On the structure of vector bundles over noncommutative tori

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M. A. Rieffel: Projective modules over higher-dimensional noncommutative tori. Canad. J. Math., 1988.

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Schrödinger representation

- **translation** $T_x f(t) = f(t-x)$ for $x \in \mathbb{R}^d$, **modulation** $M_\omega f(t) = e^{2\pi i t\cdot\omega} f(t)$ for $\omega \in \widehat{\mathbb{R}}^d$
- **time-frequency shift** $\pi(x,\omega)f(t) = M_\omega T_x f(t)$ for $(x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$

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Spectrally invariant subalgebras of $C^*(\Lambda, c)$

$$\mathcal{A}^1_s(\Lambda, c) = \{A \in \mathcal{B}(L^2(\mathbb{R})) : A = \sum_\lambda a(\lambda)\pi(\lambda), \|\mathbf{a}\|_{\ell^1_s} < \infty\}$$

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Theorem:

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Replace $v_s = (1 + |x|^2 + |\omega|^2)^{s/2}$ by a submultiplicative weight. Then $\mathcal{A}^1_v(\Lambda, c)$ and $\mathcal{A}^\infty(\Lambda, c)$ is spectrally invariant in $C^*(\Lambda, c)$ if and only if v is a Gelfand-Raikov-Shilov-weight, i.e. $\lim_{n\to\infty} v(n\lambda)^{1/n} = 1$ for all $\lambda \in \Lambda$.

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- Suppose V is a Hilbert $\mathcal{A}$-module. Then a module mapping $T : V \to V$ is **adjointable**, if there is a mapping $T^* : V \to V$ such that

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Frames for Hilbert C^*-modules

For the sake of simplicity I restrict my discussion to the case of **finitely** generated Hilbert C^*-modules.

- Let $\mathcal{A}$ be a unital C^*-algebra. A sequence $\{g_j : j = 1, ..., n\}$ in a (left) Hilbert $\mathcal{A}$-module ${}_{\mathcal{A}}V$ is called a **standard module frame** if there are positive reals C, D such that

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- Suppose ${}_{\mathcal{A}}V$ is a finitely generated Hilbert $\mathcal{A}$-module. Then any set of generators $\{g_j : j = 1, ..., n\}$ is a standard module frame. The number of the shortest frame gives the number of factors of $\mathcal{A}^n$ into which ${}_{\mathcal{A}}V$ is embeddable.
- In other words, the positive module operator $S = \sum_{j=1}^{n} \Theta^{\mathcal{A}}_{g_j,g_j}$ is invertible and the upper and lower frame bounds are given by $\|S\|^2$ and $\|S^{-1}\|^{-2}$.
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- In words, Condition (b) says that $\mathcal{A}$ acts by adjointable operators on $V_{\mathcal{B}}$ and that $\mathcal{B}$ acts by adjointable operators on ${}_{\mathcal{A}}V$,
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Recall that a unital subalgebra $\mathcal{A}$ of a unital C^*-algebra $\mathcal{B}$ with common unit is called *spectrally invariant*, if for $A \in \mathcal{A}$ with $A^{-1} \in \mathcal{B}$ one actually has that $A^{-1} \in \mathcal{A}$.

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$\mathcal{S}(\mathbb{R}^d)$ becomes a full left Hilbert $C^*(\Lambda, c)$-module ${}_\Lambda V$ w.r.t to right action on $\mathcal{S}(\mathbb{R}^d)$ and the inner product $\langle ., . \rangle_\Lambda$ when completed w.r.t. $\|g\|_\Lambda = \|\langle g, g\rangle_\Lambda\|_{\mathrm{op}}^{1/2}$.

- What C^*-algebra is Morita-Rieffel equivalent to $C^*(\Lambda, c)$?
- What operators are commuting with elements in $C^*(\Lambda, c)$?
- commutation relations:

$$\pi(\lambda)\pi(z)\pi(\lambda)^{-1}\pi(z)^{-1} = e^{2\pi i \Omega(z,\lambda)}$$

for $z = (x, \omega) \in \mathbb{R}^{2d}$ and $\lambda \in \Lambda$.

Adjoint lattice

Observe that the characters of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ are of the form $z \mapsto \chi_s(z) = e^{2\pi i \Omega(z,z')}$ for some $z' \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, where the standard symplectic form Ω of $z = (x, \omega)$ and $z' = (y, \eta)$ is defined by

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In terms of the Euclidean inner product $\langle ., . \rangle$ on $\mathbb{R}^{2d}$ the symplectic form Ω can be expressed as follows:

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There is an analogous result for the opposite C^*-algebra of $C^*(\Lambda^\circ, c)$, i.e. $C^*(\Lambda^\circ, \overline{c})$.

$$g \cdot \pi_{\Lambda^\circ}(\mathbf{b}) = \sum_{\lambda^\circ \in \Lambda^\circ} \pi(\lambda^\circ)^* g \overline{b}(\lambda^\circ) \quad \mathbf{b} \in \mathcal{S}(\Lambda^\circ), g \in \mathcal{S}(\mathbb{R}^d)$$

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Associativity condition

Poisson summation formula for the symplectic Fourier transform:

$$\sum_{\lambda\in\Lambda} F(\lambda) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \widehat{F}(\lambda^\circ).$$

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- If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$.
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Vector bundles over noncommutative tori

Theorem:

$C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$ are Morita-Rieffel equivalent. Moreover, $\mathcal{S}(\mathbb{R}^d)$ is an equivalence bimodule between $\mathcal{A}^\infty(\Lambda, c)$ and $\mathcal{A}^\infty(\Lambda^\circ, \overline{c})$.

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Frames for Hilbert and Banach spaces

Another useful reformulation of the notion of frames:
Let $g \otimes h$ be the rank one operator defined by $(g \otimes h)f = \langle f, g \rangle h$. Then

$$A \cdot I_{\mathcal{H}} \leq \sum_{j \in J} g_j \otimes g_j \leq B \cdot I_{\mathcal{H}},$$

the series converges in the strong operator topology.
Frames are of relevance because they allow the construction of (non-orthogonal) expansions.

- If $\{g_j : j \in J\}$ is a frame for $\mathcal{H}$, then $\{S^{-1} g_j : j \in J\}$ is a frame with frame bounds B^{-1}, A^{-1} $f \in \mathcal{H}$ and

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- If $\{g_j : j \in J\}$ is a frame for $\mathcal{H}$, then $\{S^{-1} g_j : j \in J\}$ is a frame with frame bounds B^{-1}, A^{-1} $f \in \mathcal{H}$ and

$$f = \sum_{j \in J} \langle f, S^{-1} g_j \rangle g_j. \quad (2)$$

Gabor frames

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Discrete reconstruction

$$
\begin{aligned}
f &= S^{-1}Sf = \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)S^{-1}g\rangle\pi(\lambda)g \\
&= SS^{-1}f = \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)S^{-1}g \\
&= S^{-1/2}SS^{-1/2}f = \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)S^{-1/2}g\rangle\pi(\lambda)S^{-1/2}g.
\end{aligned}
$$

- **canonical dual atom** $\tilde{g} := (S^{\Lambda}_{g,g})^{-1}g$
- **canonical tight atom** $h_0 := (S^{\Lambda}_{g,g})^{-1/2}g$

Transmission of signals – Denis Gabor

- Let $(a_k)_{k\in\mathbb{Z}}$ be a *word* and a *pulse* $g \in L^2(\mathbb{R})$. Then one sends
$$f(t) = \sum_{k\in\mathbb{Z}} a_k g(t-k).$$

- How to transmit a family $\mathbf{a} = (a^{(l)})_{l\in\mathbb{Z}}$ of words $a^{(l)} = (a_{k,l})_{k\in\mathbb{Z}}$? Send $a^{(l)}$ on frequency band θl, i.e.
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- Generators of Hilbert C^*-modules generate a **multi-window Gabor frame** for $L^2(\mathbb{R}^d)$

$$\|f\|_2^2 = \sum_{i=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g_i \rangle|^2.$$

- $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if

$$\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle| (1 + |\lambda|^2)^{s/2} < \infty \quad \text{for all} \quad s \geq 0.$$

- Every $f \in \mathcal{S}(\mathbb{R}^d)$ has a representation of the form:

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Modulation spaces

- **Modulation spaces** $M_s^{p,q}(\mathbb{R}^d)$

$$\|f\|_{M_s^{p,q}} = (\int (\int_{\mathbb{R}^d} |\langle f, \pi(x,\omega)g\rangle|^p (1+|x|^2+|\omega|^2)^{s/2} d)^{p/q} x d\omega$$

- $M_s^{p,q}(\mathbb{R}^d)$ is the **minimal** is invariant under time-frequency shifts and we have that $\|\pi(x,\omega)g\|_{M_s^{p,q}} = v_s(x,\omega)\|g\|_{M^{p,q}s}$.

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Theorem:

$M_s^1(\mathbb{R}^d)$ is an equivalence bimodule between $\mathcal{A}q_s(\Lambda, c)$ and $\mathcal{A}_s^1(\Lambda^\circ, \overline{c})$.

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  Then $\Theta^\Lambda_{g,g}$ is invertible if and only if $\sigma_{g,\Lambda}$ is invertible in $M^{\infty,1}(\mathbb{R}^{2d})$ with respect to twisted convolution.

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  Then $\Theta^\Lambda_{g,g}$ is invertible if and only if $\sigma_{g,\Lambda}$ is invertible in $M^{\infty,1}(\mathbb{R}^{2d})$ with respect to twisted convolution.
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Janssen-Ron-Shen duality theorem

Line bundles over noncommutative tori have a rich structure:

- $\mathcal{G}(g,\Lambda)$ is a Gabor frame with frame constants A, B for $L^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, i.e. if there exist positive constants $C, D > 0$ such that for all finite sequences $(a_{\lambda^\circ})_{\lambda^\circ\in\Lambda^\circ}$ we have that

$$C\sum_{\lambda^\circ\in\Lambda^\circ}|a_{\lambda^\circ}|^2 \le \|\sum_{\lambda^\circ\in\Lambda^\circ} a_{\lambda^\circ}\pi(\lambda^\circ)g\|_2^2 \le D\sum_{\lambda^\circ\in\Lambda^\circ}|a_{\lambda^\circ}|^2.$$

  In fact, the corresponding condition numbers are equal, i.e. $B/A = D/C$.

This fact lies at the heart of real-world applications, e.g. wireless communication.

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Consequences

The **unit sphere** of the Hilbert $C^*(\Lambda^\circ, \overline{c})$-module V_{Λ° is defined by $\mathcal{S}(V_{\Lambda^\circ}) = \{g \in V_{\Lambda^\circ} : \langle g, g\rangle_{\Lambda^\circ} = I\}$, which is the set of all **tight** Gabor frames.

$g_1(t) = (2)^{1/4} e^{-\pi t^2}$ a Gaussian, $g_2(t) = (\frac{\pi}{2})^{1/2} \frac{1}{\cosh(\pi t)}$ the hyperbolic secant and $g_3(t) = e^{-\pi |t|}$ the two-sided exponential.

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Let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$. Then $p_{g_i} = {}_\Lambda\langle g_i, g_i\rangle$ is a projection in $C^*(\Lambda, c)$ if and only if $\alpha\beta < 1$.

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