Reconstruction theorems in noncommutative Riemannian geometry

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West Coast Operator Algebra Seminar 2012
October 21, 2012

References

Published

A reconstruction theorem for almost-commutative spectral triples, Lett. Math. Phys. 100 (2012), no. 2, 181-202.

Erratum

Erratum of June 2012, incorporated into `arXiv:1101.5908v4`.

Preprint

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Dirac-type operators

Definition

Let

- (X, g) be a compact oriented Riemannian manifold,
- $\mathcal{E} \to X$ be a Hermitian vector bundle.

A *Dirac-type operator* D on $\mathcal{E} \to (X, g)$ is a symmetric first-order differential operator with D^2 Laplace-type, i.e., locally

$$D^2 = -g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \text{lower order terms.}$$

Example

- $\not{D}$ on the spinor bundle $\mathcal{S}$ for X spin;
- $d + d^*$ on $\wedge T^*X$ for general X.

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Spectral triples

Definition

A *spectral triple* is (A, H, D) for

- H a Hilbert space;
- A a unital $*$-subalgebra of $B(H)$;
- D a (densely-defined) self-adjoint operator on H such that
  - D has compact resolvent,
  - $[D, a] \in B(H)$ for all $a \in A$.

Example

For (X, g) cpct. orient. Riem., $\mathcal{E} \to X$ Herm., D Dirac-type on $\mathcal{E}$,

$$(A, H, D) := (C^\infty(X), L^2(X, \mathcal{E}), D).$$

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The reconstruction theorem

Theorem (Connes 1996, 2008; cf. Rennie–Várilly 2006)

Let (A, H, D) be a p-dimensional commutative spectral triple. Then there exists an compact oriented p-manifold X such that

$$A \cong C^\infty(X).$$

Moreover, if A'' acts on H with multiplicity $2^{\lfloor p/2 \rfloor}$, then

- *X is spin$^{\mathbb{C}}$,*
- *$H \cong L^2(X, \mathcal{S})$ for $\mathcal{S} \to X$ a spinor module,*
- *D is an ess. self-adjoint Dirac-type operator on $\mathcal{S}$.*

See [Gracia-Bondía–Várilly–Figueroa, Ch. 11] for reconstruction of the spinor bundle $\mathcal{S}$ and the Riemannian metric on X.

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Definition

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A spectral triple (A, H, D) with A commutative is called a *p-dimensional commutative spectral triple* if it satisfies the following:

Metric dimension (Weyl's Law)

$\lambda_k((D^2+1)^{-1/2}) = O(k^{-1/p})$;

Order one (First-order differential operator)

$[[D, a], b] = 0$ for all $a, b \in A$;

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Definition ctd.

Strong regularity (Smoothness, ellipticity)

- $A + [D, A] \subseteq \cap_k \operatorname{Dom} \delta^k$, where $\delta(T) := [|D|, T]$,
- $\operatorname{End}_A(\cap_k \operatorname{Dom} D^k) \subseteq \cap_k \operatorname{Dom} \delta^k$ as well;

Orientability

There exists an antisymmetric Hochschild cycle $c \in Z_p(A, A)$ such that for $\chi := \pi_D(c)$,

- $\chi = 1$ if p is odd,
- $\chi^* = \chi$, $\chi^2 = 1$, $[\chi, A] = \{0\}$, $\chi D = -D\chi$ if p even;

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Finiteness (Vector bundle)

$H_\infty := \cap_k \operatorname{Dom} D^k$ is finitely-generated projective A-module;

Absolute continuity (Integration, Hermitian metric)

The A-module H_∞ admits a Hermitian metric $(\cdot,\cdot)$ defined by

$$\langle \xi, \eta \rangle := \operatorname{Tr}^+\left((\xi,\eta)\,(D^2+1)^{-p/2} \right), \quad \forall \xi, \eta \in H_\infty.$$

Definition ctd.

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Problems and solutions

Definition is tailored to a spin$^{\mathbb{C}}$ Dirac operator on a spinor bundle:

- Orientability condition too strong in general;
- Definition too weak in general to recover Riemannian metric.

Weak orientability

There exists an antisymmetric Hochschild cycle $c \in Z_p(A, A)$ such that for $\chi := \pi_D(c)$,

- $\chi = \chi^*$, $\chi^2 = 1$, and $\chi a = a\chi$ for all $a \in A$,
- $\chi[D, a] = (-1)^{p+1}[D, a]\chi$ for all $a \in A$.

Dirac-type

For all $a \in A$, $[D, a]^2 \in A$.

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Refining the reconstruction theorem

Theorem (cf. Gracia-Bondía–Várilly–Figueroa, Ch. 11)

Let (A, H, D) be a p-dimensional (weakly orientable!) Dirac-type commutative spectral triple. Then there exist:

- *a compact oriented Riemannian p-manifold X,*
- *a Hermitian vector bundle $\mathcal{E} \to X$,*

such that, up to unitary equivalence,

$$(A, H, D) = (C^\infty(X), L^2(X, \mathcal{E}), D)$$

for D symmetric Dirac-type on $\mathcal{E}$.

The key technical lemma

Lemma

Let (A,H,D) be a p-dimensional commutative spectral triple, and let $\chi = \pi_D(c)$ be its chirality operator. Then there exists a self-adjoint $M \in \mathrm{End}_A(\cap_k \mathrm{Dom}\, D^k)$ such that for

$$D' = \begin{cases} D - M & \text{if } p \text{ is even,} \\ \chi(D-M) & \text{if } p \text{ is odd,} \end{cases}$$

(A,H,D') is an orientable p-dimensional commutative spectral triple. In particular, $\mathrm{Dom}\, D^k = \mathrm{Dom}(D')^k$ for each k with comparable Sobolev norms, and

$$(D^2+1)^{-p/2} - ((D')^2+1)^{-p/2} \in \mathcal{L}^1(H).$$

The conventional definition

Definition

Let:

- X be a compact spin n-manifold with spinor bundle $\mathcal{S}$, Dirac operator $\not{D}$, and chirality χ;
- (A_F, H_F, D_F) be a finite spectral triple.

Then the spectral triple $X \times F$ is called an *almost-commutative spectral triple*, where, e.g.,

$$X \times F := (C^\infty(X) \otimes A_F, L^2(X, \mathcal{S}) \otimes H_F, \not{D} \otimes 1 + \chi \otimes D_F)$$

when n is even.

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when n is even.

Problems

- This definition explicitly requires the base manifold to be spin.
- This definition does not accomodate "non-trivial fibrations":
  - Topologically non-trivial NCG Einstein–Yang–Mills (Boeijink–v. Suijlekom 2011);
  - Noncommutative tori for θ rational;
  - *Projective spectral triples* on compact oriented Riemannian manifolds (Zhang 2009, cf. Mathai–Melrose–Singer);
  - NCG cosmological models with topologically nontrivial coupling to matter (Ć.–Marcolli–Teh 2012);
- In particular, inner fluctuations of the metric generally break almost-commutativity!

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A concrete definition

Definition

An *almost-commutative spectral triple* is a triple of the form

$$(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D),$$

where:

- X is a compact oriented Riemannian manifold;
- $\mathcal{E}$ is a Hermitian vector bundle on X;
- D is a Dirac-type operator on $\mathcal{E}$;
- $\mathcal{A}$ is a $*$-algebra sub-bundle of $\mathrm{End}^+_{\mathrm{Cl}(X)}(\mathcal{E})$.

An abstract definition

Definition

Let (A, H, D) be a spectral triple, and let B be a central unital $*$-subalgebra of A. Then (A, H, D) is called an *abstract almost-commutative spectral triple* with base B if:

- (B, H, D) is a Dirac-type commutative spectral triple;
- A is a finitely-generated projective B-module-$*$-subalgebra of $\mathrm{End}_B(\bigcap_k \mathrm{Dom}\, D^k)$;
- $[[D, b], a] = 0$ for all $a \in A$, $b \in B$.

A reconstruction theorem for almost-commutative triples

Theorem

Let (A, H, D) be a p-dimensional abstract almost-commutative spectral triple with base B. Then, up to unitary equivalence,

$$(A, H, D) = (C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), D),$$

where

- *X is a cpct. orient. Riemannian p-maniflod with $B = C^\infty(X)$;*
- *$\mathcal{E} \to X$ is a Hermitian vector bundle;*
- *D is symmetric Dirac-type on $\mathcal{E}$;*
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Real almost-commutative spectral triples

Definition

Let (A, H, D, J) be a real spectral triple of *KO*-dimension n mod 8. Then (A, H, D, J) is called a *real almost-commutative spectral triple* if (A, H, D) is an abstract almost-commutative spectral triple with base

$$\tilde{A}_J := \{a \in A \mid Ja^*J^* = a\} .$$

- Can obtain the corresponding concrete definition.
- A manifold X admits real (almost-)commutative spectral triples of *arbitrary KO*-dimension.
- Metric dimension and *KO*-dimension are thus independent in general.

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Let (A, H, D, J) be a real spectral triple of *KO*-dimension $n \bmod 8$. Then (A, H, D, J) is called a *real almost-commutative spectral triple* if (A, H, D) is an abstract almost-commutative spectral triple with base

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