Rigid C^* tensor categories of bimodules over interpolated free group factors

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A history

Theorem (Popa, Shlyakhtenko, '03)

Given any $\Lambda-$lattice (or planar algebra), $\mathcal{P}$ there is a finite index inclusion of II_1 factors $N \subset M$ with $N, M \cong L(\mathbb{F}_\infty)$ having standard invariant $\mathcal{P}$

Theorem (Yamagami, '03)

Given a countably generated rigid C^* tensor category, $\mathcal{C}$, there is an amalgamated free product factor M and a full subcategory of bifinite bimodules over M which is equivalent to $\mathcal{C}$.

Yamagami's factors are of the form $B *_A (A \otimes Q)$ for A and B hyperfinite II_1 algebras and $Q \neq \mathbb{C}$ an arbitrary finite von Neumann algebra. This algebra has property Γ so it is not an interpolated free group factor.

Our Result

Theorem (Brothier, H., Penneys, '12)

Given a countably rigid C^* tensor category, $\mathcal{C}$, there exists a full subcategory Bim of bifinite bimodules over $M_0 \cong L(\mathbb{F}_\infty)$ which is equivalent to $\mathcal{C}$.

The main ingredient is the use of planar algebras. They will be used both in the construction of the factor, M_0, as well as the category of bimodules.

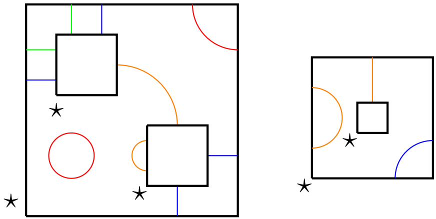
Unshaded planar tangles

Definition

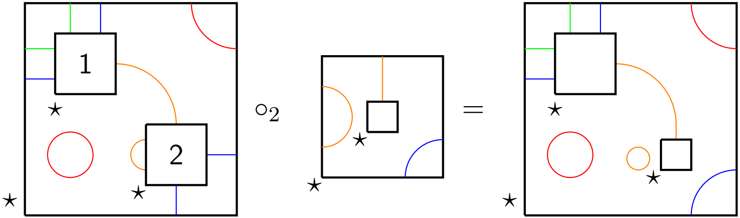
Unshaded Planar tangles consist of three pieces of data.

- An external disk or rectangle D_0
- Finitely many internal subdisks or rectangles $D_1, \ldots, D_n$
- Finitely many smoothly embedded strings whose colors are taken from a set of colors $\mathcal{L}$. Strings can not intersect each other and they intersect the disks transversely. Strings not intersecting disks are closed loops.
- Each disk comes with a distinguished marked interval.

An example



An example (contd.)



Planar algebras

Definition

Let Λ be the set of finite words on elements in $\mathcal{L}$. A planar algebra $P_\bullet$ with string labels $\mathcal{L}$ is a collection of vector spaces $(P_\alpha)_{\alpha\in\Lambda}$ with an action of planar tangles by multilinear maps satisfying:

- For each planar α_0-tangle T, whose disks $D_i(T)$ are α_i-disks, there is a multilinear map

  $$Z_T\colon \prod_{i=1}^{s} P_{\alpha_i} \to P_{\alpha_0}.$$

  If S can be obtained from T by orientation preserving diffeomorphism, then $Z_S = Z_T$.
- Composition of tangles agrees with composition of the appropriate multilinear maps (Naturality).

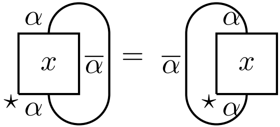
Fantastic planar algebras

Definition

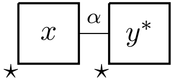
A fantastic (or factor) planar algebra is a planar algebra which satisfies the following additional properties:

- (Evaluable) P_α is finite dimensional for each $\alpha \in \Lambda$. Furthermore, $P_\emptyset \cong \mathbb{C}$ with the empty diagram identified with $1 \in \mathbb{C}$. This forces each closed loop of color a to be a scalar δ_a
- (Involutive) There is a conjugate-linear involution, $*$ taking P_α to $P_{\overline{\alpha}}$ where $\overline{\alpha}$ is the word α in the opposite order. This involution is compatible with reflection of tangles.
- (Spherical): For all $\alpha \in \Lambda$ and all $x \in P_{\alpha\overline{\alpha}}$, we have

$\mathrm{tr}(x) =$

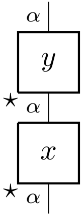


- (Positive): For every $\alpha \in \Lambda$, the map $\langle \cdot, \cdot \rangle : P_\alpha \times P_\alpha \to P_\emptyset \cong \mathbb{C}$ given by

$\langle x, y \rangle =$ 

is a positive definite inner product. Hence for all $a \in \mathcal{L}$, $\delta_a > 0$ (in fact by Jones, $\delta_a \in \{2\cos(\pi/n) : n \geq 3\} \cup [2, \infty)$).

- The operation

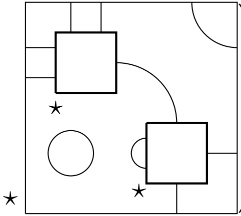
$x \cdot y =$ 

turns each $P_{\alpha\overline{\alpha}}$ into a finite dimensional C^* algebra. When diagrams are written with an α string on top and a β string on the bottom we say that they belong to the vector space $P_{\alpha \to \beta} (\cong P_{\alpha\overline{\beta}})$.

Examples

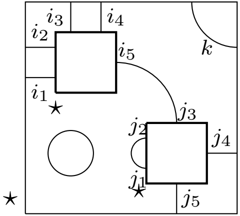
- The colored Temperely-Lieb algebra. P_α is the linear span of all planar string diagrams with no internal disks and outer color pattern α. The adjoint operation is the conjugate-linear extension of reflection of tangles.
- The noncommutative polynomial planar algebra, $\mathbb{C}\langle X_1, \ldots, X_n \rangle$ with one string color. Here, $P_k = \mathbb{C}\{X_{i_1} \cdots X_{i_k}\}$, and the adjoint operation is the conjugate-linear extension of reflecting words. The action is as follows:

Examples (contd.)

$Z\Bigg($  $\Bigg)(X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5}, X_{j_1}X_{j_2}X_{j_3}X_{j_4}X_{j_5})$

Examples (contd.)

Z



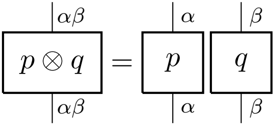
$(X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5}, X_{j_1}X_{j_2}X_{j_3}X_{j_4}X_{j_5})$

$$= n \cdot \delta_{i_5,j_3} \cdot \delta_{j_1,j_2} \cdot X_{i_1}X_{i_2}X_{i_3}X_{i_4}\left(\sum_{k=1}^{n} X_k^2\right) X_{j_4}X_{j_5}$$

It is a direct check that $P_{k\to k}$ is isomorphic to $n^k \times n^k$ matrices over the scalars.

The category of projections, Pro associated to P

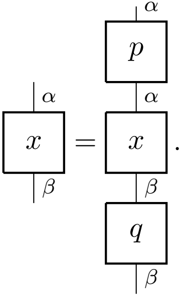
- We will now assume that $\star$ is in the lower left corner of a box unless otherwise marked.
- The objects are projections $p \in P_{\alpha \to \alpha}$ (i.e. $p = p^* = p^2$).
- We tensor objects in Pro by horizontal concatenation; e.g., if $p \in P_{\alpha \to \alpha}$ and $q \in P_{\beta \to \beta}$, then $p \otimes q \in P_{\alpha\beta \to \alpha\beta}$ is given by



$\in P_{\alpha\beta \to \alpha\beta}$.

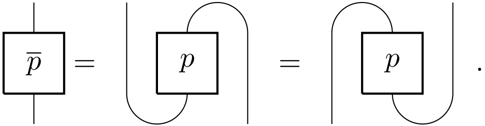
We extend the tensor product to direct sums of projections linearly.

- If $p \in P_{\alpha\to\alpha}$ and $q \in P_{\beta\to\beta}$, then elements in $\mathrm{Pro}(p,q) = q(P_{\alpha\to\beta})p$ are all $x \in P_{\alpha\to\beta}$ such that $x = qxp$, i.e.,



  We compose morphisms by vertical concatenation of elements in the planar algebra. Composition of matrices of morphisms occurs in the usual way.

- The *duality* operation on objects and morphisms is rotation by π



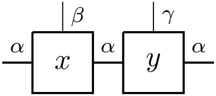
Notice that if $p \in P_{\alpha \to \alpha}$ then $\overline{p} \in P_{\overline{\alpha} \to \overline{\alpha}}$.

- The *adjoint* operation in Pro is the identity on objects. The adjoint of a 1-morphism is the same as the adjoint operation in the planar algebra $P_\bullet$.
- The key fact (due to Jones, Yamagami, Morrison, Peters and Snyder) is that given a rigid C^*- tensor category, $\mathcal{C}$, then there is a fantastic planar algebra, $\mathcal{P}$, such that $\mathcal{C}$ is equivalent to Pro.

Guionnet, Jones, Shlyakhtenko type factors

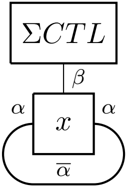
- We define the graded algebra $\mathrm{Gr}_\alpha(P_\bullet) = \bigoplus_{\beta \in \Lambda} P_{\overline{\alpha}\beta\alpha}$. Multiplication $\wedge_\alpha$ is given by

$x \wedge_\alpha y =$



- The trace is given by

$\mathrm{tr}(x) = \frac{1}{\delta_\alpha} \cdot$



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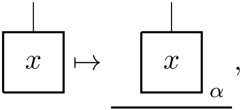
with CTL the set of colored Temperely-Lieb diagrams.

Proposition

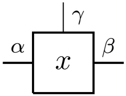
The trace tr is positive definite on $\mathrm{Gr}_\alpha(P_\bullet)$ and multiplication is bounded with respect to the inner product induced by the trace. $\mathrm{Gr}_\alpha(P_\bullet)$ completes to a II_1 factor M_α. Furthermore, $M_0 \subset M_\alpha$ is of finite Jones index for any $\alpha \in \Lambda$ where the inclusion is given by



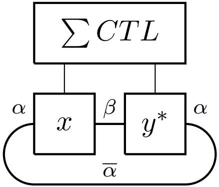
- The proof involves an "orthogonalization" procedure, i.e. passing to an algebra where the different box spaces are orthogonal.
- It is straightforward to check that in the case $P_\bullet = \mathbb{C}\langle X_1, ..., X_n\rangle$, the X_i form a free semicircular family under the trace on $\mathrm{Gr}_0(P_\bullet)$. Therefore, $M_0 \cong L(\mathbb{F}_n)$.

Bimodules over M_0

- Set $\mathrm{Gr}_{\alpha,\beta}(P) = \bigoplus_{\beta \in \Lambda} P_{\overline{\alpha}\gamma\beta}$. It is spanned by boxes of the form



  with inner product

$\langle x, y \rangle =$


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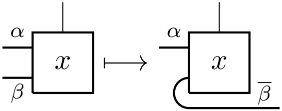
- $\mathrm{Gr}_0(P_\bullet)$ acts on the left by viewing $\mathrm{Gr}_0(P_\bullet) \subset \mathrm{Gr}_\alpha(P_\bullet)$ as above. There is an analogous right action.

Bimodules (contd.)

Proposition

Set $\mathcal{H}_{\alpha,\beta}$ to be the completion of $\mathrm{Gr}_{\alpha,\beta}(P_\bullet)$. Then the $\mathrm{Gr}_0(P)$ action extends to an M_0 action. We have an isomorphism of $M_0 - M_0$ modules $\mathcal{H}_{\alpha\beta,\emptyset} \cong \mathcal{H}_{\alpha,\overline{\beta}}$ where the mapping is given by the extension of



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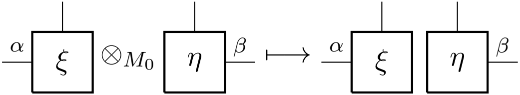
Furthermore $\mathcal{H}_{\alpha,\beta}$ is a bifinite $M_0 - M_0$ module for any $\alpha, \beta \in \Lambda$.

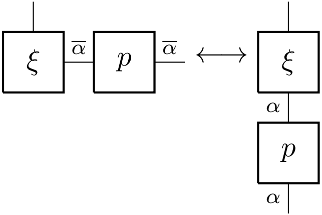
Tensor product

Proposition

We have an isomorphism $\mathcal{H}_{\alpha,\emptyset} \otimes_{M_0} \mathcal{H}_{\emptyset,\beta} \to \mathcal{H}_{\alpha,\beta}$ given by the natural extension of



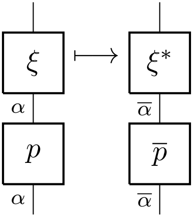
- For $p \in P_{\alpha\to\alpha}$ we consider the bimodule $(\mathcal{H}_{\emptyset,\alpha})p$. We attach p at the bottom using the following identification:



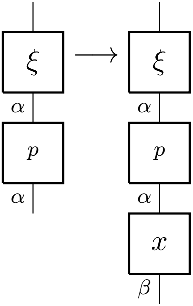
We call this bimodule H_p. Notice $\overline{p}(\mathcal{H}_{\alpha,\emptyset}) \cong H_p \cong (\mathcal{H}_{\emptyset,\overline{\alpha}})p$.

The category of bimodules Bim over M_0

- The objects are the bimodules H_p and finite direct sums thereof.
- The tensor product is the Connes fusion product. Notice that $H_p \otimes_{M_0} H_q \cong H_{p \otimes q}$.
- $\mathrm{Bim}(H_p, H_q)$ is the space of bimodule intertwiners.
- The contragredient bimodule of H_p is isomorphic to $H_{\overline{p}}$. This is achieved by reflection:

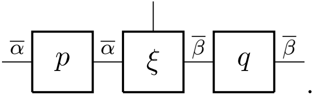


- Let $p \in P_{\alpha\to\alpha}$ and $q \in P_{\beta\to\beta}$ Notice that $x \in q(P_{\alpha\to\beta})p$ is naturally seen as an element in $\mathrm{Bim}(H_p, H_q)$ via the mapping



We claim that all such intertwiners are of this form. Frobenius reciprocity implies that such intertwiners must be the $M_0 - M_0$ central vectors of $H_{\overline{p}} \otimes_{M_0} H_q$.

- These are the $M_0 - M_0$ central vectors of $p(\mathcal{H}_{\alpha,\beta})q$, i.e. the central vectors that are diagrammatically represented as:



These are exactly the diagrams with no strings on top.

This is used to prove the following theorem:

Theorem (Brothier, H., Penneys '12)

There is an equivalence of rigid C^* categories between Bim and Pro.

Theorem (Brothier, H., Penneys '12)

$\mathcal{P}$ can always be chosen to have infinitely many different colors. This allows M_0 to always be isomorphic to $L(\mathbb{F}_\infty)$.

This is proved by writing M_0 as a compression of an amalgamated free product associated to the principal graph of $P_\bullet$

Thank you for listening!

Slides and preprint are available at:

`http://math.berkeley.edu/~mhartgla`