $(a, b) \in \mathfrak{B}^2 \mapsto \|h(b-a)h\|_{\mathfrak{A}}$ metrizes the weakly uniform topology of $\mathfrak{A}$ restricted to $\mathfrak{B}$,*
4. *The strongly uniform topology is stronger than the weakly uniform topology and restricts to the strict topology on bounded sets in $\mathfrak{A}$.*

Characterizations of Quasi-Lip-norms

Theorem (Latrémolière, 2007)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair and let:

$$\mathfrak{B} = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leq 1 \text{ and } \|a\|_{\mathfrak{A}} \leq 1\}.$$

Then the following are equivalent:

1. *L is a quasi-Lip-norm,*
2. *The set $\mathfrak{B}$ is totally bounded in the weakly uniform topology,*
3. *For some strictly positive element $h \in \mathfrak{A}$ the set $h\mathfrak{B}h$ is norm precompact,*
4. *For all strictly positive elements $h \in \mathfrak{A}$, the set $h\mathfrak{B}h$ is norm precompact.*

Finite Radius Metric Spaces

The Bounded-Lipschitz distances are particularly applicable to bounded-radii metric spaces.

Theorem (Latrémolière, 2007)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair. If the extended Monge-Kantorovich metric associated with L gives $\mathscr{S}(\mathfrak{A})$ finite radius, then it agrees with some Bounded-Lipschitz distance.

Finite Radius Metric Spaces

The Bounded-Lipschitz distances are particularly applicable to bounded-radii metric spaces.

Corollary

Let $(\mathfrak{A}, \mathsf{L})$ be a separable Lipschitz pair such that the set:

$$\mathfrak{B} = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leq 1\}$$

is norm bounded. The distance mk_{L} metrizes the weak* topology on $\mathscr{S}(\mathfrak{A})$ if and only if there exists some strictly positive element $h \in \mathfrak{A}$ such that the set $h\mathfrak{B}h$ is norm precompact.

Some Examples

Example (Latrémolière, 2007)

Let (X, m) be a locally compact separable metric space and let $\mathfrak{K}$ be the C*-algebra of compact operators on a separable Hilbert space. Let $\mathfrak{A} = C_0(X, \mathfrak{K})$.

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Then $\max\{\mathsf{L}, \|\cdot\|_{\mathfrak{A}}\}$ is a quasi-Lip-norm on $\mathfrak{A}$.

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Then $\max\{\mathsf{L}, \|\cdot\|_{\mathfrak{A}}\}$ is a quasi-Lip-norm on $\mathfrak{A}$.

This includes some examples of C*-crossed-products $C_0(Y) \rtimes_\alpha \mathbb{Z}$, with (Y, m) locally compact separable metric, α free and proper, and $H_3\left(Y/_\alpha, \mathbb{Z}\right) = 0$.

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This can be generalized to more interesting continuous fields of compact operators, thus including more examples of C*-crossed-products.

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Then $\max\{\mathsf{L}, \|\cdot\|_{\mathfrak{A}}\}$ is a quasi-Lip-norm on $\mathfrak{A}$.

This also includes the C*-algebra of the noncommutative space-time of Doplicher and Roberts, as shown by Rieffel.

Some Examples: Proof for some fields of compacts

Proof.

1. Let $\mathscr{H}$ be a separable Hilbert space, $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathscr{H}$ and P_n the projection on $\operatorname{span}\{e_0,\ldots,e_n\}$ for all $n \in \mathbb{N}$.

$\square$

Some Examples: Proof for some fields of compacts

Proof.

1. Let $\mathscr{H}$ be a separable Hilbert space, $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathscr{H}$ and P_n the projection on $\mathrm{span}\{e_0,\ldots,e_n\}$ for all $n \in \mathbb{N}$.
2. Let $f \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}(f) \leq 1$. For all $p,q \in \mathbb{N}$ we have:

$$\sup\left\{\frac{\langle (f(x)-f(y))e_p, e_q\rangle}{m(x,y)} : x,y \in X, x \neq y\right\} \leq \mathsf{L}(f) \leq 1$$

and $\|f\|_{\mathfrak{K}} \leq 1$.

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and $\|f\|_{\mathfrak{K}} \leq 1$.
3. Since $P_n\mathfrak{A}P_n = C_0(\mathfrak{A}, \mathfrak{M}_n)$, the set $P_n\{f \in \mathfrak{A} : \max\{\mathsf{L}(f), \|f\|_{\mathfrak{A}}\} \leq 1\}P_n$ is compact by Arzéla-Ascoli.

$\square$

Bellissard-Marcolli-Reihani spectral triples

Let α be an action of $\mathbb{Z}$ on $\mathfrak{A}$ and $(\mathfrak{A}, \mathscr{H}_\pi, D)$ be a sprectral triple on $\mathfrak{A}$ such that:

$$\mathsf{L} : a \in \mathfrak{A} \longmapsto \|[D, \pi(a)]\|_{\mathscr{H}}.$$

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A natural idea to define a spectral triple on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is to use the representation on $\ell^2(\mathbb{Z}, \mathscr{H})$:

$$\Pi \left| \begin{array}{rcl} a \in \mathfrak{A} & \longmapsto & [n \in \mathbb{Z} \mapsto \pi(\alpha^n(a))] \\ U \in \mathfrak{A} \rtimes_\alpha \mathbb{Z} & \longmapsto & u \otimes 1 \end{array} \right.$$

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Problem: If the action α is not quasi-isometric, then this construction does not lead to a spectral triple.

Bellissard-Marcolli-Reihani spectral triples

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Bellissard,Marcolli and Reihini suggests to replace $\mathfrak{A}$ be the space of a metric bundle. As only the orbit of one metric is important, we work on $\mathfrak{A} \otimes c_0(\mathbb{Z})$.
Choosing the right differential calculus on $c_0(\mathbb{Z})$, so that we get in particular a finite diameter metric on $\mathbb{Z}$, one can build a spectral triple on $(\mathfrak{A} \otimes c_0(\mathbb{Z})) \rtimes_\beta \mathbb{Z}$ by modifying appropriately the above idea. This gives examples of bounded quantum metric spaces.

GPS

The need to deal with the extended Monge-Kantorovich metric

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1. The Bounded-Lipschitz metrics only capture local metric information. This is a problem for Gromov-Hausdorff convergence and does not fit well many natural examples, for instance from physics.
2. The Bounded-Lipschitz metrics framework and the compact metric space framework are not part of a larger coherent theory. For instance, $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leq 1\}$ is never norm bounded in the compact setting (as it contains $\mathbb{R}1_{\mathfrak{A}}$.

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3. This is a problem which we should somehow understand!

Dobrushin-tightness

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Theorem (Prohorov, 1956)

Let X be a locally compact Hausdorff space. The weak closure of a set $\mathscr{P}$ of Borel probability measures on X is a subset of $\mathscr{S}(C_0(X))$ if and only if $\mathscr{P}$ is tight, meaning:*

$$\forall\varepsilon > 0 \exists K \subseteq X \text{ compact} \quad \forall \mathbb{P} \in \mathscr{P} \quad \mathbb{P}(X \setminus K) < \varepsilon.$$

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Problem

There are weak* compact subsets of $\mathscr{S}(C_0(X))$ for which the restriction of the extended Monge-Kantorovich metric is still ill-behaved.

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Question

What is a metric notion of tightness?

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Let (X, m) be a metric space and mk be the associated extended Monge-Kantorovich metric on $\mathscr{S}(C_0(X))$. Let $\mathscr{P} \subseteq \mathscr{S}(C_0(X))$. Then the topology of $(\mathscr{P}, \mathsf{mk})$ is the weak topology if $\mathscr{P}$ is Dobrushin-tight, namely, if there exists $x_0 \in X$ such that:*

$$\lim_{r\to\infty} \sup_{\mathbb{P}\in\mathscr{P}} \int_{\{x\in X: d(x_0,x)\geq r\}} d(x_0, x)\, d\mathbb{P}(x) = 0.$$

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Remark: Dobrushin-tightness does not depend on the choice of a base point x_0 thanks to the triangle inequality.

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Problem: This is a mix of local behavior and behavior at infinity.

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$$\lim_{r\to\infty} \sup_{\mathbb{P}\in\mathscr{P}} \int_{\{x\in X : d(x_0,x)\geq r\}} d(x_0, x)\, d\mathbb{P}(x) = 0.$$

Problem: The function $x \mapsto d(x_0, x)$ is not in $C_0(X)$ — and we have no equivalent in the noncommutative world for this function.

Locality and Going to Infinity in C*-algebras

Our requirements:

1. The extended Monge-Kantorovich metric should be well-behaved on sets of states which have, somehow, a controlled behavior "at infinity". So we seek a notion of convergence at infinity within a C*-algebra.

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Both these notions are ill-defined within the general context of noncommutative C*-algebras.

Our Solution

We isolate a set of commuting observables large enough.

Topographic quantum spaces

Definition (Latrémolière, 2012)

A topographic quantum space $(\mathfrak{A}, \mathfrak{M})$ is a C*-algebra $\mathfrak{A}$ and an Abelian C*-subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ containing an approximate unit for $\mathfrak{A}$.

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Our terminology is inspired by the following example. Let X be a locally compact metric space, and let h be a strictly positive function on X, seen as an "altitude function": we see the level sets of h as drawing a topographic map on X. Our definition is more general to handle non-σ-compact spaces.

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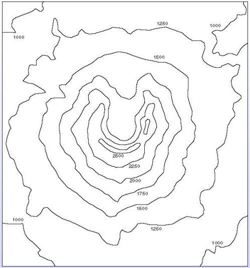

Figure: A topographic map of Saint-Helens volcano

Tight Sets of States

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Tight Sets of States

A topographic quantum space allows us to define local concepts as well as control behavior at infinity. For any Abelian C*-algebra, let $\sigma(\mathfrak{M})$ be its Gel'fand spectrum. For any space X, let $\mathcal{K}(X)$ be the directed set of all the compact subsets of X (inclusion is dual order).
An important first example of the use of topography is the concept of tight sets of states.

Definition (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathfrak{M})$ be a topographic quantum space. A subset $\mathscr{K}$ of $\mathscr{S}(\mathfrak{A})$ is tight when:

$$\forall \varepsilon > 0 \quad \exists Q \in \mathcal{K}(\sigma(\mathfrak{M})) \quad \forall K \in \mathcal{K}(\sigma(\mathfrak{M})) \\ (Q \subseteq K) \implies \sup\{\varphi(1_{\mathfrak{A}} - \chi_K)\} < \varepsilon.$$

Weak* Compactness and Tightness

As an analogue of the commutative setting:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathfrak{M})$ be a topographic quantum metric space. A subset $\mathscr{K}$ of $\mathscr{S}(\mathfrak{A})$ is tight if and only if its weak closure is a weak* compact subset of $\mathscr{S}(\mathfrak{A})$.*

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Proof.

Assume $\mathscr{K}$ is tight. Let $(\varphi_\alpha)_{\alpha \in I}$ be a net in $\mathscr{K}$ weak* converging to $\psi \in \mathfrak{A}^*$. Note that ψ is a continuous positive linear functional of norm at most 1, so it is sufficient to show that $\|\psi\|_{\mathfrak{A}^*} \geq 1$.

$\square$

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Proof.

Let $a \in \mathfrak{M}$ with $\|a\|_{\mathfrak{A}} \leq 1$. Then for all $\alpha \in I$ and $K \in \mathcal{K}(\sigma(\mathfrak{M}))$:

$$\begin{aligned} |1 - \psi(a)| &\leq |1 - \varphi_\alpha(a)| + |\varphi_\alpha(a) - \psi(a)| \\ &\leq |\varphi_\alpha(1_{u\mathfrak{A}} - \chi_K)| + |\varphi_\alpha(\chi_K - a)| + |\varphi_\alpha(a) - \psi(a)|. \end{aligned}$$

$\square$

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Proof.

Since $(\mathfrak{A}, \mathfrak{M})$ is a topographic quantum space, we can build an approximate unit $(a_\beta)_{\beta \in J}$ for $\mathfrak{A}$ in $\mathfrak{sa}(\mathfrak{M})$ and a net $(K_\beta)_{\beta \in J}$ such that for all $\beta \in J$, we have $a_\beta \chi_{K_\beta} = \chi_{K_\beta}$ and $0 \leq a_\beta \leq 1$. Moreover, for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ there exists $\beta \in J$ with $K = K_\beta$. (This uses Urysohn's lemma and some care.) $\square$

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Proof.

Let $\epsilon > 0$ and $\varepsilon = \min\{\epsilon, 1\}$. Since $\mathscr{K}$ is tight, there exists $\beta \in J$ such that:

$$\sup_{\varphi \in \mathscr{K}} \varphi(1_{\mathfrak{u}\mathfrak{A}} - \chi_{K_\beta}) \leq \frac{1}{9}\varepsilon^2 \leq \frac{1}{3}\varepsilon.$$

$\square$

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Proof.

Moreover for all $\alpha \in I$, by Cauchy-Schwarz's Inequality, we have:

$$|\varphi_\alpha(\chi_{K_\beta} - a_\beta)| = |\varphi_\alpha(\chi_{K_\beta} a_\beta - a_\beta)| \leq \sqrt{\varphi_\alpha(1_{\mathfrak{uA}} - \chi_{K_\beta})\varphi_\alpha(a_\beta^2)}$$
$$\leq \sqrt{\varphi_\alpha(1_{\mathfrak{uA}} - \chi_{K_\beta})} \leq \frac{1}{3}\varepsilon.$$

$\square$

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Proof.

Hence there exists $\beta \in J$ such that for all $\alpha \in I$ we have:

$$|1 - \psi(a_\beta)| \leq \frac{2}{3}\varepsilon + |\varphi_\alpha(a_\beta)) - \psi(a_\beta)|.$$

□

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Proof.

Last, by weak convergence and since $a_\beta \in \mathfrak{A}$, there exists $\alpha \in I$ such that $|\varphi_\alpha(a_\beta) - \psi(a_\beta)| \leq \frac{1}{3}\varepsilon$. Thus:

$$\forall \varepsilon > 0 \exists \beta \in J \quad |1 - \psi(a_\beta)| \leq \varepsilon \leq \epsilon,$$

so ψ is a state of $\mathfrak{A}$ since $(a_\beta)_{\beta \in J}$ is an approximate identity of $\mathfrak{A}$. $\square$

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Proof.

Conversely, assume that the weak* closure of $\mathscr{K}$ is a weak* compact subset of $\mathscr{S}(\mathfrak{A})$. Since $\mathscr{K}$ is a subset of its closure, it is enough to assume $\mathscr{K}$ is weak* compact — as a subset of a tight set is also tight.

$\square$

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Proof.

Let $(a_\alpha)_{\alpha \in I}$ and $(K_\alpha)_{\alpha \in I}$ be our approximate unit. For all $a \in \mathfrak{A}$ we define

$$\Theta_{\mathscr{K}}(a) : \varphi \in \mathscr{K} \longmapsto \varphi(a).$$

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Proof.

The map $\Theta_{\mathscr{K}}$ takes elements of $\mathfrak{A}$ to complex-valued weak* continuous functions on $\mathscr{K}$, and, since $\mathscr{K}$ consists of states, i.e. positive linear functionals, $\Theta_{\mathscr{K}}$ is increasing on $\mathfrak{sa}(\mathfrak{A})$.

$\square$

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Proof.

On the other hand, since for all $\alpha, \beta \in I$ with $\alpha \succ \beta$ we have $a_\alpha a_\beta = a_\beta$, we see that for all $x \in \sigma(\mathfrak{M})$, if $a_\beta(x) \neq 0$ then $a_\alpha(x) = 1$. Since $0 \leq a_\alpha, a_\beta \leq 1_{\mathfrak{u}\mathfrak{A}}$, we see that $(a_\alpha)_{\alpha \in I}$ is an increasing net in $\mathfrak{sa}(\mathfrak{A})$. $\square$

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Proof.

Therefore, $(\Theta_{\mathscr{K}}(1_{\mathfrak{uA}} - a_\alpha))_{\alpha \in I}$ is a net of continuous functions on the compact $\mathscr{K}$ pointwise decreasing and pointwise convergent to the continuous function 0 on $\mathscr{K}$. Hence by Dini's theorem, $(\Theta_{\mathscr{K}}(1_{\mathfrak{uA}} - a_\alpha))_{\alpha \in I}$ uniformly converges to 0. $\square$

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Proof.

In other words:

$$\lim_{\alpha \in I} \sup\{\varphi(1_{\mathfrak{u}\mathfrak{A}} - a_\alpha) : \varphi \in \mathscr{K}\} = 0.$$

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Proof.

Now, let $\varepsilon > 0$. Let $\alpha \in I$ such that $\sup\{|\varphi(1_{u\mathfrak{A}} - a_\alpha)| : \varphi \in \mathscr{K}\} \leq \varepsilon$. By definition of a_α, we have $\chi_{K_\beta} \geq a_\alpha$ for any $\beta \in I, \beta \succ \alpha$. Fix such a $\beta \in I, \beta \succ \alpha$. Then for any $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $K_\beta \subseteq K$ we have

$$1_{u\mathfrak{A}} - \chi_K \leq 1_A - \chi_{K_\beta} \leq 1_{u\mathfrak{A}} - a_\alpha$$

$\square$

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Proof.

Hence:

$$\sup\{\varphi(1_{\mathfrak{uA}} - \chi_K) : \varphi \in \mathscr{K}\} \leq \varepsilon,$$

as desired. $\square$

The Local State Space

While the state space is not well-behaved for the extended Monge-Kantorovich metric, a smaller set has better properties.

Definition (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathfrak{M})$ be a topographic quantum space. Let $K \in \mathcal{K}(\sigma(\mathfrak{M}))$. We define:

$$\mathscr{S}(\mathfrak{A}|K) = \{\varphi \in \mathscr{S}(\mathfrak{A}) : \varphi(\chi_K) = 1\}.$$

The local state space of $(\mathfrak{A}, \mathfrak{M})$ is:

$$\mathscr{S}(\mathfrak{A}|\mathfrak{M}) = \bigcup\{\mathscr{S}(\mathfrak{A}|K) : K \in \mathcal{K}(\sigma(\mathfrak{M}))\},$$

and its elements are called local states of $(\mathfrak{A}, \mathfrak{M})$.

The Local State Space is Norm-Dense

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Proof.

Let $\varphi \in \mathscr{S}(\mathfrak{A})$ and $\epsilon > 0$. Let $\varepsilon = \min\left\{\frac{1}{2}\epsilon, \frac{1}{2}\right\}$. Since $\{\varphi\}$ is weak* compact in $\mathscr{S}(\mathfrak{A})$, it is a tight set, so, there exists $K_0 \in \mathcal{K}(\sigma(\mathfrak{M}))$ such that for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $K_0 \subseteq K$, we have $1 - \varepsilon^2 \leq \varphi(\chi_K) \leq 1$. Since $0 \leq \varepsilon \leq \frac{1}{2}$, we thus have $|1 - \varphi(\chi_K)| \leq \varepsilon^2 \leq \varepsilon$ as well. $\square$

The Local State Space is Norm-Dense

Theorem (Latrémolière, 2012)

The local state space $\mathscr{S}(\mathfrak{A}|\mathfrak{M})$ of a topographic quantum space is norm dense in $\mathscr{S}(\mathfrak{A})$.

Proof.

Let $K \in \mathcal{K}(\sigma(\mathfrak{M}))$, with $K_0 \subseteq K$. Let $\psi_K : a \in \mathfrak{A} \mapsto \varphi(\chi_K)^{-1}\varphi(\chi_K a \chi_K)$ (note that $\varphi(\chi_K) > 0$). Then by construction, $\psi_K(\chi_K) = 1$ and ψ_K is a positive functional on $\mathfrak{u}A$, with $\|\psi_K\|_{\mathfrak{A}^*} = \psi_K(1_{\mathfrak{u}\mathfrak{A}}) = \psi_K(\chi_K) = 1$. Hence $\psi_K \in \mathscr{S}(\mathfrak{A}|K)$. $\square$

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The local state space $\mathscr{S}(\mathfrak{A}|\mathfrak{M})$ of a topographic quantum space is norm dense in $\mathscr{S}(\mathfrak{A})$.

Proof.

On the other hand, let $a \in \mathfrak{A}$ be given. Then:

$$\begin{aligned}
|\varphi(a) - \psi_K(a)| &\leq |\varphi(a) - \varphi(\chi_K)^{-1}\varphi(a)| \\
&\quad + \varphi(\chi_K)^{-1}|\varphi(a) - \varphi(\chi_K a \chi_K)| \\
&\leq \frac{\varepsilon}{1-\varepsilon}|\varphi(a)| \\
&\quad + \frac{1}{1-\varepsilon}\sqrt{\varphi(1-\chi_K)}\|a\|_{\mathfrak{A}}
\end{aligned}$$

$\square$

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The local state space $\mathscr{S}(\mathfrak{A}|\mathfrak{M})$ of a topographic quantum space is norm dense in $\mathscr{S}(\mathfrak{A})$.

Proof.

Hence, for all $a \in \mathfrak{A}$:

$$\begin{aligned} |\varphi(a) - \psi_K(a)| &\leq \frac{2\varepsilon}{1-\varepsilon}\|a\|_{\mathfrak{A}} \\ &\leq \epsilon\|a\|_{\mathfrak{A}}. \end{aligned}$$

Hence $(\psi_K)_{K\in\mathcal{K}(\sigma(\mathfrak{M}))}$ converges to φ in norm. ∎

Lipschitz triples

We now define objects which share the signature of quantum locally compact metric spaces by bringing together our two base notions:

Definition (Latrémolière, 2012)

A Lipschitz triple $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ and a topographic quantum space $(\mathfrak{A}, \mathfrak{M})$.

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Definition (Latrémolière, 2012)

A Lipschitz triple $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is regular when for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$, the set $\mathscr{S}(\mathfrak{A}|K)$ has finite diameter for the extended Monge-Kantorovich metric mk_{L} associated with $(\mathfrak{A}, \mathsf{L})$.

An Alternative Expression for mk_{L}

Our description of quantum locally compact metric spaces relies on an alternative description of the extended Monge-Kantorovich metric.

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Definition (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair. Then, for any state $\mu \in \mathscr{S}(\mathfrak{A})$, we define:

$$\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu) = \{a \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A}) : \mathsf{L}(a) \leq 1, \mu(a) = 0\}.$$

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Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair. For any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, we have:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \{|\varphi(a) - \psi(a)| : a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)\}.$$

Tame Sets of States

The key concept of this talk is the following noncommutative analogue of Dobrushin tight sets of states.

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Definition (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a Lipschitz triple. A subset $\mathscr{K}$ of $\mathscr{S}(\mathfrak{A})$ is tame when there exists $\mu \in \mathscr{S}(\mathfrak{A}|\mathfrak{M})$ such that:

$$\lim_{K \in \mathcal{K}(\sigma(\mathfrak{M}))} \sup\{|\varphi(a - \chi_K a \chi_K)| : \varphi \in \mathscr{K}, a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)\} = 0.$$

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Example

For any $K \in \mathcal{K}(\sigma(\mathfrak{M}))$, the set $\mathscr{S}(\mathfrak{A}|K)$ is tame in $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$.

Tame Implies Tight

A fundamental property of tame sets is:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a Lipschitz triple. A tame subset $\mathscr{K}$ of $\mathscr{S}(\mathfrak{A})$ is tight.

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Proof.

Let $\mathscr{K}$ be a tame subset of $\mathscr{S}(\mathfrak{A})$. Let $\varepsilon > 0$. Let $U \subseteq \sigma(\mathfrak{M})$ be a nonempty open set with compact closure and $x \in U$. Since $\{x\}$ is compact, there exists by Urysohn's Lemma for locally compact Hausdorff spaces a continuous function $f \in \mathfrak{sa}(\mathfrak{M})$ with $0 \leq f \leq 1_{\mathfrak{uA}}, f(x) = 1$ and $f\chi_{\sigma(\mathfrak{M})\setminus U} = 0$. Let $g = 1_{\mathfrak{uA}} - f \in \mathfrak{sa}(\mathfrak{u}M)$ and note that $(1_{\mathfrak{uA}} - \chi_K)g = 1_{\mathfrak{uA}} - \chi_K$ for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $U \subseteq K$.

$\square$

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Proof.

Now, since $\{a \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ is norm dense in $\mathfrak{sa}(\mathfrak{u}\mathfrak{A})$, there exists $b \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A})$ with $\mathsf{L}(b) < \infty$ and $\|g - b\|_{\mathfrak{u}\mathfrak{A}} < \frac{1}{3}\varepsilon$. Let $\varphi \in \mathscr{S}(\mathfrak{A})$ and set $\lambda = \max\{\mathsf{L}(b), 1\} \in [1, \infty) \subseteq \mathbb{R}$. Let $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $U \subseteq K$.

$\square$

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Proof.

Then:

$$\begin{aligned}|\varphi(1_A - \chi_K)| &= |\varphi((1_{u\mathfrak{A}} - \chi_K)g)| \\ &= |\varphi(g - \chi_K g)| = |\varphi(g - \chi_K g \chi_K)| \\ &\leq |\varphi(b - \chi_K b \chi_K)| + |\varphi(b - g)| \\ &+ |\varphi(\chi_K(g - b)\chi_K)| \\ &\leq \lambda|\varphi(\lambda^{-1}b - \chi_K \lambda^{-1} b \chi_K)| + \frac{2}{3}\varepsilon.\end{aligned}$$

□

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Proof.

Since $\mathscr{K}$ is tame, there exists $K_0 \in \mathcal{K}(\sigma(\mathfrak{M}))$ and $\mu \in \mathscr{S}(\mathfrak{A}|\mathfrak{M})$ so that for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $K_0 \subseteq K$ we have:

$$\sup\{|\varphi(a - \chi_K a \chi_K)| : \varphi \in \mathscr{K} \text{ and } a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)\} \leq \frac{1}{3\lambda}\varepsilon.$$

$\square$

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Proof.

Thus, since $\mathsf{L}(\lambda^{-1}b) \leq 1$, we conclude that for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $\overline{U} \cup K_0 \subseteq K$ we have:

$$\sup\{\varphi(1_{\mathfrak{u}\mathfrak{A}} - \chi_K) : \varphi \in \mathscr{K}\} \leq \varepsilon,$$

as desired. $\square$

Regularity and Tame Sets

Regularity brings two important results:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a regular Lipschitz triple. Then a subset $\mathscr{K}$ of $\mathscr{S}(\mathfrak{A})$ is tame if and only if for all local states μ of $(\mathfrak{A}, \mathfrak{M})$, we have:

$$\lim_{K \in \mathcal{K}(\sigma(\mathfrak{M}))} \sup\{|\varphi(a - \chi_K a \chi_K)| : \varphi \in \mathscr{K}, a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)\} = 0.$$

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Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a regular Lipschitz triple. Then a tame subset of $\mathscr{S}(\mathfrak{A})$ lies within some (finite radius) closed ball centered at some (and therefore all) local state.

We are now ready to define the concept at the core of this work:

Definition (Latrémolière, 2012)

A quantum locally compact metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a regular Lipschitz triple such that, for all tame subset $\mathscr{K}$ of $\mathscr{S}(\mathfrak{A})$, the topology of $(\mathscr{K}, \mathsf{mk}_{\mathsf{L}})$ is the weak* topology restricted to $\mathscr{K}$.

Quantum Locally Compact Metric Spaces

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A quantum locally compact metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is separable when $\mathfrak{A}$ is separable.

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This notion is a different from the notion of W^*-metric, introduced by Kuperberg and Weaver.

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A quantum locally compact metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is separable when $\mathfrak{A}$ is separable.

This definition presents some obvious difficulty: one would not expect to be able to describe all tame sets, beyond the given definition. A C*-algebraic characterization is certainly desirable.

Metric Structure on the Topography

As an important object for the study of Gromov-Hausdorff convergence, we note:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a quantum locally compact metric space. For any state μ of $\mathfrak{M}$, let $\mathscr{S}(\mathfrak{A}|\mu)$ be the set of all states of $\mathfrak{A}$ whose restriction to $\mathfrak{M}$ is μ. For any $\mu, \nu \in \mathscr{S}(\mathfrak{M})$, we set:

$$\mathsf{m}(\mu,\nu) = \inf\{\mathsf{mk}_{\mathsf{L}}(\varphi,\psi) : \varphi \in \mathscr{S}(\mathfrak{A}|\mu), \psi \in \mathscr{S}(\mathfrak{A}|\nu)\}.$$

Then m is a distance on $\mathscr{S}(\mathfrak{M})$ and $(\mathfrak{M}, \mathfrak{M}, \mathsf{m})$ is a quantum locally compact metric space. Moreover, the topology of $(\sigma(\mathfrak{M}), \mathsf{m})$ is the standard (weak) topology.*

The Topographic-Uniform Topology

The strategy is to find a topology on C*-algebras which can be used to characterize quantum locally compact metric space.

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Definition (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathfrak{M})$ be a topographic quantum space. The topographic-uniform topology is the locally convex topology generated by the set of seminorms:

$$\mathsf{P}_K : a \in \mathfrak{A} \longmapsto \sup\{|\varphi(a)| : \varphi \in \mathscr{S}(\mathfrak{A}|K)\},$$

for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$.

Comparison of our Topologies

Our new topology was hidden in the Bounded-Lipschitz study, as the following shows:

Theorem (Latrémolière, 2012)

The topographic-uniform topology of a topographic quantum space $(\mathfrak{A}, \mathfrak{M})$ agrees on norm-bounded subsets of $\mathfrak{A}$ with the weakly uniform topology.

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The topographic-uniform topology of a topographic quantum space $(\mathfrak{A}, \mathfrak{M})$ agrees on norm-bounded subsets of $\mathfrak{A}$ with the weakly uniform topology.

The topographic uniform topology is however usually weaker than the weakly uniform topology.

Characterization of quantum locally compact metric spaces

Our main result is the following characterization of quantum locally compact metric spaces.

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a Lipschitz triple. The following are equivalent:

1. $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ *is a quantum locally compact metric space,*
2. *There exists a local state μ such that $\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$ is totally bounded in the topographic uniform topology,*
3. *For all local states μ the set $\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$ is totally bounded in the topographic uniform topology.*

Proof of the main theorem

Proof of Main Theorem.

Assume that $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a quantum locally compact metric space. Let μ be a local state. Let $K \in \mathcal{K}(\sigma(\mathfrak{M}))$. For any $a \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A})$ we define:

$$\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a) : \varphi \in \mathscr{S}(\mathfrak{A}|K) \longmapsto \varphi(a).$$

The map $\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a)$ is continuous from the weak* topology on $\mathscr{S}(\mathfrak{A}|K)$ to $\mathbb{R}$ for all $a \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A})$. Let $C(\mathscr{S}(\mathfrak{A}|K))$ be the real Banach space of real valued weak* continuous functions on $\mathscr{S}(\mathfrak{A}|K)$, with the supremum norm denoted by $\|\cdot\|_{C(\mathscr{S}(\mathfrak{A}|K))}$.

$\square$

Proof of the main theorem

Proof of Main Theorem.

Now, since $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a regular Lipschitz triple, the extended Monge-Kantorovich metric mk_{L} restricted to $\mathscr{S}(\mathfrak{A}|K)$ is in fact a metric, and there exists $r \in \mathbb{R}$ such that $\mathsf{mk}_{\mathsf{L}}(\varphi, \mu) \leq r$ for all $\varphi \in \mathscr{K}$ by regularity. Since μ is local, there exists $K' \in \mathcal{K}(\sigma(\mathfrak{M}))$ with $K \subseteq K'$ and $\mu \in \mathscr{S}(\mathfrak{A}|K')$, the latter being bounded for mk_{L}.

So, for all $a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$ we have:

$$\begin{aligned}\|\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a)\|_{C(\mathscr{S}(\mathfrak{A}|K))} &= \sup\{|\varphi(a)| : \varphi \in \mathscr{S}(\mathfrak{A}|K)\}\\ &= \sup\{|\varphi(a) - \mu(a)| : \varphi \in \mathscr{S}(\mathfrak{A}|K)\}\\ &\leq r.\end{aligned}$$

$\square$

Proof of the main theorem

Proof of Main Theorem.

Since $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a quantum locally compact metric space and $\mathscr{S}(\mathfrak{A}|K)$ is tame, the topology induced by mk_{L} on $\mathscr{S}(\mathfrak{A}|K)$ is the relative topology induced by the weak* topology. On the other hand, by definition of mk_{L}, we have, for all $a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$, that:

$$\begin{aligned} |\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a)(\varphi) - \Theta_{\mathscr{S}(\mathfrak{A}|K)}(a)(\psi)| &= |\varphi(a) - \psi(a)| \\ &\leq \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) \end{aligned}$$

so $\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a)$ is 1-Lipschitz on $(\mathscr{S}(\mathfrak{A}|K), \mathsf{mk}_{\mathsf{L}})$.

$\square$

Proof of the main theorem

Proof of Main Theorem.

Thus, the set:

$$\Theta_{\mathscr{S}(\mathfrak{A}|K)}(\mathfrak{L}_1(\mathfrak{A},\mathsf{L},\mathfrak{M})) = \{\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a) : a \in \mathfrak{L}_1(\mathfrak{A},\mathsf{L},\mu)\}$$

is an equicontinuous, bounded subset of $C(\mathscr{S}(\mathfrak{A}|K))$, and $\mathscr{S}(\mathfrak{A}|K)$ is weak* compact, so by the Arzéla-Ascoli theorem, $\Theta_{\mathscr{S}(\mathfrak{A}|K)}(\mathfrak{L}_1(\mathfrak{A},\mathsf{L},\mathfrak{M}))$ is totally bounded for $\|\cdot\|_{C(\mathscr{S}(\mathfrak{A}|K))}$. Now, we have:

$$\|\Theta_{\mathscr{S}(\mathfrak{A}|K)}(a)\|_{C(\mathscr{S}(\mathfrak{A}|K))} = \mathsf{P}_K(a)$$

for all $a \in \mathfrak{u}\mathfrak{A}$, so we conclude that $\mathfrak{L}_1(\mathfrak{A},\mathsf{L},\mathfrak{M})$ is totally bounded for P_K as desired. So $\mathfrak{L}_1(\mathfrak{A},\mathsf{L},\mu)$ is ***tu***-totally bounded. $\square$

Proof of the main theorem

Proof of Main Theorem.

Conversely, assume that $\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$ is ***tu***-totally bounded for some local state μ. First, since for all $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ the set $\chi_K \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mathfrak{M})\chi_K$ is totally bounded in $\mathfrak{A}^{**}$, hence bounded, in norm, the Lipschitz triple $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a regular. Let $\mathscr{K}$ be a $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$-tame subset of $\mathscr{S}(\mathfrak{A})$. The extended Monge-Kantorovich metric mk_{L} induces a finer topology on $\mathscr{K}$ than the weak* topology. It is thus sufficient to show that the weak* topology is finer than the metric topology induced by mk_{L}. $\square$

Proof of the main theorem

Proof of Main Theorem.

Let $(\varphi_\alpha)_{\alpha\in I}$ be a net in $\mathscr{K}$ indexed by a directed set I, and weak* convergent to φ in $\mathscr{K}$. The set $\Phi = \{\varphi_\alpha, \varphi : \alpha \in I\}$ is weak* compact by construction and tame, as it is a subset of a tame set.

We start with the key role than tameness plays in this proof. Since Φ is tame, there exists $K \in \mathcal{K}(\sigma(\mathfrak{M}))$ such that:

$$\sup\{|\psi(a - \chi_K a \chi_K)| : \psi \in \Phi, a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)\} \leq \frac{1}{9}\varepsilon.$$

$\square$

Proof of the main theorem

Proof of Main Theorem.

Since $\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is totally bounded for P_K, there exists a finite subset $F \subseteq \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$ such that:

$$\forall a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu) \quad \exists f(a) \in F \qquad \mathsf{P}_K(f(a) - a) \leq \frac{1}{9}\varepsilon.$$

We then have by definition that:

$$|\psi(\chi_K a \chi_K) - \psi(\chi_K f(a) \chi_K)| \leq \mathsf{P}_K(a - f(a))$$

for all $\psi \in \mathscr{S}(\mathfrak{A})$ and $a \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A})$.

$\square$

Proof of the main theorem

Proof of Main Theorem.

Thus for all $\psi \in \Phi$ and $a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$:

$$\begin{aligned}|\psi(a) - \psi(f(a))| \leq& |\psi(a) - \psi(\chi_K a \chi_K)| \\ &+ |\psi(\chi_K a \chi_K) - \psi(\chi_K f(a) \chi_K)| \\ &+ |\psi(\chi_K f(a) \chi_K - f(a))| \leq \frac{1}{3}\varepsilon.\end{aligned}$$

Let $\omega \in I$ be chosen so that if $\alpha \succ \omega$ and $b \in F$ then $|\varphi_\alpha(b) - \varphi(b)| \leq \frac{1}{3}\varepsilon$. This is of course possible since I is directed and F is finite.

$\square$

Proof of the main theorem

Proof of Main Theorem.

Now, for any $a \in \mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)$ and $\alpha \in I$ with $\alpha \succ \omega$, we have:

$$\begin{aligned} |\varphi_\alpha(a) - \varphi(a)| &\leq |\varphi_\alpha(a) - \varphi_\alpha(f(a))| \\ &+ |\varphi_\alpha(f(a)) - \varphi(f(a))| \\ &+ |\varphi(f(a)) - \varphi(a)| \\ &\leq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

Hence $\mathsf{mk}_{\mathsf{L}}(\varphi_\alpha, \varphi) \leq \varepsilon$ for $\alpha \succ \omega$, thus proving that the net $(\varphi_\alpha)_{\alpha \in I}$ converges to φ in for mk_{L} as desired. $\square$

Characterization of quantum locally compact metric spaces, 2

A theorem more amenable to C*-algebraic techniques is given by:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a Lipschitz triple. The following are equivalent:

1. *$(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a quantum locally compact metric space,*
2. *There exists a local state μ such that for all $a, b \in \mathfrak{M}$ with compact supports, the set $a\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)b$ is totally bounded in norm,*
3. *For all local states μ and for all $a, b \in \mathfrak{M}$ with compact supports, the set $a\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)b$ is totally bounded in norm.*

Characterization of quantum locally compact separable metric spaces

The separable case allows for a particularly nice characterization.

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a separable Lipschitz triple. The following are equivalent:

1. $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ *is a quantum locally compact separable metric space,*
2. *There exists a local state μ and a strictly positive element $h \in \mathfrak{A}$ such that the set $h\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)h$ is totally bounded in norm,*
3. *There exists a strictly positive element $h \in \mathfrak{A}$ such that for all local states μ the set $h\mathfrak{L}_1(\mathfrak{A}, \mathsf{L}, \mu)h$ is totally bounded in norm.*

Locally compact metric spaces

As a first step, the classical picture does fit in our framework:

Theorem (Latrémolière, 2012)

Let (X, m) be a locally compact metric space and let $C_0(X)$ be the C-algebra of complex valued continuous functions on X vanishing at infinity. Let Lip_m be the Lipschitz seminorm:*

$$\mathsf{Lip}_m : f \in C_0 \longmapsto \sup \left\{ \frac{|f(y) - f(x)|}{m(y,x)} : x, y \in X \text{ and } x \neq y \right\}.$$

The Lipschitz triple $(C_0(X), \mathsf{Lip}_m, C_0(X))$ is a quantum locally compact metric space.

Compact Quantum Metric Spaces

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair with $\mathfrak{A}$ unital.

An Abelian C*-subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ is a topography for $\mathfrak{A}$ if and only if $\mathfrak{M}$ is unital with the same unit as $\mathfrak{A}$.

Compact Quantum Metric Spaces

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair with $\mathfrak{A}$ unital.
TFAE:

1. $(\mathfrak{A}, \mathsf{L})$ is a compact quantum metric space,
2. $(\mathfrak{A}, \mathsf{L}, \mathbb{C}1_{u\mathfrak{A}})$ is a quantum locally compact metric space,
3. For all Abelian C*-subalgebras $\mathfrak{M}$ of $\mathfrak{A}$ with $1_{u\mathfrak{A}} \in \mathfrak{M}$, the Lipschitz triple $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a quantum locally compact separable metric space,
4. There exists an Abelian C*-subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ with $1_{u\mathfrak{A}} \in \mathfrak{M}$ such that the Lipschitz triple $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a quantum locally compact metric space,
5. There exists a strictly positive element $h \in \mathfrak{A}$ such that the Lipschitz triple $(\mathfrak{A}, \mathsf{L}, C^*(h))$ is a quantum locally compact metric space.

Bounded Quantum Metric Spaces

Our previous work on Bounded-Lipschitz distances led us to the tentative definition for a Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ to be a noncompact, bounded quantum separable metric space:

Boundedness Condition

$\mathfrak{L} = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leq 1\}$ is norm bounded,

Local Boundedness Condition

There exists a strictly positive element $h \in \mathfrak{A}$ such that $h\mathfrak{L}h$ is totally bounded in norm.

Bounded Quantum Metric Spaces

Again, our framework is compatible:

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair with $\mathfrak{A}$ not unital. Then the following are equivalent:

1. *$(\mathfrak{A}, \mathsf{L})$ satisfies the Boundedness Condition and the Local Boundedness Condition,*
2. *There exists a strictly positive element h such that $(\mathfrak{A}, \mathsf{L}, C^*(h))$ is a quantum locally compact separable metric space whose state space has finite diameter for the extended Monge-Kantorovich metric associated with $(\mathfrak{A}, \mathsf{L})$.*

Compact Operators C*-algebra

Theorem (Latrémolière, 2012)

Let $\mathfrak{K}$ be the C-algebra of compact operators on a separable Hilbert space $\mathcal{H}$. Let $(\zeta_n)_{n\in\mathbb{N}}$ be a Hilbert basis for $\mathcal{H}$. Let P_n be the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}\{\zeta_0,\ldots,\zeta_n\}$ for all $n\in\mathbb{N}$. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $\alpha>0$ such that for all $n\in\mathbb{N}$ we have $\omega_n\geq\alpha$. Define, for any $n\in\mathbb{N}$:*

$$\mathsf{L} : T\in\mathfrak{K}\longmapsto \sup\{\omega_n^{-1}\|P_n a P_n\|_{\mathfrak{K}} : n\in\mathbb{N}\}.$$

Let $\mathfrak{D}$ be the C-subalgebra of $\mathfrak{K}$ consisting of compact operators which are diagonal in $(\zeta_n)_{n\in\mathbb{N}}$, i.e. the C*-algebra generated by $(P_n)_{n\in\mathbb{N}}$. Note that $\mathfrak{D}$ is Abelian.*
Then $(\mathfrak{K},\mathsf{L},\mathfrak{D})$ is a quantum locally compact metric space.

Moyal Planes

Let $\theta \in \mathbb{R}, \theta > 0$ and:

$$\sigma_\theta : (p_1, q_1), (p_2, q_2) \in \mathbb{R}^2 \longmapsto \exp(2i\pi\theta(p_1 q_2 - p_2 q_1)).$$

Let $\mathfrak{M}_\theta = C^*(\mathbb{R}^2, \sigma_\theta)$. This is a Moyal Plane.

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The C*-algebra $\mathfrak{M}_\theta$ is *-isomorphic to the C*-algebra of compact operators on a separable Hilbert space.
Let f_{00} be the density of the centered Gaussian distribution on $\mathbb{R}^2$ of variance $\sqrt{\theta}$ and $f_{nm} = z^n f_{00} z^m$ for all $n, m \in \mathbb{N}$ and the product is taken in $\mathfrak{M}_\theta$. Then $(f_{nm})_{n,m\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R}^2)$. Let $\mathfrak{D}_\theta = C^*(f_{nn} : n \in \mathbb{N})$ and note $(\mathfrak{M}_\theta, \mathfrak{D}_\theta)$ is a topographic quantum space.

A Spectral Triple on Moyal Planes

Using the σ_θ-twisted convolution product, one can represent $\mathfrak{M}_\theta$ on $L^2(\mathbb{R}^2)$, and thus on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. Gayral, Gracia-Bondia, Iochum, Schücker and Varilly introduced the isospectral spectral triple on $\mathfrak{M}_\theta$ based on this representation, by setting:

$$D = -i\sqrt{2} \begin{bmatrix} 0 & \frac{\partial}{\partial \overline{z}} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix}$$

Let $\mathsf{L}_\theta : a \in \mathfrak{M}_\theta \mapsto \|[D, a]\|$.

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Let $\mathsf{L}_\theta : a \in \mathfrak{M}_\theta \mapsto \|[D,a]\|$. Using computations from Cagnache, D'Andrea, Martinneti and Wallet, we prove:

Theorem (Latrémolière, 2012)

The Lipschitz triple $(\mathfrak{M}_\theta, \mathsf{L}_\theta, \mathfrak{D}_\theta)$ is a quantum locally compact separable metric space for all $\theta \geq 0$