Hessian estimates for the sigma-2 equation in dimension three

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(joint work with Yu Yuan)

We derive an *interior a priori* Hessian estimate for the σ_2 equation

$$\sigma_2 \left(D^2 u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{1}$$

in dimension three, where λ_i are the eigenvalues of the Hessian D^2u.

Theorem 1. *Let u be a smooth solution to (1) on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \leq C(3) \exp \left[C(3) \max_{B_R(0)} |Du|^3 / R^3 \right].$$

In the 1950's, Heinz [H] derived a Hessian bound for the two dimensional Monge-Ampère equation, $\sigma_2(D^2u) = \lambda_1\lambda_2 = \det(D^2u) = 1$, which is equivalent to (2) with $n = 2$ and $\Theta = \pm\pi/2$. In the 1970's Pogorelov [P] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$.

By Trudinger's [T] gradient estimates for σ_k equations, we can bound D^2u in terms of u as

$$|D^2u(0)| \leq C(3) \exp \left[C(3) \max_{B_{2R}(0)} |u|^3 / R^6 \right].$$

We attack (1) via its special Lagrangian equation form

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{2}$$

with $n = 3$ and $\Theta = \pi/2$. Equation (2) stems from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, Du\,(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $\left(1 + \sqrt{-1}\lambda_1\right) \cdots \left(1 + \sqrt{-1}\lambda_n\right)$ is constant Θ, and it is special if and only if $(x, Du\,(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17].

In dimensions two and three, the special Lagrangian equations (2) can be expressed as

$$\cos\Theta(\sigma_1 - \sigma_3) + \sin\Theta(\sigma_2 - 1) = 0.$$

1. Outline of Proof

Heuristically, the proof breaks into the following three steps. We may assume by scaling that u is a solution on $B_4(0) \subset \mathbb{R}^3$.

Step 1) Choose a function $b(D^2u(x))$ such that, with respect to the induced metric on the graph $(x, Du\,(x))$, b satisfies the (weak) Jacobi inequality

$$\triangle_g b \geq |\nabla_g b|_g^2.$$

We choose

$$b(D^2u) = \ln\sqrt{1+\lambda_{\max}^2}$$

where $\lambda_{\max}$ is the largest eigenvalue of the Hessian D^2u.

Step 2) By Michael and Simon's mean value and Sobolev inequalities for minimal surfaces, it follows from Step 1 (choosing appropriate exponents) that

$$b(0) \leq C(3)\left[\int_{B_2} \varphi^2 \left|\nabla_g b\right|^2 V dx + \int_{B_2} b \left|\nabla_g \varphi\right|^2 V dx\right]$$

where V is the volume element on the graph, and φ is an appropriately chosen test function.

Step 3) Finally, we show that the integrals in Step 2 may be bounded by $\|Du\|_{L^\infty(B_4)}$. In fact, Step 1) implies that

$$\int_{B_2} \varphi^2 \left|\nabla_g b\right|^2 dv_g \leq C(3) \int_{B_2} \left|\nabla_g \varphi\right|^2 dv_g.$$

The identity

$$g^{ii}V = \sigma_1 - \lambda_i$$

yields an estimate

$$\int_{B_2} \left|\nabla_g \varphi\right|^2 V dx \leq C(3) \int_{B_2} \triangle u dx \leq \|Du\|_{L^\infty(B_2)}.$$

Further integration by parts, using the above identity, completes the estimate.

2. Questions

1) Estimates for $\sigma_2\left(D^2u\right) = 1$ when $n \geq 4$? The special Lagrangian structure is available only when $n = 2$ or 3, so a challenge is to replace the mean value and Sobolev inequalities in Step 2.

2) Estimates for $\sigma_2\left(D^2u\right) = f(x, u, Du)$? What conditions on $f(x, u, Du)$? The mean curvature is not given by a simple expression in terms of derivatives of f, so this generalization does not follow immediately.

3) Similar estimates for special Lagrangian equations (2) with larger phase $|\Theta| \geq \pi/2$ in dimension three are obtained in [WY]. The challenging problem is to derive estimates for (2) with lower phase, in particular the equation $\sigma_1 = \sigma_3$ corresponding to $\Theta = 0$.

References

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