Special Lagrangian Equations

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This is to certify that I have examined this copy of a doctoral dissertation by

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Abstract

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In joint work with Yu Yuan, we prove a priori interior estimates for special Lagrangian equations. In particular, we show interior a priori Hessian estimates for solutions of the σ_2-equation in dimension three

$$\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1.$$

We show Hessian estimates for large phase special Lagrangian equations in dimension three, and give sharper estimates for dimension two. We also show gradient estimates for large phase special Lagrangian equations. In joint work with Jingyi Chen and Yu Yuan, we show Hessian estimates for convex solutions in arbitrary dimension. In chapter two, we discuss Monge-Ampère equations in the context of calibrated Lagrangian geometry.

TABLE OF CONTENTS

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DEDICATION

to Henry Aaron and Levi Micah

0.1 Introduction

In this thesis, we discuss two fully nonlinear elliptic equations, both of which arise in minimal surface theory and Lagrangian geometry. The special Lagrangian equation (1.1) was given in the groundbreaking work of Harvey and Lawson in 1982. Solutions to (1.1) describe calibrated minimal Lagrangian surfaces in complex space. Such special Lagrangian surfaces have importance in the study of string theory. The first chapter introduces the special Lagrangian equation, and then describes work of the author and Yuan, giving a priori interior Hessian estimates for various classes of solutions to (1.1).

In the first section of Chapter 1, the equation (1.1) is introduced and given geometric context. We then state our estimates and a Liouville type theorem, and give these estimates historical context and an heuristic outline of proof. We then discuss some more preliminaries on our way to proving each result.

Much of Chapter 1 draws from papers by the author and Yuan. In [51] we prove estimates for the sigma-2 equation; [50] contains estimates for large phase equations in three dimensions, and also the gradient estimates; [49] gives estimates for special Lagrangian equations in two dimensions; the Liouville result is contained in [52]. The section on convex solutions is from the paper with Chen and Yuan [10]. We have attempted to unify the presentation of the above papers by identifying common arguments and also by adding introductory material at the beginning of Chapter 1.

The Monge-Ampère equation (2.1), though much older, has in the last eight years been implicated in the study of minimal surfaces, calibrated geometry, and string theory. The main contribution of the second chapter is that (2.1) describes calibrated Lagrangian volume maximizing surfaces in a pseudo-Euclidean space. It makes sense then, to think of the equation (2.1) as a special Lagrangian equation. In fact, (2.1) can be connected to the equation (1.1) by looking at a family of metrics, each accompanying an elliptic equation which describes calibrated submanifolds. Chapter 2 is a version of [48].

Chapter 1
THE SPECIAL LAGRANGIAN EQUATION

1.1 Introduction

The main focus of this thesis is the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{1.1}$$

where λ_i are the eigenvalues of the Hessian D^2u. This fully nonlinear elliptic equation stems from the special Lagrangian geometry [19]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is constant Θ. Such a graph is special if and only if $(x, Du(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ ([19], Theorem 2.3, Proposition 2.17.)

Equation (1.1) may be rewritten as

$$\cos\Theta \sum_{1\leq 2k+1\leq n} (-1)^k \sigma_{2k+1}(D^2u) + \sin\Theta \sum_{0\leq 2k\leq n} (-1)^{k+1}\sigma_{2k}(D^2u) = 0 \tag{1.2}$$

where $\sigma_k(D^2u)$ is the k-th elementary symmetric polynomial on the eigenvalues of D^2u. In particular, we see that equation (1.1) with $n=3$ and $|\Theta| = \pi/2$ also takes the form

$$\sigma_2\left(D^2u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1. \tag{1.3}$$

Also when $n=3$, and $|\Theta| = \pi$ or $\Theta = 0$, (1.1) takes the form

$$\triangle u = \det D^2 u. \tag{1.4}$$

1.2 The Special Lagrangian Calibration

In this section we offer a derivation of the special Lagrangian equation (1.1). To this end, we will discuss the notions of calibrations, Lagrangian submanifolds, and Lagrangian phase. The following discussion is abbreviated and tailored for the purpose of deriving (1.1). A more thorough and more general geometric discussion is available in the seminal paper of Harvey and Lawson [19], or other more recent works such as [26].

1.2.1 Calibrations

Let (X^n, g) be a Riemannian manifold of dimension n. A closed p-form φ is called a calibration if φ has the property that

$$\varphi(E_1, ..., E_p) \leq 1$$

for any (local) orthonormal set $E_1, ..., E_p$ of sections of TX. An oriented p-submanifold M is said to be calibrated by φ (also called a φ-submanifold) if φ is a calibration such that

$$\varphi(E_1, ..., E_p) = 1$$

for all oriented orthonormal frames $E_1, ..., E_p$ for TM. Equivalently, the restriction of φ to M equals the volume form induced on M, that is

$$\varphi|_M = Vol_M.$$

We may conclude that any calibrated submanifold is homologically volume minimizing through the following argument. Let M be a φ-submanifold, and let M' be a homologous submanifold. There exists a $(p+1)$-current N such that $\partial N = M \cup M'$, with appropriate orientation. By Stokes' Theorem

$$0 = \int_N d\varphi = \int_M \varphi - \int_{M'} \varphi$$

thus

$$Vol(M) = \int_M Vol_M = \int_M \varphi = \int_{M'} \varphi \leq \int_{M'} Vol_{M'} = Vol(M').$$

1.2.2 Lagrangian Manifolds

Let $J : \mathbb{C}^n \to \mathbb{C}^n$ be the map defined by multiplication by $\sqrt{-1}$. This $\mathbb{C}^n$-linear map induces an $\mathbb{R}$-linear map $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$. The Kähler form ω is the two form defined by

$$\omega(V, W) = g(JV, W),$$

where g is the standard metric on $\mathbb{R}^n \times \mathbb{R}^n$. An n-plane P in $\mathbb{R}^n \times \mathbb{R}^n$ is called Lagrangian if any of the following equivalent conditions hold:

- $\omega(V, W) = 0$ for all V, W which lie in P
- JP is perpendicular to P
- If $\{e_i\}$ is an orthonormal basis for P, then $\{e_i, Je_i\}$ is an orthonormal basis for $\mathbb{R}^n \times \mathbb{R}^n$
- Viewed as vectors in $\mathbb{C}^n$, any orthonormal basis $\{e_i\}$ for P is a unitary basis for $\mathbb{C}^n$.

Given an oriented orthonormal basis $\{e_i\}$, we can associate a unitary matrix using this last condition. It can be shown that the determinant of this matrix does not depend on choice of basis. We then define the *Lagrangian phase* $\Theta(P)$ to be the argument of this S^1 -valued determinant.

An n-submanifold $M \subset \mathbb{C}^n$ is Lagrangian if the tangent space T_xM is Lagrangian for each $x \in M$. We can locally define the phase $\Theta(x)$ along a Lagrangian submanifold M by taking $\Theta(T_xM)$. We note that this phase function is unique modulo $2\pi k$ once we choose complex coordinates for $\mathbb{C}^n$.

Lemma 1.2.1. *Any C^2 Lagrangian submanifold is locally represented by a gradient graph $(x, Du\,(x))$ over its tangent plane.*

Proof. Locally, we may represent M as a graph of an n-valued function over its tangent plane, which we identify with $\mathbb{R}^n$, that is $M = (x + \sqrt{-1}F(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$.

Taking coordinates x, the tangent space is spanned by the vectors

$$\begin{aligned}\partial_1 &= (1, 0, ..., 0) + \sqrt{-1}(\partial_1 F^1, \partial_1 F^2, ..., \partial_1 F^n),\\ \partial_2 &= (0, 1, ..., 0) + \sqrt{-1}(\partial_2 F^1, \partial_2 F^2, ..., \partial_2 F^n), \quad \text{etc.}\end{aligned}$$

The Lagrangian condition implies that $J\partial_i \perp \partial_j$, that is, the innner product satisfies

$$\begin{aligned}&\langle((-\partial_i F^1, -\partial_i F^2, ..., -\partial_i F^n) + \sqrt{-1}(0, ..., 1, ...),\\ &(0, .., 1, ...) + \sqrt{-1}(\partial_j F^1, \partial_j F^2, ..., \partial_j F^n)\rangle = 0,\end{aligned}$$

and thus

$$-\partial_i F^j + \partial_j F^i = 0.$$

It follows that the 1-form $\sum_{i=1}^n F^i dx^i$ is closed, so locally there exists some u such that $\sum_{i=1}^n F^i dx^i = du$, that is, $F = Du$. $\square$

Conversely, it is easy to see that any gradient graph $(x, Du(x))$ is a Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. Given such a graph we compute the phase $\Theta(x)$ via the following. The tangent space is spanned by the vectors

$$\begin{aligned} \partial_1 &= (1,0,...,0) + \sqrt{-1}(u_{11}, u_{12}, ..., u_{1n}), \\ \partial_2 &= (0,1,...,0) + \sqrt{-1}(u_{21}, u_{22}, ..., u_{2n}), \quad \text{etc.} \end{aligned} \tag{1.5}$$

At any point x, we may diagonalize $D^2u(x)$. This orthogonalizes the tangent vectors $\{\partial_i\}$, which we can then normalize to get the following orthonormal basis:

$$\begin{aligned} \partial_1 &= \frac{(1,0,...,0) + \sqrt{-1}(\lambda_1, ..., 0)}{|1+\sqrt{-1}\lambda_1|}, \\ \partial_2 &= \frac{(0,1,...,0) + \sqrt{-1}(0, \lambda_2, ..., 0)}{|1+\sqrt{-1}\lambda_2|}, \quad \text{etc.} \end{aligned} \tag{1.6}$$

The unitary matrix corresponding to this orthonormal basis is given by

$$\begin{pmatrix} \frac{1+\sqrt{-1}\lambda_1}{|1+\sqrt{-1}\lambda_1|} & & & \\ & \frac{1+\sqrt{-1}\lambda_2}{|1+\sqrt{-1}\lambda_2|} & & \\ & & \cdots & \\ & & & \frac{1+\sqrt{-1}\lambda_n}{|1+\sqrt{-1}\lambda_n|} \end{pmatrix}.$$

Defining $\theta_i \in (-\pi/2, \pi/2)$ by

$$e^{\sqrt{-1}\theta_i} = \frac{1+\sqrt{-1}\lambda_i}{|1+\sqrt{-1}\lambda_i|},$$

or equivalently, $\theta_i = \arctan \lambda_i$, the determinant of this unitary matrix is given by

$$e^{\sqrt{-1}\theta_1 + ... + \sqrt{-1}\theta_n} = e^{\sqrt{-1}\Theta}$$

where

$$\Theta = \sum \arctan \lambda_i.$$

The connection between this phase and minimal surfaces was given by Harvey and Lawson:

Theorem (Harvey-Lawson [19]). *A Lagrangian graph $x+\sqrt{-1}Du(x) \subset \mathbb{C}^n$ is calibrated by the form*

$$\Omega_\theta = \mathrm{Re}\{e^{-\sqrt{-1}\theta} dz_1 \wedge ... \wedge dz_n\}$$

if and only if the phase is constant θ.

The proof of this fact consists of the verification of two algebraic facts: First that the complex valued form $dz_1 \wedge ... \wedge dz_n$ evaluates to $e^{\sqrt{-1}\Theta}$ on any oriented orthonormal basis for the tangent space to the Lagrangian graph, with Θ defined as above. Secondly, $dz_1 \wedge ... \wedge dz_n$ evaluates to a value ς with $|\varsigma| < 1$ on any orthonormal basis for an n-plane which is not Lagrangian. See Chapter 2 for more details, as well as a pseudo-Euclidean analogue of this result.

It follows then that the gradient graph determines a volume minimizing surface in $\mathbb{C}^n$, if and only if u satisfies the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta.$$

Examining the above derivation, we see that we took an orthogonal frame, normalized it to get (1.5), and then took its complex determinant, which lies in S^1. It is clear however, that we only are concerned with the argument of this determinant, so it isn't necessary to normalize the vectors in order to check the phase. The special Lagrangian condition also takes the form

$$\arg\det\left(I + \sqrt{-1}D^2u\right) = \Theta. \tag{1.7}$$

We may compute directly

$$\det\left(I + \sqrt{-1}D^2u\right) = \prod_{i=1}^{n}\left(1+\sqrt{-1}\lambda_i\right) = (1-\sigma_2+\sigma_4...) + \sqrt{-1}(\sigma_1 - \sigma_3 + \sigma_5...) \tag{1.8}$$

where σ_k are the elementary symmetric polynomials on the eigenvalues of D^2u. Thus

$$\Theta = \arctan\frac{(\sigma_1 - \sigma_3 + \sigma_5...)}{(1-\sigma_2+\sigma_4...)}$$

and (1.7) can be reduced to

$$\tan\Theta = \frac{(\sigma_1 - \sigma_3 + \sigma_5...)}{(1-\sigma_2+\sigma_4...)} \tag{1.9}$$

which is equivalent to (1.2).

1.3 Summary of Results

Our main results concern a priori estimates for this equation. We list these in the following.

Theorem 1 (Warren-Yuan [51]). *Let u be a smooth solution to (1.3) on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\max_{B_R(0)}|Du|^3/R^3\right].$$

By Trudinger's [43] gradient estimates for σ_k equations, we can bound D^2u in terms of the solution u in $B_{2R}(0)$ as

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\max_{B_{2R}(0)}|u|^3/R^6\right].$$

Also in dimension three,

Theorem 2 (Warren-Yuan [50]). *Let u be a smooth solution to (1.1) with $|\Theta| \geq \pi/2$ and $n = 3$ on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\left(\cot\frac{|\Theta| - \pi/2}{3}\right)^2\max_{B_R(0)}|Du|^7/R^7\right],$$

and also

$$|D^2u(0)| \leq C(3)\exp\left\{C(3)\exp\left[C(3)\max_{B_R(0)}|Du|^3/R^3\right]\right\}.$$

One immediate consequence of the above estimates is a Liouville type result for global solutions with quadratic growth to (1.3), namely any such a solution must be quadratic (cf. [56], [58]. Another consequence is the regularity (analyticity) of the C^0 viscosity solutions to (1.3) or (1.1) with $n = 3$ and $|\Theta| \geq \pi/2$.

In order to link the dependence of Hessian estimates in the above theorems to the potential u itself, we have the following gradient estimate which holds in general dimensions.

Theorem 3 (Warren-Yuan [50]). *Let u be a smooth solution to (1.1) with $|\Theta| \geq (n-2)\frac{\pi}{2}$ on $B_{3R}(0) \subset \mathbb{R}^n$. Then we have*

$$\max_{B_R(0)}|Du| \leq C(n)\left[\operatorname*{osc}_{B_{3R}(0)}\frac{u}{R} + 1\right]. \tag{1.10}$$

In two dimensions, the following estimates are sharper than previously known estimates.

Theorem 4 (Warren-Yuan[49]). *Let u be a smooth solution to (1.1) with $n = 2$ on $B_R(0) \subset \mathbb{R}^2$. Then the following both hold*

$$\left|D^2u(0)\right| \leq C(2) \exp\left[C(2) \max_{B_R(0)} |Du|^2/R^2\right], \tag{1.11}$$

or

$$\left|D^2u(0)\right| \leq C(2) \exp\left[C(2) \frac{1}{|\sin\Theta|^{3/2}} \max_{B_R(0)} |Du|/R\right]. \tag{1.12}$$

In arbitrary dimension, for convex solutions, we have

Theorem 5 (Chen-Warren-Yuan [10]). *Let u be a smooth convex solution to (1.1) on a ball $B_R(0) \subset \mathbb{R}^n$. Then we have*

$$|D^2u(0)| \leq C(n) \exp\left\{C(n) \left[\operatorname*{osc}_{B_R(0)} u^{3n-2}\right] / R^{6n-4}\right\},$$

where $C(n)$ is some uniform dimensional constant.

Observe that all eigenvalues λ_i are positive if the phase Θ is very large

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \geq (n-1)\pi/2.$$

Then a direct consequence of Theorem 5 is

Corollary 1. *Let u be a smooth solution to (1.1) with $|\Theta| \geq (n-1)\pi/2$ on $B_R(0) \subset \mathbb{R}^n$. Then we have*

$$|D^2u(0)| \leq C(n) \exp\left\{C(n) \left[\operatorname*{osc}_{B_R(0)} u^{3n-2}\right] / R^{6n-4}\right\}.$$

As an application of our a priori estimates in Corollary 1, we have the following regularity result

Theorem 6. *Any C^0 viscosity solution to (1.1) with $|\Theta| \geq (n-1)\pi/2$ is analytic inside the domain of the solution.*

We also have the following Liouville type result.

Theorem 7 (Warren-Yuan[52]). *Let u be a smooth solution to the special Lagrangian equation*

$$\sum_{i=1}^{n} \arctan \lambda_i = c \quad on \quad \mathbb{R}^n, \tag{1.13}$$

where $\lambda_i s$ are the eigenvalues of the Hessian $D^2u\,(x)$. Suppose that

$$3 + (1-\varepsilon)\lambda_i^2\,(x) + 2\lambda_i\,(x)\,\lambda_j\,(x) \geq 0 \tag{1.14}$$

for all i, j, x and any small fixed $\varepsilon > 0$; and the gradient $\nabla u(x)$ satisfies

$$|\nabla u(x)| \leq \delta\,(n)\,|x| \tag{1.15}$$

for large $|x|$ and any fixed $\delta\,(n) < 1/\sqrt{n-1}$. Then u must be a quadratic polynomial.

1.3.1 Discussion of Results

Estimates

In the 1950's, Heinz [21] derived a Hessian bound for the two dimensional Monge-Ampère equation, $\sigma_2(D^2u) = \lambda_1\lambda_2 = \det(D^2u) = 1$, which is equivalent to (1.1) with $n = 2$ and $\Theta = \pm\pi/2$, see also Pogorelov [36] for Hessian estimates for these equations including (1.1) with $|\Theta| > \pi/2$ and $n = 2$. Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k equations ($k \geq 2$) were derived by Pogorelov [38] and Chou-Wang [11] respectively using the Pogorelov technique. Urbas [44] [45] obtained (pointwise) Hessian estimates in terms of certain integrals of the Hessian, for σ_k equations. Also, the estimates of Bao and Chen [2] for special Lagrangian equation (1.1) with $n = 3$ and $\Theta = \pi$, are in terms of integrals of the Hessian.

A Hessian bound for (1.1) with $n = 2$ also follows from an earlier work by Gregori [17], where Heinz's Jacobian estimate was extended to get a gradient bound in terms of the heights of the two dimensional minimal surfaces with any codimension. (Observe that gradient estimates for the minimal Lagrangian surfaces are then Hessian estimates for the special Lagrangian equation (1.1)). Using an integral method developed for codimension one minimal graphs, a gradient estimate for general dimensional and codimensional minimal graphs with certain constraints on the gradients themselves was obtained by Wang [47], applying to the special Lagrangian equation when pairs of eigenvalues satisfy $|\lambda_i \lambda_j| \leq 1$. Although it is not clear whether the exponential dependence in our estimates (1.11) and (1.12) is sharp, still it is sharper than the double exponential dependence on Du by Heinz ([21], Theorem 2), ([20], p.263, p.255) and Gregori ([17], Theorem 1), when applied to the special Lagrangian equation of dimension two. On the other hand, like our nonuniform estimate (1.12), Heinz's estimate deteriorates as Θ goes to 0. Estimates in [47] only apply to (1.1) with $|\Theta| = \pi/2$ and $n = 2$, that is, $\lambda_1 \lambda_2 = 1$, but give the same order of dependence as (1.12). The gradient estimate for the codimension one minimal surface equation of Bombieri-De Giorgi-Miranda [4] (see also [41] [5] [27]) is by now classic.

In the 1970's Pogorelov [38] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$; those irregular solutions also serve as counterexamples for cubic and higher order symmetric σ_k equations (cf. [46]). We see that while the regularity for quadratic special Lagrangian equations (those involving only σ_1 and σ_2) has been established, it is unclear what to expect when σ_3 becomes involved. We first found estimates for the σ_2 equation in dimension three, and then used some related but different methods to obtain Theorem 2. So while regularity does not hold for convex solutions of $\sigma_3 = 1$, we have regularity for convex solutions of equations such as $\sigma_3 = \sigma_1$, in dimension three. Whether or not regularity holds for phase 0 solutions is an important open question.

We note that using the method of Pogorelov [38], one can construct a solution of the equation $\sigma_2 = 1$, which, like the counterexamples for Monge-Ampère equations, fails to be C^2 along an axis. This function $u = \left|x^2 + y^2\right|^{1/2} \cosh z$ is a subsolution along $|x| = |y| = 0$, but fails to be a viscosity supersolution. One can check this by noting that this solution can be "touched from below" by quadratics satisfying $\sum \arctan \lambda_i > \pi/2$.

A well-known result (cf. Corollary 3 in [30]) states that all $W^{2,n}$ strong solutions to any (possibly degenerate) elliptic equation are also C^0 viscosity solutions. From this and Theorem 6 we see that all $W^{2,n}$ strong solutions to (1.1) with $|\Theta| \geq (n-1)\,\pi/2$ are regular. Previously in [2], the regularity was shown for any $W^{2,3+}$ convex strong solution to $\Delta u = \det D^2u$, which is (1.1) with $\Theta = \pi$ and $n = 3$.

There are several ways by now to establish the existence and uniqueness of C^0 viscosity solutions to the Dirichlet problem of special Lagrangian equation (1.1); see the remarks in the end of the section on convex solutions.

Liouville Results

By Fu's classification result [15], any global solution to (1.1) on $\mathbb{R}^2$ is either quadratic or harmonic; a harmonic function with any linear growth condition on the gradient is certainly quadratic. In fact, global solutions to any strictly large phase ($|\Theta| > (n-2)\frac{\pi}{2}$) equation are quadratic by the uniqueness result of Yuan [58]. Note that given any phase strictly less than large ($|\Theta| < (n-2)\frac{\pi}{2}$) we can create a global solution which is not quadratic, namely the sum of a global harmonic function in two variables and quadratic polynomials in the remaining variables. The resulting special Lagrangian graph then splits into a product of planes and the graph of a phase 0 two-dimensional graph. It is unclear whether or not there exist global solutions which do not decompose in this way. A global solution with critical phase $|\Theta| = (n-2)\frac{\pi}{2}$ which is not a polynomial cannot decompose in this way, so we would be very interested to find or construct one.

In the case $n = 3$, other Liouville-Bernstein type results hold true for (1.1) under the following conditions respectively: $\lambda_i \geq -K$ [56]; $\lambda_i\lambda_j \geq -K$ [54]; or $c = \pi$ and the solution is strictly convex with quadratic growth [3]. In the general case ($n \geq 4$), Liouville-Bernstein type results are valid for (1.1) with the assumptions that $c = k\pi$ and the solution is convex with linear growth [6]; with the almost convex assumption $\lambda_i \geq -\varepsilon(n)$ [56]; with the semi-convex assumption $\lambda_i \geq -\frac{1}{\sqrt{3}} + \gamma$ everywhere, or with the ("equivalent") assumption $|\lambda_i| \leq \sqrt{3} - \gamma'$ everywhere [54]; or with the assumption $\lambda_i\lambda_j \geq -1 - \varepsilon(n)$ [54]. (It is straightforward that any convex solution with a bounded Hessian to (1.1) is a quadratic polynomial, by the well-known C^α Hessian estimate of Evans-Krylov-Safanov for now the convex elliptic equation (1.1); see also ([53], p. 217–218) for a different approach via the iteration argument of [22].)

The more general "convexity" condition (1.14) does not alone lead to any Hessian bound for the solutions to (1.1), but does guarantee a Jacobi inequality for the volume element V, as is discussed in the following section.

Once a Hessian bound for solutions to (1.1) is available, the "standard" blow-down process from the geometric measure theory will show that the global solution is a quadratic polynomial, provided certain convexity conditions like (1.14) or others are available in the *whole* process (for $n \geq 4$). (Unlike [25], we could not generalize the iteration argument in [22] to get a Liouville type result for now the larger image set (1.14) of the corresponding harmonic Gauss map to the Lagrangian Grassmanian.) The simple constraints $|\lambda_i| \leq K$ like $|\lambda_i| \leq 1$ or $|\lambda_i| \leq \sqrt{3} - \gamma$ are easily shown to be available in the blow-down process. An *extra* effort is needed to justify that the constraints (1.14) or others like $\lambda_i \lambda_j \geq const$ are preserved under the $C^{1,\alpha}$ convergence of the scaling process $u_k(x) = u(kx)/k^2$. Taking advantage of the single elliptic equation (1.1), we apply the $W^{2,\delta}$ estimates for solutions in terms the supremum norm of the solution to extract a $W^{2,\delta}$ sub-convergent sequence, as in [55]. Then we extract another subsequence with the Hessians converging almost everywhere. This justifies that the constraints (1.14) are preserved in the above blow-down process.

Actually, Theorem 7 holds true for $n = 3$ without any growth condition like (1.15). The condition (1.14) implies $\lambda_i \lambda_j \geq -K$, so as in [54] we can find a bound on the Hessian (possibly for a new potential), and then draw the conclusion. Note that the boundedness on the Hessian alone for $n = 3$ is enough for one to run the blow-down process to obtain a Liouville type result; see ([13], Theorem 5.4) of Fischer-Colbrie. In general dimension $n \geq 4$, we derive yet another Liouville-Bernstein type result for the solutions to (1.1) with the bounded Hessian satisfying weaker constraints (1.103); see Theorem 9 (given in the section on cones). One consequence of Theorem 9 coupled with the De Giorgi-Allard ε-regularity theory is an improvement of the above mentioned Liouville-Bernstein type result in [54], namely, any global solution to (1.1) with $\lambda_i \geq -\frac{1}{\sqrt{3}} - \varepsilon(n)$ everywhere or $|\lambda_i| \leq \sqrt{3} + \varepsilon'(n)$ everywhere is a quadratic polynomial (for $n \geq 4$). The argument is identical to the one in [56] with Proposition 2.1 there replaced by Proposition 1.11.1 here.

1.3.2 Heuristics for the Estimates

Heuristically, our proofs draw inspiration from the gradient estimate for the codimension one minimal surface equation. In particular, the outline of the proof of Theorem 1 is reminiscent of Trudinger's [41] (see also [16], chapter 16) proof of the gradient estimate for the minimal surface equation. Our proofs of estimates for the large phase, convex and dimension two special Lagrangian equations require an isoperimetric argument that was used in Bombieri-De Giorgi-Miranda's [4] proof. The technique used to obtain the key estimates for the Liouville Theorem is inspired by Korevaar's [27] proof of the gradient estimate.

The heuristic ideas for Hessian estimates are as follows. First we find a function b which is subharmonic, so that b at any point is bounded by its integral over a ball around the point on the minimal surface by Michael-Simon's mean value inequality [32]. This special choice of b needs to be not only subharmonic, but even stronger, to satisfy a Jacobi inequality of the form

$$\triangle_g b \geq \epsilon \left|\nabla_g b\right|^2$$

with (preferably) $\epsilon = \frac{1}{n}$, but any $\epsilon > 0$ will do. Coupled with Sobolev inequalities for functions both with and without compact support, this Jacobi inequality leads to a bound on the integral of b by the volume of the ball on the minimal surface. Taking advantage of the divergence form of the volume element of the minimal Lagrangian graph, we bound the volume in terms of the height of the special Lagrangian graph, which is the gradient of the solution to equation (1.1).

A standard choice (cf. [41]) for the function b is the volume element $\ln V$. However, for the σ_2 equation, we find that $\ln V$ does not appear to satisfy any type of Jacobi inequality. Instead we are led to the function $b = \ln \sqrt{1 + \lambda_{\max}^2}$. There is obviously the need for some extra work to deal with the somewhat strange choice of function, which is not expected to be more regular than Lipschitz. Once this wrangling is accomplished, some fortunate algebraic structure in the equation allows us to take advantage of the divergence form of the volume element of the minimal Lagrangian graph.

For three dimensional large phase equations, we use this non-smooth function b as well, but we find that the cubic structure of generic three dimensional special Lagrangian equations does not lend itself to the same sort of integral bounds we were able to use for the critical case $\theta = \pi/2$. Instead we must build Sobolev inequalities for noncompactly supported functions. The rotation down to the critical case allows us to obtain the isoperimetric inequalities necessary to do this. (See the sections on Lewy-Yuan rotations and large phase equations.) Combining with the estimate for the critical case, it turns out we can make estimates in a uniform way.

While we can use the volume element in two dimensions, we find that the Jacobi type inequality deteriorates as θ becomes small. Fortunately, the $\theta = 0$ equation is just the classical Laplace equation, which is well understood. For very small theta, we can rotate to this equation and interpolate a uniform estimate.

The choice $b = \ln V$ works for the estimates in the Liouville result [52], but only with the unpleasant technical assumption $\delta(n) < 1/\sqrt{n-1}$. This assumption reflects the limitation of our pointwise arguments; the assumption is necessary for us to push the Bernstein-Pogorelov-Korevaar technique to obtain a Hessian estimate for special Lagrangian equations; see Lemma 1.11.2.

Lewy-Yuan rotation is also used along with a relative isoperimetric inequality to get another key ingredient in our proof of the Hessian estimates, namely a Sobolev inequality for functions without compact support. This works whenever there is a lower bound on the Hessian, in particular if the potential is convex or when the phase is super critical.

What Theorem 5 says is that the geometry of the special Lagrangian graph with convex potential is simple, namely the induced metric is quasi-isometric to the flat one up to a factor of the height (or the oscillation of the potential). The idea of our arguments is thus to retrieve this simple geometry. By a Lewy type rotation, the geometry of the Lagrangian graph is already simple in the rotated coordinate system. Consequently, the subharmonic volume element, in the original coordinates, of the special Lagrangian graph with convex potential is actually a subsolution to a uniformly elliptic equation. Therefore the volume element is bounded pointwise by its integral on the minimal graph. (Here one avoids the "harder" mean value inequality for the non-uniformly elliptic Laplace-Beltrami operator on minimal surfaces.) Using a relative isoperimetric inequality on the rotated coordinate plane, we derive a Sobolev inequality for functions without compact support on the Lagrangian graph. The Lewy rotation also leads to a bound of the volume by the height of the Lagrangian graph, or the gradient of the potential. Lastly, the gradient of any convex function is dominated by it oscillation.

As for the gradient estimates, we adapt Trudinger's method [T2] for σ_k equations to (1.1) with the critical phase $\Theta = (n-2)\,\pi/2$. Gradient estimates for (1.1) with larger phase $\Theta > (n-2)\,\pi/2$ are straightforward consequences of the observation that the Hessians of solutions have lower bound depending on the phase Θ. In order to obtain the uniform gradient estimates independent of the phase Θ, we make use of the Lewy-Yuan rotation, which links the corresponding estimates to the ones in the case of the critical phase.

Now only some technical obstacles remain for Hessian estimates for (1.1) with large phase $|\Theta| \geq (n-2)\,\pi/2$ and $n \geq 4$. Yet further new ideas are lacking for us to handle the special Lagrangian equation (1.1) with general phases in dimension three and higher, including (1.4) corresponding to $\Theta = 0$ and $n = 3$. The challenging regularity problem for sigma-2 equations in dimension four and higher also still remains open to us.

1.4 More about the Special Lagrangian Equations

1.4.1 Induced Metric

Given a Lagrangian graph $x+\sqrt{-1}Du\,(x)\subset\mathbb{C}^n$, we can compute the induced metric on the coordinate frame (1.5) given above

$$g_{ij}=\delta_{ij}+\sum_k u_{ik}u_{kj}$$

or

$$g=I+D^2u\left(D^2u\right)^T.$$

When D^2u is diagonalized,

$$g=\begin{pmatrix}1+\lambda_1^2 & & \\ & \cdots & \\ & & 1+\lambda_n^2\end{pmatrix}. \tag{1.16}$$

The Second Fundamental Form

We compute the second fundamental form on the induced graph. Taking (1.5) as a frame for our tangent space, we define a frame for the normal space by

$$n_1=J\partial_1=(-u_{11,}-u_{12},...,-u_{1n},1,0,...,0),\text{ etc.}$$

The second fundamental form II is a two tensor with values in the normal bundle. By using the metric on $\mathbb{R}^n\times\mathbb{R}^n$, we can also think of this as a tensor which takes two tangent vectors and one normal vector, with components

$$h_{ijk}=\langle II(\partial_i,\partial_j),n_k\rangle=-\langle\nabla_{\partial_j}n_k,\partial_i\rangle$$

using the Weingarten equation. This is easy to compute

$$-\langle\nabla_{\partial_j}n_k,\partial_i\rangle=-(-u_{kj1},-u_{kj2},...,-u_{kjn},0,...0)\cdot\partial_i=u_{kji}.$$

It will be convenient to diagonalize D^2u and normalize these components at a point p, dividing by the length of ∂_i,∂_j, and n_k, that is

$$h_{ijk}=\frac{u_{ijk}}{\sqrt{g_{ii}g_{jj}g_{kk}}}=\sqrt{g^{ii}g^{jj}g^{kk}}u_{ijk}.$$

Volume Element

It is easy to compute the volume element

$$V = \sqrt{\det g} = \prod_{i=1}^{n} \sqrt{1+\lambda_i^2} = \prod_{i=1}^{n} \left|1+\sqrt{-1}\lambda_i\right| = \left|\prod_{i=1}^{n}\left(1+\sqrt{-1}\lambda_i\right)\right| \tag{1.17}$$
$$= \left|\det\left(I+\sqrt{-1}D^2u\right)\right| = \left|(1-\sigma_2+\sigma_4 ...)+\sqrt{-1}(\sigma_1-\sigma_3+\sigma_5 ...)\right|.$$

Thus the complex number $\det\left(I+\sqrt{-1}D^2u\right)$ is given by $Ve^{\sqrt{-1}\Theta}$, or equivalently

$$\ln\det\left(I+\sqrt{-1}D^2u\right) = \ln V + \sqrt{-1}\Theta. \tag{1.18}$$

Using (1.9) one can also see that the volume element takes a somewhat nicer form, by continuing with (1.17):

$$V = \sqrt{(1-\sigma_2+\sigma_4 ...)^2 + (\sigma_1-\sigma_3+\sigma_5 ...)^2} =$$
$$\sqrt{(\cot^2\Theta+1)(\sigma_1-\sigma_3+\sigma_5 ...)^2} =$$
$$= |\csc\Theta|\,|(\sigma_1-\sigma_3+\sigma_5 ...)|$$
$$= |\sec\Theta|\,|(1-\sigma_2+\sigma_4 ...)|.$$

By differentiating the minimal surface equation (1.21) and performing a long computation, one gets the standard formula for $\triangle_g \ln V$; see for example the computation by Yuan in Lemma 2.1 in [56]. (The general formula for minimal submanifolds of any dimension or codimension originates in Simons [40], pg. 90.) The Laplace-Beltrami for the volume element will be a central piece of our Hessian estimates.

Lemma 1.4.1 (Yuan [56]). *Let $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ be a minimal surface. Suppose that the Hessian D^2u is diagonalized at p. Then*

$$\triangle_g \ln V = \sum_{a,b,c=1}^{n} h_{abc}^2 \left(1+\lambda_b\lambda_c\right). \tag{1.19}$$

Linearized Operators and Relation to Sigma Equations

Since (1.1) is a minimal surface equation it follows that the coordinates (x, Du) are harmonic with respect to the Laplace Beltrami operator on the surface

$$\triangle_g = \frac{1}{\sqrt{\det g}} \sum_{i,j}^{n} \partial_i \left(\sqrt{\det g}\, g^{ij}\partial_j\right),$$

which reduces to

$$\triangle_g = \sum_{i,j}^{n} g^{ij}\partial_{ij} \tag{1.20}$$

because the surface is minimal (cf [35]). It follows that for each k

$$\sum_{i,j}^{n} g^{ij}\partial_{ij}u_k = 0. \tag{1.21}$$

To see ellipticity, differentiate with respect to the eigenvalues

$$D_\lambda \sum_{i=1}^{n} \arctan\lambda_i = \left(\frac{1}{1+\lambda_1^2}, \ldots, \frac{1}{1+\lambda_n^2}\right).$$

and we have that (1.1) is an elliptic equation. We also observe that the inverse of the induced metric g is the linearized operator to (1.1).

Differentiating (1.18) in λ-space, considering (1.8), we get

$$\frac{D_\lambda\left[(1-\sigma_2+\sigma_4\ldots)+\sqrt{-1}(\sigma_1-\sigma_3+\sigma_5\ldots)\right]}{Ve^{\sqrt{-1}\Theta}} = \frac{D_\lambda V}{V} + \sqrt{-1}D_\lambda\Theta.$$

Multiplying by V and setting imaginary parts equal, and recalling that $\Theta = \sum_{i=1}^{n}\arctan\lambda_i$, we have

$$VD_\lambda\Theta = -\sin\Theta D_\lambda(1-\sigma_2+\sigma_4\ldots) + \cos\Theta D_\lambda(\sigma_1-\sigma_3+\sigma_5\ldots). \tag{1.22}$$

Examining the relation (1.22) between the minimal surface form of the special Lagrangian equation (1.1) and the sigma form of the equation (1.2) shows the following "conformality" equality.

Lemma 1.4.2. *The linearized operator to (1.2) is equal to the volume element V times the linearized operator to (1.1).*

Geometrically, the linearized operator lies normal to the level set determined by the equations. When level sets of two different equations coincide, the normals are then parallel up to a "conformal" factor. Note that each of the equations (1.1) corresponds to one equation of the form (1.2), whereas the equation (1.2) generally arises as n disconnected level sets, each corresponding to a different phase. For example, the equation (1.4) corresponds to three different phases, $\pi, 0$, and $-\pi$. As we will see below, the phase π gives the level set corresponding to convex solutions and the phase $-\pi$ the level set corresponding to concave solutions. The middle phase 0 gives a level set of "saddle" solutions.

We see that the signs of the real and imaginary parts of (1.8) are determined by the phase Θ. For example, when $n=3$, suppose $\Theta \in (\pi/2, 3\pi/2)$. Then the complex number (1.8) with argument Θ lies in the left half plane, so has negative real part, and it follows that $1-\sigma_2<0$, or

$$\sigma_2 > 1.$$

We also note that for large phase equations $\Theta \geq (n-2)\pi/2$, we must have $\sigma_{n-1}>0$, by the following calculation. To begin

$$\sigma_{n-1} = \sigma_n\left(\frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_n}\right).$$

Only one eigenvalue may be negative (see next subsection on large phase), assume that λ_n is negative (otherwise the observation is trivial). Then σ_n is negative, and we must show that the sum of the reciprocals is also negative. Taking $\theta_i = \arctan \lambda_i$, we have for some $\delta \geq 0$,

$$\sum_{i=1}^{n} \theta_i = (n-2)\frac{\pi}{2} + \delta,$$

$$\theta_n + \sum_{i=1}^{n-1} \theta_i = (n-1)\frac{\pi}{2} - \frac{\pi}{2} + \delta$$

$$\left(\frac{\pi}{2} + \theta_n\right) = \sum_{i=1}^{n-1}\left(\frac{\pi}{2} - \theta_i\right) + \delta.$$

Now $\tan(\pi/2+\theta_n) = -1/\lambda_n$, and $\tan(\pi/2-\theta_i) = \cot\theta_i = 1/\lambda_i.$ Using properties of the tangent function, we have

$$\tan(\pi/2+\theta_n) = \tan\left(\sum_{i=1}^{n-1}\left(\frac{\pi}{2}-\theta_i\right)+\delta\right) > \sum_{i=1}^{n-1}\tan\left(\frac{\pi}{2}-\theta_i\right) + \tan\delta,$$

$$-\frac{1}{\lambda_n} > \sum_{i=1}^{n-1}\frac{1}{\lambda_i} + \tan\delta,$$

$$0 > \sum_{i=1}^{n}\frac{1}{\lambda_i} + \tan\delta.$$

Large Phase and Very Large Phase

We define large phase, and very large phase, to be the values Θ such that, respectively

$$\Theta \geq \frac{\pi}{2}(n-2),$$

$$\Theta \geq \frac{\pi}{2}(n-1).$$

Equations with large phase have several properties which make them vulnerable to our attacks. A special Lagrangian equation is convex if and only if it is large phase , as is demonstrated by Yuan [58]. Also, strictly large phase puts the condition of a lower bound on the Hessian of any solution, as follows. Suppose

$$\sum_{i=1}^{n} \arctan \lambda_i = \frac{\pi}{2}(n-2) + \delta.$$

Since

$$\sum_{i=1}^{n-1} \arctan \lambda_i \leq \frac{\pi}{2}(n-1)$$

it follows that $\arctan \lambda_n \geq \delta - \frac{\pi}{2}$, or $\lambda_n \geq \tan(\delta - \frac{\pi}{2})$. Taking $\delta = \frac{\pi}{2}$ we see that very large phase equations admit only convex solutions. A similar argument also shows that a solution to a large phase equation is 2-convex, that is, the sum of any two eigenvalues is nonnegative. Finally, any solution with Hessian bounded below by a fixed lower bound can be rotated to a solution with controlled Hessian [58], as we will see in our section on rotations. This means geometrically that the rotated solution satisfies a uniformly elliptic equation.

We note that the equation (1.1) is odd symmetric, which follows from the simple fact that arctan is an odd function. So the study of any equation of negative phase can be reduced to studying equations of positive phase. For this reason we will treat large and very large negative phases simply as large and very large phases.

1.5 More Geometric Preliminaries

1.5.1 Lewy-Yuan Rotation

Geometrically, a given Lagrangian surface $\mathfrak{M}$ in $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ can be parametrized as a gradient graph over a large number of Lagrangian planes. There is nothing special about which plane we choose. Choosing a different Lagrangian plane may change the phase of the graph by a constant along the whole graph. We can use this to our advantage. The easiest way to change the phase is to rotate by the same angle in each complex plane, as in the following proposition.

The Lewy rotation was a transformation first used in 1937 [28], to rotate convex solutions of Monge-Ampère equations to functions with bounded Hessians. (See also [34] for its use in a proof of the classical Bernstein Theorem.) In our terminology, Lewy's transformation amounts to a phase change of $-n\pi/2$. Yuan [58] then showed that this transformation can be used for a continuum of phases to obtain solutions of special Lagrangian equations with bounded Hessian. The following proposition is essentially given in ([58] p. 1356, see also. [56].)

Proposition 1.5.1. *Let u be a smooth solution to (1.1) with $\Theta = (n-2)\pi/2 + \delta$ on $B_R(0) \subset \mathbb{R}^n$. Then the special Lagrangian surface $\mathfrak{M} = (x, Du(x))$ can be represented as a gradient graph $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ of the new potential $\bar{u}$ satisfying (1.1) with phase $\Theta = (n-2)\pi/2$ in a domain containing a ball of radius*

$$\bar{R} \geq \frac{R}{2\cos(\delta/n)}.$$

Proof. To obtain the new representation, take a $U(n)$ rotation of $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$: $\bar{z} = e^{-\sqrt{-1}\delta/n} z$ with $z = x + \sqrt{-1}y$ and $\bar{z} = \bar{x} + \sqrt{-1}\bar{y}$. Because $U(n)$ rotation preserves the length and complex structure, $\mathfrak{M}$ is still a special Lagrangian submanifold with the parametrization

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{n} + Du(x)\sin\frac{\delta}{n} \\ D\bar{u} = -x\sin\frac{\delta}{n} + Du(x)\cos\frac{\delta}{n} \end{cases}.$$

In order to show that this parametrization is that of a gradient graph over $\bar{x}$, we must first show that $\bar{x}(x)$ is a diffeomorphism onto its image. This is accomplished by showing that

$$\left|\bar{x}(x^\alpha) - \bar{x}(x^\beta)\right| \geq \frac{1}{2\cos\delta/n}\left|x^\alpha - x^\beta\right| \tag{1.23}$$

for any x^α, x^β. We assume by translation that $x^\beta = 0$ and $Du\left(x^\beta\right) = 0$. Now $0 < \delta < \pi$, and $\theta_i > \delta - \frac{\pi}{2}$, so $u + \frac{1}{2}\cot\delta|x|^2$ is convex, and we have

$$\begin{aligned}
\left|\bar{x}\left(x^\alpha\right) - \bar{x}\left(x^\beta\right)\right|^2 &= \left|\bar{x}\left(x^\alpha\right)\right|^2 = \left|x^\alpha \cos\frac{\delta}{n} + Du\left(x^\alpha\right)\sin\frac{\delta}{n}\right|^2 \\
&= \left|x^\alpha\left(\cos\frac{\delta}{n} - \cot\delta\sin\frac{\delta}{n}\right) + \left[Du\left(x^\alpha\right) + x^\alpha\cot\delta\right]\sin\frac{\delta}{n}\right|^2 \\
&= |x^\alpha|^2\left[\frac{\sin\frac{(n-1)\delta}{n}}{\sin\delta}\right]^2 + \left|Du\left(x^\alpha\right) + x^\alpha\cot\delta\right|^2\sin^2\frac{\delta}{n} \\
&+ 2\frac{\sin\frac{(n-1)\delta}{n}\sin\frac{\delta}{n}}{\sin\delta}\left\langle x^\alpha, Du\left(x^\alpha\right) + x\cot\delta\right\rangle \\
&\geq |x^\alpha|^2\left(\frac{1}{2\cos\frac{\delta}{n}}\right)^2.
\end{aligned}$$

It follows that $\mathfrak{M}$ is a special Lagrangian graph over $\bar{x}$. The Lagrangian graph is the gradient graph of a potential function $\bar{u}$ (cf. [19], Lemma 2.2), that is, $\mathfrak{M} = \left(\bar{x}, D\bar{u}\left(\bar{x}\right)\right)$. The eigenvalues $\bar{\lambda}_i$ of the Hessian $D^2\bar{u}$ are determined by

$$\bar{\theta}_i = \arctan\bar{\lambda}_i = \theta_i - \frac{\delta}{n} \in \left(-\frac{\pi}{2} + \frac{(n-1)\,\delta}{n}, \frac{\pi}{2} - \frac{\delta}{n}\right). \tag{1.24}$$

Then

$$\sum_{i=1}^{n}\bar{\theta}_i = \frac{(n-2)\,\pi}{2},$$

that is, $\bar{u}$ satisfies the special Lagrangian equation (1.1) of phase $(n-2)\,\pi/2$. The lower bound on $\bar{R}$ follows immediately from (1.23). □

1.5.2 Relative Isoperimetric Inequality

This relative isoperimetric inequality is needed in the proof of Theorems 2 and 5 to prove a key ingredient, namely a Sobolev inequality for functions without compact support. The proposition is proved from the following classical relative isoperimetric inequality for balls.

Lemma 1.5.1. *Let A and A^c be disjoint measurable sets such that $A \cup A^c = B_1(0) \subset \mathbb{R}^n$. Then*

$$\min\{|A|, |A^c|\} \leq C(n)\,|\partial A \cap \partial A^c|^{n/n-1}. \tag{1.25}$$

Proof. See for example ([29], Theorem 5.3.2.). □

Proposition 1.5.2. *Let $\Omega_1 \subset \Omega_2 \subset B_\rho \subset \mathbb{R}^n$. Suppose that $dist(\Omega_1, \partial\Omega_2) \geq 2$, also A and A^c are disjoint measurable sets such that $A \cup A^c = \Omega_2$. Suppose that*

$$|A \cap \Omega_1| \leq (1-\varepsilon)\,|\Omega_1|,$$

for $\varepsilon \in (0, 1/2]$, then

$$|A \cap \Omega_1| \leq \frac{C(n)}{\varepsilon}\rho^n\,|\partial A \cap \partial A^c \cap \Omega_2|^{\frac{n}{n-1}}.$$

Proof. Define a continuous function on Ω_1

$$\xi(x) = \frac{|A \cap B_1(x)|}{|B_1|}.$$

Case 1. $\xi(x_*) = 1 - \varepsilon$ for some $x_* \in \Omega_1$. From the classical relative isoperimetric inequality for balls above, we have

$$\varepsilon|B_1| = \min\{|A \cap B_1(x)|, |A^c \cap B_1(x)| \leq C(n)\,|\partial A \cap \partial A^c \cap B_1(x_*)|^{\frac{n}{n-1}} \leq C(n)|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

It then follows

$$|A \cap B_1| \leq |\Omega_1| < |B_\rho| = C(n)\rho^n|B_1| \leq \frac{1}{\varepsilon}C(n)\rho^n|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

Case 2.1. $\xi(x) < 1 - \varepsilon$ for all $x \in \Omega_1$. Cover Ω_1 by $C(n)\rho^n$ unit balls $B_1(x_i)$ for some uniform constant $C(n)$. Note that all these balls are inside Ω_2. By the classical relative isoperimetric inequality for balls again,

$$|A \cap B_1(x_i)| \leq \frac{1-\varepsilon}{\varepsilon}\min\{|A \cap B_1(x_i)|, |A^c \cap B_1(x_i)|\} \leq \frac{1-\varepsilon}{\varepsilon}C(n)|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

Summing this inequality over this cover, we get

$$|A \cap \Omega_1| \leq \sum_{i=1}^{C(n)\rho^n} |A \cap B_1(x_i)| \leq \frac{1}{\varepsilon}C(n)\rho^n\,|\partial A \cap \partial A^c|^{\frac{n}{n-1}},$$

then the conclusion of the proposition follows.

Case 2.2. $\xi(x) > 1-\varepsilon$ for all $x \in \Omega_1$. Again cover Ω_1 by $C(n)\rho^n$ unit balls $B_1(x_i)$, lying inside Ω_2. As before,

$$|A^c \cap B_1(x_i)| = \min\{|A \cap B_1(x_i)|, |A^c \cap B_1(x_i)|\} \leq C(n)|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

Summing this inequality over the cover, we get

$$|A^c \cap \Omega_1| \leq \sum_{i=1}^{C(n)\rho^n} |A^c \cap B_1(x_i)| \leq C(n)\rho^n |\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

By the hypothesis,

$$|A \cap \Omega_1| \leq \frac{1-\varepsilon}{\varepsilon}|A^c \cap \Omega_1|,$$

thus

$$|A \cap \Omega_1| \leq \frac{C(n)}{\varepsilon}\rho^n |\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

□

Remark. Considering dumbbell type regions, we see that the order of dependence on ρ is sharp in Proposition 1.5.2.

We note that in the two dimensional case, the analogous isoperimetric inequality is stronger, see the proof of Theorem 4 for details.

1.6 Proof of Theorem 1: The Sigma-2 Equation in Dimension Three

In this section, we derive an *interior a priori* Hessian estimate for the σ_2 equation

$$\sigma_2\left(D^2u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$$

in dimension three, where λ_i are the eigenvalues of the Hessian D^2u. We attack (1.3) via its special Lagrangian equation form

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta$$

with $n = 3$ and $\Theta = \pi/2$.

1.6.1 Jacobi Inequality

Lemma 1.6.1. *Let u be a smooth solution to (1.1). Suppose that the Hessian D^2u is diagonalized and the eigenvalue λ_1 is distinct from all other eigenvalues of D^2u at point p. Set $b_1 = \ln\sqrt{1+\lambda_1^2}$ near p. Then we have at p*

$$|\nabla_g b_1|^2 = \sum_{k=1}^{n} \lambda_1^2 h_{11k}^2 \tag{1.26}$$

and

$$\triangle_g b_1 =$$

$$(1+\lambda_1^2)h_{111}^2 + \sum_{k>1}\left(\frac{2\lambda_1}{\lambda_1-\lambda_k} + \frac{2\lambda_1^2\lambda_k)}{\lambda_1-\lambda_k}\right)h_{kk1}^2 \tag{1.27}$$

$$+\sum_{k>1}\left[1+\frac{2\lambda_1}{\lambda_1-\lambda_k}+\frac{\lambda_1^2\left(\lambda_1+\lambda_k\right)}{\lambda_1-\lambda_k}\right]h_{11k}^2 \tag{1.28}$$

$$+\sum_{k>j>1} 2\lambda_1\left[\frac{1+\lambda_k^2}{\lambda_1-\lambda_k}+\frac{1+\lambda_j^2}{\lambda_1-\lambda_j}+(\lambda_j+\lambda_k)\right]h_{kj1}^2. \tag{1.29}$$

Proof. We first compute the derivatives of the smooth function b_1 near p. We may implicitly differentiate the characteristic equation

$$\det(D^2u - \lambda_1 I) = 0$$

near any point where λ_1 is distinct from the other eigenvalues. Then we get at p

$$\partial_e \lambda_1 = \partial_e u_{11},$$
$$\partial_{ee}\lambda_1 = \partial_{ee} u_{11} + \sum_{k>1} 2\frac{(\partial_e u_{1k})^2}{\lambda_1 - \lambda_k},$$

with arbitrary unit vector $e \in \mathbb{R}^n$.

Thus we have (1.26) at p

$$|\nabla_g b_1|^2 = \sum_{k=1}^n g^{kk}\left(\frac{\lambda_1}{1+\lambda_1^2}\partial_k u_{11}\right)^2 = \sum_{k=1}^n \lambda_1^2 h_{11k}^2,$$

where we used the notation $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$.

From

$$\partial_{ee} b_1 = \partial_{ee} \ln\sqrt{1+\lambda_1^2} = \frac{\lambda_1}{1+\lambda_1^2}\partial_{ee}\lambda_1 + \frac{1-\lambda_1^2}{(1+\lambda_1^2)^2}(\partial_e \lambda_1)^2,$$

we conclude that at p

$$\partial_{ee} b_1 = \frac{\lambda_1}{1+\lambda_1^2}\left[\partial_{ee} u_{11} + \sum_{k>1} 2\frac{(\partial_e u_{1k})^2}{\lambda_1 - \lambda_k}\right] + \frac{1-\lambda_1^2}{(1+\lambda_1^2)^2}(\partial_e u_{11})^2,$$

and

$$\begin{aligned}\triangle_g b_1 &= \sum_{\gamma=1}^n g^{\gamma\gamma}\partial_{\gamma\gamma} b_1 \\ &= \sum_{\gamma=1}^n g^{\gamma\gamma}\frac{\lambda_1}{1+\lambda_1^2}\left(\partial_{\gamma\gamma} u_{11} + \sum_{k>1} 2\frac{(u_{1k\gamma})^2}{\lambda_1 - \lambda_k}\right) + \sum_{\gamma=1}^n \frac{1-\lambda_1^2}{(1+\lambda_1^2)^2} g^{\gamma\gamma} u_{11\gamma}^2. \end{aligned} \tag{1.30}$$

Next we substitute the fourth order derivative terms $\partial_{\gamma\gamma} u_{11}$ in the above by lower order derivative terms. Differentiating the minimal surface equation (1.21) $\sum_{\alpha,\beta=1}^n g^{\alpha\beta} u_{j\alpha\beta} = 0$, we obtain

$$\begin{aligned}\triangle_g u_{ij} &= \sum_{\alpha,\beta=1}^n g^{\alpha\beta} u_{ji\alpha\beta} = \sum_{\alpha,\beta=1}^n -\partial_i g^{\alpha\beta} u_{j\alpha\beta} = \sum_{\alpha,\beta,\gamma,\delta=1}^n g^{\alpha\gamma}\partial_i g_{\gamma\delta} g^{\delta\beta} u_{j\alpha\beta} \\ &= \sum_{\alpha,\beta=1}^n g^{\alpha\alpha} g^{\beta\beta}(\lambda_\alpha + \lambda_\beta) u_{\alpha\beta i} u_{\alpha\beta j}, \end{aligned} \tag{1.31}$$

where we used

$$\partial_i g_{\gamma\delta} = \partial_i(\delta_{\gamma\delta} + \sum_{\varepsilon=1}^n u_{\gamma\varepsilon} u_{\varepsilon\delta}) = u_{\gamma\delta i}(\lambda_\gamma + \lambda_\delta)$$

with diagonalized D^2u. Plugging (1.31) with $i=j=1$ in (1.30), we have at p

$$\begin{aligned}
\triangle_g b_1 = & \frac{\lambda_1}{1+\lambda_1^2}\left[\sum_{\alpha,\beta=1}^{n} g^{\alpha\alpha}g^{\beta\beta}(\lambda_\alpha+\lambda_\beta)u_{\alpha\beta 1}^2 + \sum_{\gamma=1}^{n}\sum_{k>1} 2\frac{u_{1k\gamma}^2}{\lambda_1-\lambda_k}g^{\gamma\gamma}\right] \\
& + \sum_{\gamma=1}^{n}\frac{1-\lambda_1^2}{\left(1+\lambda_1^2\right)^2} g^{\gamma\gamma}u_{11\gamma}^2 \\
& = \lambda_1 \sum_{\alpha,\beta=1}^{n}(\lambda_\alpha+\lambda_\beta)h_{\alpha\beta 1}^2 + \sum_{k>1}\sum_{\gamma=1}^{n}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{\gamma k 1}^2 + \sum_{\gamma=1}^{n}\left(1-\lambda_1^2\right)h_{11\gamma}^2,
\end{aligned}$$

where we used the notation $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$. Regrouping those terms $h_{\heartsuit\heartsuit 1}$, $h_{11\heartsuit}$, and $h_{\heartsuit\clubsuit 1}$ in the last expression, we have

$$\begin{aligned}
\triangle_g b_1 = & \left(1-\lambda_1^2\right)h_{111}^2 + \sum_{\alpha=1}^{n}2\lambda_1\lambda_\alpha h_{\alpha\alpha 1}^2 + \sum_{k>1}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{kk1}^2 \\
& + \sum_{k>1}2\lambda_1(\lambda_k+\lambda_1)h_{k11}^2 + \sum_{k>1}\left(1-\lambda_1^2\right)h_{11k}^2 + \sum_{k>1}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{1k1}^2 \\
& + \sum_{k>j>1}2\lambda_1(\lambda_j+\lambda_k)h_{jk1}^2 + \sum_{\substack{j,k>1,\\ j\neq k}}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{jk1}^2.
\end{aligned}$$

After simplifying the above expression, we have the second formula in Lemma 1.6.1 ☐

Lemma 1.6.2. *Let u be a smooth solution to (1.1) with $n=3$ and $\Theta \geq \pi/2$. Suppose that the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the Hessian D^2u satisfy $\lambda_1 > \lambda_2$ at point p. Set*

$$b_1 = \ln\sqrt{1+\lambda_{\max}^2} = \ln\sqrt{1+\lambda_1^2}.$$

Then we have at p

$$\triangle_g b_1 \geq \frac{1}{3}|\nabla_g b_1|^2. \tag{1.32}$$

Proof . We assume that the Hessian D^2u is diagonalized at point p.

Step 1. Recall $\theta_i = \arctan\lambda_i \in (-\pi/2, \pi/2)$ and $\theta_1+\theta_2+\theta_3 = \Theta \geq \pi/2$. It is easy to see that $\theta_1 \geq \theta_2 > 0$ and $\theta_i+\theta_j \geq 0$ for any pair. Consequently $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_i+\lambda_j \geq 0$ for any pair of distinct eigenvalues. It follows that (1.29) in the formula for $\triangle_g b_1$ is positive, then from (1.27) and (1.28) we have the inequality

$$\triangle_g b_1 \geq \lambda_1^2\left(h_{111}^2 + \sum_{k>1}\frac{2\lambda_k}{\lambda_1-\lambda_k}h_{kk1}^2\right) + \lambda_1^2\sum_{k>1}\left(1+\frac{2\lambda_k}{\lambda_1-\lambda_k}\right)h_{11k}^2. \tag{1.33}$$

Combining (1.33) and (1.26) gives

$$\triangle_g b_1 - \frac{1}{3}|\nabla_g b_1|^2 \geq$$
$$\lambda_1^2 \left(\frac{2}{3} h_{111}^2 + \sum_{k>1} \frac{2\lambda_k}{\lambda_1 - \lambda_k} h_{kk1}^2 \right) + \lambda_1^2 \sum_{k>1} \frac{2(\lambda_1 + 2\lambda_k)}{3(\lambda_1 - \lambda_k)} h_{11k}^2. \tag{1.34}$$

Step 2. We show that the last term in (1.34) is nonnegative. Note that $\lambda_1 + 2\lambda_k \geq \lambda_1 + 2\lambda_3$. We only need to show that $\lambda_1 + 2\lambda_3 \geq 0$ in the case that $\lambda_3 < 0$ or equivalently $\theta_3 < 0$. From $\theta_1 + \theta_2 + \theta_3 = \Theta \geq \pi/2$, we have

$$\frac{\pi}{2} > \theta_3 + \frac{\pi}{2} = \left(\frac{\pi}{2} - \theta_1\right) + \left(\frac{\pi}{2} - \theta_2\right) + \Theta - \frac{\pi}{2} \geq 2\left(\frac{\pi}{2} - \theta_1\right).$$

It follows that

$$-\frac{1}{\lambda_3} = \tan\left(\theta_3 + \frac{\pi}{2}\right) > 2\tan\left(\frac{\pi}{2} - \theta_1\right) = \frac{2}{\lambda_1},$$

then

$$\lambda_1 + 2\lambda_3 > 0. \tag{1.35}$$

Step 3. We show that the first term in (1.34) is nonnegative by proving

$$\frac{2}{3} h_{111}^2 + \frac{2\lambda_2}{\lambda_1 - \lambda_2} h_{221}^2 + \frac{2\lambda_3}{\lambda_1 - \lambda_3} h_{331}^2 \geq 0. \tag{1.36}$$

We only need to show it for $\lambda_3 < 0$. Directly from the minimal surface equation (1.21)

$$h_{111} + h_{221} + h_{331} = 0,$$

we bound

$$h_{331}^2 = (h_{111} + h_{221})^2 \leq \left(\frac{2}{3} h_{111}^2 + \frac{2\lambda_2}{\lambda_1 - \lambda_2} h_{221}^2 \right) \left(\frac{3}{2} + \frac{\lambda_1 - \lambda_2}{2\lambda_2} \right).$$

It follows that

$$\frac{2}{3} h_{111}^2 + \frac{2\lambda_2}{\lambda_1 - \lambda_2} h_{221}^2 + \frac{2\lambda_3}{\lambda_1 - \lambda_3} h_{331}^2 \geq$$
$$\left(\frac{2}{3} h_{111}^2 + \frac{2\lambda_2}{\lambda_1 - \lambda_2} h_{221}^2 \right) \left[1 + \frac{2\lambda_3}{\lambda_1 - \lambda_3} \left(\frac{3}{2} + \frac{\lambda_1 - \lambda_2}{2\lambda_2} \right) \right].$$

The last term becomes

$$1+\frac{2\lambda_3}{\lambda_1-\lambda_3}\left(\frac{3}{2}+\frac{\lambda_1-\lambda_2}{2\lambda_2}\right)=\frac{\sigma_2}{(\lambda_1-\lambda_3)\,\lambda_2}>0.$$

The above inequality is from the observation

$$\operatorname{Re}\prod_{i=1}^{3}\left(1+\sqrt{-1}\lambda_i\right)=1-\sigma_2\leq 0$$

for $3\pi/2>\theta_1+\theta_2+\theta_3=\Theta\geq\pi/2$. Therefore (1.36) holds.

We have proved the pointwise Jacobi inequality (1.32) in Lemma 1.6.2. □

Lemma 1.6.3. *Let u be a smooth solution to (1.1) with $n=3$ and $\Theta\geq\pi/2$. Suppose that the ordered eigenvalues $\lambda_1\geq\lambda_2\geq\lambda_3$ of the Hessian D^2u satisfy $\lambda_2>\lambda_3$ at point p. Set*

$$b_2=\frac{1}{2}\left(\ln\sqrt{1+\lambda_1^2}+\ln\sqrt{1+\lambda_2^2}\right).$$

Then b_2 satisfies at p

$$\triangle_g b_2\geq 0. \tag{1.37}$$

Further, suppose that $\lambda_1\equiv\lambda_2$ in a neighborhood of p. Then b_2 satisfies at p

$$\triangle_g b_2\geq\frac{1}{3}\left|\nabla_g b_2\right|^2. \tag{1.38}$$

Proof. We assume that Hessian D^2u is diagonalized at point p. We may use Lemma 1.6.1 to obtain expressions for both $\triangle_g\ln\sqrt{1+\lambda_1^2}$ and $\triangle_g\ln\sqrt{1+\lambda_2^2}$, whenever the eigenvalues of D^2u are distinct. From (1.27), (1.28), and (1.29), we have

$$\begin{aligned}
&\triangle_g\ln\sqrt{1+\lambda_1^2}+\triangle_g\ln\sqrt{1+\lambda_2^2}= \qquad (1.39)\\
&(1+\lambda_1^2)h_{111}^2+\sum_{k>1}\frac{2\lambda_1(1+\lambda_1\lambda_k)}{\lambda_1-\lambda_k}h_{kk1}^2+\sum_{k>1}\left[1+\lambda_1^2+2\lambda_1\left(\frac{1+\lambda_1\lambda_k}{\lambda_1-\lambda_k}\right)\right]h_{11k}^2\\
&\quad+2\lambda_1\left[\frac{1+\lambda_3^2}{\lambda_1-\lambda_3}+\frac{1+\lambda_2^2}{\lambda_1-\lambda_2}+(\lambda_3+\lambda_2)\right]h_{321}^2\\
&+(1+\lambda_2^2)h_{222}^2+\sum_{k\neq 2}\frac{2\lambda_2(1+\lambda_2\lambda_k)}{\lambda_2-\lambda_k}h_{kk2}^2+\sum_{k\neq 2}\left[1+\lambda_2^2+2\lambda_2\left(\frac{1+\lambda_2\lambda_k}{\lambda_2-\lambda_k}\right)\right]h_{22k}^2\\
&\quad+2\lambda_2\left[\frac{1+\lambda_3^2}{\lambda_2-\lambda_3}+\frac{1+\lambda_1^2}{\lambda_2-\lambda_1}+(\lambda_3+\lambda_1)\right]h_{321}^2.
\end{aligned}$$

The function b_2 is symmetric in λ_1 and λ_2, thus b_2 is smooth even when $\lambda_1 = \lambda_2$, provided that $\lambda_2 > \lambda_3$. We simplify (1.39) to the following, which holds by continuity wherever $\lambda_1 \geq \lambda_2 > \lambda_3$.

$$2 \triangle_g b_2 =$$

$$(1+\lambda_1^2)h_{111}^2 + (3+\lambda_2^2+2\lambda_1\lambda_2)h_{221}^2 + \left(\frac{2\lambda_1}{\lambda_1-\lambda_3} + \frac{2\lambda_1^2\lambda_3}{\lambda_1-\lambda_3}\right)h_{331}^2 \tag{1.40}$$

$$+ (3+\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + (1+\lambda_2^2)h_{222}^2 + \left(\frac{2\lambda_2}{\lambda_2-\lambda_3} + \frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\right)h_{332}^2 \tag{1.41}$$

$$+ \left[\frac{3\lambda_1-\lambda_3+\lambda_1^2(\lambda_1+\lambda_3)}{\lambda_1-\lambda_3}\right]h_{113}^2 + \left[\frac{3\lambda_2-\lambda_3+\lambda_2^2(\lambda_2+\lambda_3)}{\lambda_2-\lambda_3}\right]h_{223}^2 \tag{1.42}$$

$$+2\left[1+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1+\frac{\lambda_1\left(1+\lambda_3^2\right)}{\lambda_1-\lambda_3}+\frac{\lambda_2\left(1+\lambda_3^2\right)}{\lambda_2-\lambda_3}\right]h_{123}^2. \tag{1.43}$$

Using the relations $\lambda_1 \geq \lambda_2 > 0$, $\lambda_i+\lambda_j > 0$, and $\sigma_2 \geq 1$ derived in the proof of Lemma 1.6.2, we see that (1.43) and (1.42) are nonnegative. We only need to justify the nonnegativity of (1.40) and (1.41) for $\lambda_3 < 0$. From the minimal surface equation (1.21), we know

$$h_{332}^2 = (h_{112}+h_{222})^2 \leq \left[(\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2h_{222}^2\right]\left(\frac{1}{\lambda_1^2+2\lambda_1\lambda_2}+\frac{1}{\lambda_2^2}\right).$$

It follows that

$$\begin{aligned}
(1.41) &\geq (\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2h_{222}^2 + \frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}h_{332}^2 \\
&\geq \left[(\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2h_{222}^2\right]\left[1+\frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\left(\frac{1}{\lambda_1^2+2\lambda_1\lambda_2}+\frac{1}{\lambda_2^2}\right)\right].
\end{aligned}$$

The last term becomes

$$\begin{aligned}
&\frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\left(\frac{\lambda_2-\lambda_3}{2\lambda_2^2\lambda_3}+\frac{1}{\lambda_1^2+2\lambda_1\lambda_2}+\frac{1}{\lambda_2^2}\right) \\
&= \frac{\lambda_2}{\lambda_2-\lambda_3}\left[\frac{\sigma_2}{\lambda_1\lambda_2} - \frac{\lambda_3}{(\lambda_1+2\lambda_2)}\right] \geq 0.
\end{aligned}$$

Thus (1.41) is nonnegative. Similarly (1.40) is nonnegative. We have proved (1.37).

Next we prove (1.38), still assuming D^2u is diagonalized at point p. Plugging in $\lambda_1 = \lambda_2$ into (1.40), (1.41), and (1.42), we get

$$2 \triangle_g b_2 \geq$$

$$\lambda_1^2\left(h_{111}^2+3h_{221}^2+\frac{2\lambda_3}{\lambda_1-\lambda_3}h_{331}^2\right)$$
$$+\ \lambda_1^2\left(3h_{112}^2+h_{222}^2+\frac{2\lambda_3}{\lambda_1-\lambda_3}h_{332}^2\right)$$
$$+\lambda_1^2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)\left(h_{113}^2+h_{223}^2\right).$$

Differentiating the eigenvector equations in the neighborhood where $\lambda_1\equiv\lambda_2$

$$\left(D^2u\right)U=\frac{\lambda_1+\lambda_2}{2}U,\ \left(D^2u\right)V=\frac{\lambda_1+\lambda_2}{2}V,\ \text{and}\ \left(D^2u\right)W=\lambda_3W,$$

we see that $u_{11e}=u_{22e}$ for any $e\in\mathbb{R}^3$ at point p. Using the minimal surface equation (1.21), we then have

$$h_{11k}=h_{22k}=-\frac{1}{2}h_{33k}$$

at point p. Thus

$$\triangle_g b_2\geq\lambda_1^2\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)h_{111}^2+2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)h_{112}^2+\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)h_{113}^2\right].$$

The gradient $|\nabla_g b_2|^2$ has the expression at p

$$|\nabla_g b_2|^2=\sum_{k=1}^3 g^{kk}\left(\frac{1}{2}\frac{\lambda_1}{1+\lambda_1^2}\partial_k u_{11}+\frac{1}{2}\frac{\lambda_2}{1+\lambda_2^2}\partial_k u_{22}\right)^2=\sum_{k=1}^3\lambda_1^2h_{11k}^2.$$

Thus at p

$$\triangle_g b_2-\frac{1}{3}|\nabla_g b_2|^2\geq$$
$$\lambda_1^2\left\{\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)-\frac{1}{3}\right]h_{111}^2+\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)-\frac{1}{3}\right]h_{112}^2+\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}-\frac{1}{3}\right)h_{113}^2\right\}$$
$$\geq 0,$$

where we again used $\lambda_1+2\lambda_3>0$ from (1.35). We have proved (1.38) of Lemma 1.6.3. $\square$

The following is the main ingredient of the proof of Theorem 1, and is also useful in the next section on large phase equations.

Proposition 1.6.1. *Let u be a smooth solution to the special Lagrangian equation (1.1) with $n=3$ and $\Theta\geq\pi/2$ on $B_4(0)\subset\mathbb{R}^3$. Set*

$$b=\max\left\{\ln\sqrt{1+\lambda_{\max}^2},\ K\right\}$$

with $K = 1 + \ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)}$. *Then* b *satisfies the integral Jacobi inequality*

$$\int_{B_4} -\langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g \geq \frac{1}{3} \int_{B_4} \varphi \left|\nabla_g b\right|^2 dv_g \tag{1.44}$$

for all non-negative $\varphi \in C_0^\infty(B_4)$.

Proof. If $b_1 = \ln\sqrt{1+\lambda_{\max}^2}$ is smooth everywhere, then the pointwise Jacobi inequality (1.32) in Lemma 1.6.2 already implies the integral Jacobi (1.44). It is known that $\lambda_{\max}$ is always a Lipschitz function of the entries of the Hessian D^2u. Now u is smooth in x, so $b_1 = \ln\sqrt{1+\lambda_{\max}^2}$ is Lipschitz in terms of x. If b_1 (or equivalently $\lambda_{\max}$) is not smooth, then the first two largest eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ coincide, and $b_1(x) = b_2(x)$, where $b_2(x)$ is the average $b_2 = \left(\ln\sqrt{1+\lambda_1^2} + \ln\sqrt{1+\lambda_2^2}\right)/2$. We prove the integral Jacobi inequality (1.44) for a possibly singular $b_1(x)$ in two cases. Set

$$S = \{x \mid \lambda_1(x) = \lambda_2(x)\}.$$

Case 1. S has measure zero. For small $\tau > 0$, let

$$\begin{aligned} \Omega &= B_4 \backslash \{x \mid b_1(x) \leq K\} = B_4 \backslash \{x \mid b(x) = K\} \\ \Omega_1(\tau) &= \{x \mid b(x) = b_1(x) > b_2(x) + \tau\} \cap \Omega \\ \Omega_2(\tau) &= \{x \mid b_2(x) \leq b(x) = b_1(x) < b_2(x) + \tau\} \cap \Omega. \end{aligned}$$

Now $b(x) = b_1(x)$ is smooth in $\overline{\Omega_1(\tau)}$. We claim that $b_2(x)$ is smooth in $\overline{\Omega_2(\tau)}$. We know $b_2(x)$ is smooth wherever $\lambda_2(x) > \lambda_3(x)$. If (the Lipschitz) $b_2(x)$ is not smooth at $x_* \in \overline{\Omega_2(\tau)}$, then

$$\begin{aligned} \ln\sqrt{1+\lambda_3^2} = \ln\sqrt{1+\lambda_2^2} &\geq \ln\sqrt{1+\lambda_1^2} - 2\tau \\ &\geq \ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)} + 1 - 2\tau, \end{aligned}$$

by the choice of K. For small enough τ, we have $\lambda_2 = \lambda_3 > \tan\left(\frac{\Theta}{3}\right)$ and a contradiction

$$(\theta_1 + \theta_2 + \theta_3)(x_*) > \Theta.$$

Note that

$$\int_{B_4} - \langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g = \int_{\Omega} - \langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g$$

$$= \lim_{\tau \to 0^+} \left[\int_{\Omega_1(\tau)} - \langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g + \int_{\Omega_2(\tau)} - \langle \nabla_g \varphi, \nabla_g (b_2 + \tau) \rangle_g \, dv_g \right].$$

By the smoothness of b in $\Omega_1(\tau)$ and b_2 in $\Omega_2(\tau)$, and also inequalities (1.32) and (1.37), we have

$$\begin{aligned}
&\int_{\Omega_1(\tau)} - \langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g + \int_{\Omega_2(\tau)} - \langle \nabla_g \varphi, \nabla_g (b_2 + \tau) \rangle_g \, dv_g \\
&= \int_{\partial\Omega_1(\tau)} -\varphi \, \partial_{\gamma_g^1} b \, dA_g + \int_{\Omega_1(\tau)} \varphi \, \triangle_g \, b_1 dv_g \\
&+ \int_{\partial\Omega_2(\tau)} -\varphi \partial_{\gamma_g^2} (b_2 + \tau) \, dA_g + \int_{\Omega_2(\tau)} \varphi \, \triangle_g \, (b_2 + \tau) \, dv_g \\
&\geq \int_{\partial\Omega_1(\tau)} -\varphi \, \partial_{\gamma_g^1} b \, dA_g + \int_{\partial\Omega_2(\tau)} -\varphi \partial_{\gamma_g^2} (b_2 + \tau) \, dA_g + \frac{1}{3} \int_{\Omega_1(\tau)} \varphi \left| \nabla_g b_1 \right|^2 dv_g,
\end{aligned}$$

where γ_g^1 and γ_g^2 are the outward co-normals of $\partial\Omega_1(\tau)$ and $\partial\Omega_2(\tau)$ with respect to the metric g.

Observe that if b_1 is not smooth on any part of $\partial\Omega \backslash \partial B_4$, which is the K-level set of b_1, then on this portion $\partial\Omega \backslash \partial B_4$ is also the K-level set of b_2, which is smooth near this portion. Applying Sard's theorem, we can perturb K so that $\partial\Omega$ is piecewise C^1. Applying Sard's theorem again, we find a subsequence of positive τ going to 0, so that the boundaries $\partial\Omega_1(\tau)$ and $\partial\Omega_2(\tau)$ are piecewise C^1.

Then, we show the above boundary integrals are non-negative. The boundary integral portion along $\partial\Omega$ is easily seen non-negative, because either $\varphi = 0$, or $-\partial_{\gamma_g^1} b \geq 0$, $-\partial_{\gamma_g^2} (b_2 + \tau) \geq 0$ there. The boundary integral portion in the interior of Ω is also non-negative, because there we have

$$b = b_2 + \tau \quad (\text{and } b \geq b_2 + \tau \ \text{ in } \Omega_1(\tau))$$

$$-\partial_{\gamma_g^1} b \ - \partial_{\gamma_g^2} (b_2 + \tau) = \partial_{\gamma_g^2} b \ - \partial_{\gamma_g^2} (b_2 + \tau) \geq 0.$$

Taking the limit along the (Sard) sequence of τ going to 0, we obtain $\Omega_1(\tau) \to \Omega$ up to a set of measure zero, and

$$\begin{aligned}
&\int_{B_4} - \langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g \\
&= \int_\Omega - \langle \nabla_g \varphi, \nabla_g b \rangle_g \, dv_g \geq \frac{1}{3} \int_\Omega |\nabla_g b|^2 \, dv_g \\
&= \frac{1}{3} \int_{B_4} |\nabla_g b|^2 \, dv_g.
\end{aligned}$$

Case 2. S has positive measure. The discriminant

$$\mathcal{D} = (\lambda_1 - \lambda_2)^2 (\lambda_2 - \lambda_3)^2 (\lambda_3 - \lambda_1)^2$$

is an analytic function in B_4, because the smooth u is actually analytic (cf. [33], p. 203). So $\mathcal{D}$ must vanish identically. Then we have either $\lambda_1(x) = \lambda_2(x)$ or $\lambda_2(x) = \lambda_3(x)$ at any point $x \in B_4$. In turn, we know that $\lambda_1(x) = \lambda_2(x) = \lambda_3(x) = \tan\left(\frac{\Theta}{3}\right)$ and $b = K > b_1(x)$ at every "boundary" point of S inside B_4, $x \in \partial S \cap \mathring{B}_4$. If the "boundary" set ∂S has positive measure, then $\lambda_1(x) = \lambda_2(x) = \lambda_3(x) = \tan\left(\frac{\Theta}{3}\right)$ everywhere by the analyticity of u, and (1.44) is trivially true. In the case that ∂S has zero measure, $b = b_1 > K$ is smooth up to the boundary of every component of $\{x \mid b(x) > K\}$. By the pointwise Jacobi inequality (1.38), the integral inequality (1.44) is also valid in case 2. $\square$

1.6.2 Proof of Theorem 1

We assume that $R = 4$ and u is a solution on $B_4 \subset \mathbb{R}^3$ for simplicity of notation. By scaling $v(x) = u\left(\frac{R}{4}x\right) / \left(\frac{R}{4}\right)^2$, we still get the estimate in Theorem 1. Without loss of generality, we assume that the continuous Hessian D^2u sits on the convex branch of $\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1\}$ containing $(1,1,1)/\sqrt{3}$, then u satisfies (1.1) with $n = 3$ and $\Theta = \pi/2$. By symmetry this also covers the concave branch corresponding to $\Theta = -\pi/2$.

Step 1. By the integral Jacobi inequality (1.44) in Proposition 1.6.1, b is subharmonic in the integral sense, then b^3 is also subharmonic in the integral sense on the minimal surface

$\mathfrak{M} = (x, Du)$:

$$\int -\left\langle \nabla_g \varphi, \nabla_g b^3 \right\rangle_g dv_g = \int -\left\langle \nabla_g \left(3b^2\varphi\right) - 6b\varphi\nabla_g b, \nabla_g b\right\rangle_g dv_g \tag{1.45}$$

$$\geq \int \left(\varphi b^2 \left|\nabla_g b\right|^2 + 6b\varphi \left|\nabla_g b\right|^2\right) dv_g \geq 0 \tag{1.46}$$

for all non-negative $\varphi \in C_0^\infty$, approximating b by smooth functions if necessary.

Applying Michael-Simon's mean value inequality ([32], Theorem 3.4) to the Lipschitz subharmonic function b^3, we obtain

$$b(0) \leq C(3) \left(\int_{\mathfrak{B}_1 \cap \mathfrak{M}} b^3 dv_g\right)^{1/3} \leq C(3) \left(\int_{B_1} b^3 dv_g\right)^{1/3},$$

where $\mathfrak{B}_r$ is the ball with radius r and center $(0, Du(0))$ in $\mathbb{R}^3 \times \mathbb{R}^3$, and B_r is the ball with radius r and center 0 in $\mathbb{R}^3$. Choose a cut-off function $\varphi \in C_0^\infty(B_2)$ such that $\varphi \geq 0$, $\varphi = 1$ on B_1, and $|D\varphi| \leq 1.1$, we then have

$$\left(\int_{B_1} b^3 dv_g\right)^{1/3} \leq \left(\int_{B_2} \varphi^6 b^3 dv_g\right)^{1/3} = \left(\int_{B_2} \left(\varphi b^{1/2}\right)^6 dv_g\right)^{1/3}.$$

Applying the Sobolev inequality on the minimal surface $\mathfrak{M}$ ([32], Theorem 2.1) or ([1], Theorem 7.3) to $\varphi b^{1/2}$, which we may assume to be C^1 by approximation, we obtain

$$\left(\int_{B_2} \left(\varphi b^{1/2}\right)^6 dv_g\right)^{1/3} \leq C(3) \int_{B_2} \left|\nabla_g \left(\varphi b^{1/2}\right)\right|^2 dv_g.$$

Splitting the integrand as follows

$$\begin{aligned} \left|\nabla_g \left(\varphi b^{1/2}\right)\right|^2 = \left|\frac{1}{2b^{1/2}}\varphi\nabla_g b + b^{1/2}\nabla_g\varphi\right|^2 &\leq \frac{1}{2b}\varphi^2 \left|\nabla_g b\right|^2 + 2b\left|\nabla_g \varphi\right|^2 \\ &\leq \frac{1}{2}\varphi^2 \left|\nabla_g b\right|^2 + 2b\left|\nabla_g \varphi\right|^2, \end{aligned}$$

where we used $b \geq 1$, we get

$$\begin{aligned} b(0) &\leq C(3) \int_{B_2} \left|\nabla_g \left(\varphi b^{1/2}\right)\right|^2 dv_g \\ &\leq C(3) \left(\int_{B_2} \varphi^2 \left|\nabla_g b\right|^2 dv_g + \int_{B_2} b\left|\nabla_g \varphi\right|^2 dv_g\right) \\ &\leq \underbrace{C(3)\left\|Du\right\|_{L^\infty(B_2)}}_{\text{Step 2}} + C(3) \underbrace{\left[\left\|Du\right\|^2_{L^\infty(B_3)} + \left\|Du\right\|^3_{L^\infty(B_4)}\right]}_{\text{step 3}}. \end{aligned}$$

Step 2. By (1.44) in Proposition 1.6.1, b satisfies the Jacobi inequality in the integral sense:

$$3 \triangle_g b \geq |\nabla_g b|^2 .$$

Multiplying both sides by the above non-negative cut-off function $\varphi \in C_0^\infty(B_2)$, then integrating, we obtain

$$\begin{aligned}
\int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g &\leq 3 \int_{B_2} \varphi^2 \triangle_g b dv_g \\
&= -3 \int_{B_2} \langle 2\varphi \nabla_g \varphi, \nabla_g b \rangle dv_g \\
&\leq \frac{1}{2} \int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g + 18 \int_{B_2} |\nabla_g \varphi|^2 dv_g.
\end{aligned}$$

It follows that

$$\int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g \leq 36 \int_{B_2} |\nabla_g \varphi|^2 dv_g. \tag{1.47}$$

Observe the ("conformality") identity (Lemma 1.4.2):

$$\left(\frac{1}{1+\lambda_1^2}, \frac{1}{1+\lambda_2^2}, \frac{1}{1+\lambda_3^2} \right) V = (\sigma_1 - \lambda_1,\ \sigma_1 - \lambda_2,\ \sigma_1 - \lambda_3)$$

where we used the identity $V = \prod_{i=1}^{3} \sqrt{(1+\lambda_i^2)} = \sigma_1 - \sigma_3$ with $\sigma_2 = 1$. We then have

$$\begin{aligned}
|\nabla_g \varphi|^2 dv_g &= \sum_{i=1}^{3} \frac{(D_i \varphi)^2}{1+\lambda_i^2} V dx = \sum_{i=1}^{3} (D_i \varphi)^2 (\sigma_1 - \lambda_i) dx \\
&\leq 2.42 \triangle u \ dx.
\end{aligned} \tag{1.48}$$

Thus

$$\begin{aligned}
\int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g &\leq C(3) \int_{B_2} \Delta u \ dx \\
&\leq C(3) \|Du\|_{L^\infty(B_2)} .
\end{aligned}$$

Step 3. By (1.48), we get

$$\int_{B_2} b |\nabla_g \varphi|^2 dv_g \leq C(3) \int_{B_2} b \triangle u \ dx.$$

Choose another cut-off function $\psi \in C_0^\infty(B_3)$ such that $\psi \geq 0$, $\psi = 1$ on B_2, and $|D\psi| \leq 1.1$. We have

$$\begin{aligned}
\int_{B_2} b \triangle u dx \leq \int_{B_3} \psi b \triangle u dx &= \int_{B_3} -\langle bD\psi + \psi Db, Du\rangle \, dx \\
&\leq \|Du\|_{L^\infty(B_3)} \int_{B_3} \left(b\,|D\psi| + \psi\,|Db|\right) dx \\
&\leq C\,(3)\,\|Du\|_{L^\infty(B_3)} \int_{B_3} \left(b + |Db|\right) dx.
\end{aligned}$$

Now

$$b = \max\left\{\ln\sqrt{1+\lambda_{\max}^2},\ K\right\} \leq \lambda_{\max} + K < \lambda_1 + \lambda_2 + \lambda_3 + K = \Delta u + K,$$

where $\lambda_2 + \lambda_3 > 0$ follows from $\arctan\lambda_2 + \arctan\lambda_3 = \frac{\pi}{2} - \arctan\lambda_1 > 0$. Hence

$$\int_{B_3} bdx \leq C(3)(1 + \|Du\|_{L^\infty(B_3)}).$$

And we have left to estimate $\int_{B_3} |Db|\,dx$:

$$\begin{aligned}
\int_{B_3} |Db|\,dx &\leq \int_{B_3} \sqrt{\sum_{i=1}^{3} \frac{(D_i b)^2}{(1+\lambda_i^2)} \left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)\left(1+\lambda_3^2\right)}\,dx \\
&= \int_{B_3} |\nabla_g b|\,V dx \\
&\leq \left(\int_{B_3} |\nabla_g b|^2\,V dx\right)^{1/2} \left(\int_{B_3} V dx\right)^{1/2}.
\end{aligned}$$

Repeating the "Jacobi" argument from Step 2, we see

$$\int_{B_3} |\nabla_g b|^2\,V dx \leq C\,(3)\,\|Du\|_{L^\infty(B_4)}\,.$$

Then by the Sobolev inequality on the minimal surface $\mathfrak{M}$, we have

$$\int_{B_3} V dx = \int_{B_3} dv_g \leq \int_{B_4} \phi^6 dv_g \leq C\,(3)\left(\int_{B_4} |\nabla_g \phi|^2\,dv_g\right)^3,$$

where the non-negative cut-off function $\phi \in C_0^\infty(B_4)$ satisfies $\phi = 1$ on B_3, and $|D\phi| \leq 1.1$. Applying the conformality equality (1.48) again, we obtain

$$\int_{B_4} |\nabla_g \phi|^2\,dv_g \leq C\,(3) \int_{B_4} \Delta u\ dx \leq C\,(3)\,\|Du\|_{L^\infty(B_4)}\,.$$

Thus we get

$$\int_{B_3} V dx \leq C(3) \|Du\|^3_{L^\infty(B_4)}$$

and

$$\int_{B_3} |Db| \, dx \leq C(3) \|Du\|^2_{L^\infty(B_4)}.$$

In turn, we obtain

$$\int_{B_2} b |\nabla_g \varphi|^2 \, dv_g \leq C(3) \left[K \|Du\|_{L^\infty(B_3)} + \|Du\|^2_{L^\infty(B_3)} + \|Du\|^3_{L^\infty(B_4)} \right].$$

Finally collecting all the estimates in the above three steps, we arrive at

$$\begin{aligned} \lambda_{\max}(0) &\leq \exp\left[C(3) \left(\|Du\|_{L^\infty(B_4)} + \|Du\|^2_{L^\infty(B_4)} + \|Du\|^3_{L^\infty(B_4)} \right) \right] \\ &\leq C(3) \exp\left[C(3) \|Du\|^3_{L^\infty(B_4)} \right]. \end{aligned}$$

This completes the proof of Theorem 1.

1.7 Proof of Theorem 2: Large Phase Special Lagrangian Equations in Dimension Three

1.7.1 Proof of Theorem 2

As in the proof of Theorem 1, we assume that $R = 8$ and u is a solution on $B_8 \subset \mathbb{R}^3$ for simplicity of notation. By scaling $v(x) = u\left(\frac{R}{8}x\right) / \left(\frac{R}{8}\right)^2$, we still get the estimate in Theorem 2. We consider the cases when $\Theta = \pi/2 + \delta$ for $\delta \in (0, \pi)$. The cases $\Theta < -\pi/2$ follow by symmetry.

1.7.2 Proof of Phase Dependent Estimate

Step 1. As preparation for the proof of Theorem 2, we take the phase $\pi/2$ representation $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in Proposition 1.5.1 for the original special Lagrangian graph $\mathfrak{M} = (x, Du(x))$ with $x \in B_8$. The "critical" representation is

$$\begin{cases} \bar{x} = x \cos \frac{\delta}{3} + Du(x) \sin \frac{\delta}{3} \\ D\bar{u} = -x \sin \frac{\delta}{3} + Du(x) \cos \frac{\delta}{3} \end{cases} . \tag{1.49}$$

Define

$$\bar{\Omega}_r = \bar{x}(B_r(0)).$$

Then we have from (1.23)

$$\operatorname{dist}(\bar{\Omega}_1, \partial\bar{\Omega}_5) \geq \frac{4}{2\cos\delta/3} > 2. \tag{1.50}$$

We see from (1.49) that $|\bar{x}| \leq \rho$ for $\bar{x} \in \bar{\Omega}_8$ with

$$\rho = 8 \cos \frac{\delta}{3} + \|Du\|_{L^\infty(B_8)} \sin \frac{\delta}{3} \tag{1.51}$$

and that $|D\bar{u}(\bar{x})| \leq \kappa$ (for $\bar{x} \in \bar{\Omega}_8$) with

$$\kappa = 8 \sin \frac{\delta}{3} + \|Du\|_{L^\infty(B_8)} \cos \frac{\delta}{3}. \tag{1.52}$$

The eigenvalues of the new potential $\bar{u}$ satisfy (1.24) and the interior Hessian bound by Theorem 1

$$\left|D^2\bar{u}(\bar{x})\right| \leq C(3) \exp\left[C(3)\,\kappa^3\right]$$

for $\bar{x} \in \bar{\Omega}_5$. It follows that the induced metric on $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in $\bar{x}$ coordinates is bounded on $\bar{\Omega}_5$ by

$$d\bar{x}^2 \leq \bar{g}(\bar{x}) \leq \mu(\kappa, \delta) d\bar{x}^2, \tag{1.53}$$

where

$$\mu(\kappa, \delta) = \min\left\{1 + C(3)\exp\left[C(3)\kappa^3\right], \left[1 + \left(\cot\frac{\delta}{3}\right)^2\right]\right\}. \tag{1.54}$$

Step 2. Relying on the above set-up and the relative isoperimetric inequality in Proposition 1.5.1, we proceed with the following Sobolev inequality for functions without compact support.

Proposition 1.7.1. *Let u be a smooth solution to (1.1) with $\Theta = \pi/2 + \delta$ on $B_5(0) \subset \mathbb{R}^3$. Let f be a smooth positive function on the special Lagrangian surface $\mathfrak{M} = (x, Du(x))$. Then*

$$\left[\int_{B_1} \left|(f - \iota)^+\right|^{3/2} dv_g\right]^{2/3} \leq C(3)\rho^4 \mu(\kappa, \delta) \int_{B_5} \left|\nabla_g (f - \iota)^+\right| dv_g,$$

where ρ, κ, and μ were defined in (1.51), (1.52), and (1.54); also $\iota = \int_{B_5(0)} f dx$.

Proof. Step 2.1. Let $M = ||f||_{L^\infty(B_1)}$. We may assume $\iota < M$. By Sard's theorem, the level set $\{x| \ f(x) = t\}$ is C^1 for almost all t. We first show that for all such $t \in [\iota, M]$,

$$|\{x| \ f(x) > t\} \cap B_1|_g \leq C(3)\rho^6 \left[\mu(\kappa, \delta) |\{x| \ f(x) = t\} \cap B_5|_g\right]^{3/2}. \tag{1.55}$$

(Here $|\ |_g$ and $|\ |_{\bar{g}}$ denote the area or volume with respect to the induced metric; $|\ |$ denotes the same with respect to the Euclidean metric as in Lemma 1.5.1 and Proposition 1.5.2 .)

From $t > \int_{B_5} f dx$, it follows that $|\{x| \ f(x) > t\} \cap B_1| < 1$ and consequently

$$|\{x| \ f(x) \leq t\} \cap B_1| > |B_1| - 1 > 1. \tag{1.56}$$

Now we use instead the coordinates for $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ given by the Lewy-Yuan rotation (1.49). Let

$$A_t = \{\bar{x}| \ f(\bar{x}) > t\} \cap \bar{\Omega}_5,$$

where we are treating f as a function on the special Lagrangian surface $\mathfrak{M}$. From (1.56) we see that $\left|A_t^c \cap \bar{\Omega}_1\right| > c(3) > 0$, and thus

$$|A_t| \leq \left(1 - c(3)\rho^{-3}\right) \left|\bar{\Omega}_1\right|.$$

Applying Proposition 1.5.2 (see section 1.5) with (1.50) and (1.51), we see that

$$\left|A_t \cap \bar{\Omega}_1\right| \le C(3)\rho^6 \left|\partial A \cap \partial A^c \cap \bar{\Omega}_5\right|^{\frac{3}{2}}.$$

Then we have from (1.53)

$$\begin{aligned}\left|A_t \cap \bar{\Omega}_1\right|_{\bar{g}} &\le [\mu(\kappa,\delta)]^{3/2}\left|A_t \cap \bar{\Omega}_1\right| \\ &\le C(3)\rho^6\,[\mu(\kappa,\delta)]^{3/2}\,|\partial A_t \cap \partial A_t^c|^{3/2} \\ &\le C(3)\rho^6\,[\mu(\kappa,\delta)]^{3/2}\,|\partial A_t \cap \partial A_t^c|_{\bar{g}}^{3/2}.\end{aligned}$$

We have the desired isoperimetric inequality (now given in the new coordinates for $\mathfrak{M}$) which holds for $\iota < t < M$

$$\left|A_t \cap \bar{\Omega}_1\right|_{\bar{g}} \le C(3)\rho^6\left[\mu(\kappa,\delta)\,|\partial A_t \cap \partial A_t^c|_{\bar{g}}\right]^{3/2},$$

or equivalently (1.81) in the original coordinates.

Step 2.2. With this isoperimetric inequality in hand, the following proof is standard (cf. [29] Theorem 5.3.1).

$$\begin{aligned}\left[\int_{B_1}\left|(f-\iota)^+\right|^{3/2}dv_g\right]^{2/3} &= \left(\int_0^{M-\iota}\left|\{x|\ f(x)-\iota>t\}\cap B_1\right|_g dt^{3/2}\right)^{2/3} \\ &\le \int_0^{M-\iota}\left|\{x|\ f(x)-\iota>t\}\cap B_1\}\right|_g^{2/3}dt \\ &\le C(3)\rho^4\mu(\kappa,\delta)\int_\iota^M |\{x|\ f(x)=t\}\cap B_5|_g dt \\ &\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_5}\left|\nabla_g(f-\iota)^+\right|dv_g,\end{aligned}$$

where the last inequality followed from the the coarea formula; the second inequality from (1.55); and the first inequality from the Hardy-Littlewood-Polya inequality for any nonnegative, nonincreasing integrand $\eta(t)$:

$$\left[\int_0^T \eta(t)^q\,dt^q\right]^{1/q} \le \int_0^T \eta(t)\,dt. \tag{1.57}$$

This H-L-P inequality (with $q>1$) is proved by noting that $s\eta(s) \le \int_0^s \eta(t)\,dt$ and integrating the inequality

$$q\,[s\eta(s)]^{q-1}\eta(s) \le q\left[\int_0^s \eta(t)\,dt\right]^{q-1}\eta(s) = \frac{d}{ds}\left[\int_0^s \eta(t)\,dt\right]^q.$$

The proposition is thus proved. $\square$

Step 3. We continue the proof of Theorem 2. As in the proof of Theorem 1, we take

$$b = \max\left\{\ln\sqrt{1+\lambda_{\max}^2},\ K_\Theta\right\} \quad \text{with} \quad K_\Theta = 1+\ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)}.$$

Based on Proposition 1.6.1, a calculation similar to (1.46) shows that the Lipschitz function $[(b-\iota)^+]^{3/2}$ is weakly subharmonic, where $\iota = \int_{B_5(0)} b dx$. We apply Michael and Simon's mean value inequality ([32] Theorem 3.4) to obtain

$$\begin{aligned}(b-\iota)^+(0) &\le C(3)\left[\int_{B_1} \left|(b-\iota)^+\right|^{3/2} dv_g\right]^{2/3} \\ &\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_5} \left|\nabla_g(b-\iota)^+\right| dv_g,\end{aligned}$$

where the second inequality follows from Proposition 1.7.1, approximating $(b-\iota)^+$ by smooth functions if necessary. Thus

$$\begin{aligned}b(0) &\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_5} |\nabla_g b|\, dv_g + \int_{B_5} b dx \\ &\le C(3)\rho^4\mu(\kappa,\delta)\left(\int_{B_5} |\nabla_g b|^2\, dv_g\right)^{1/2}\left(\int_{B_5} V dx\right)^{1/2} + \int_{B_5} V dx \\ &\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_6} V dx \qquad (1.58)\end{aligned}$$

where we have used the Jacobi inequality in Proposition 1.6.1, and a similar calculation leading to (1.47) in the proof of Theorem 1.

Step 4. We finish the proof of Theorem 2 by bounding $\int_{B_6} V dx$. Observe

$$V = \left|\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_3\right)\right| = \frac{\sigma_2 - 1}{|\cos\Theta|} > 0.$$

We control the integral of σ_2 in the following.

$$\begin{aligned}\int_{B_r} \sigma_2 dx &= \int_{B_r} \frac{1}{2}\left[(\triangle u)^2 - \left|D^2 u\right|^2\right] dx \\ &= \frac{1}{2}\int_{B_r} \operatorname{div}\left[\left(\triangle u I - D^2 u\right) Du\right] dx \\ &= \frac{1}{2}\int_{\partial B_r} \left\langle\left(\triangle u I - D^2 u\right) Du, \nu\right\rangle dA,\end{aligned}$$

where ν is the outward normal of B_r. Diagonalizing D^2u, we see easily that

$$\Delta u I - D^2 u = \begin{bmatrix} \lambda_2+\lambda_3 & & \\ & \lambda_3+\lambda_1 & \\ & & \lambda_1+\lambda_2 \end{bmatrix} > 0$$

as $\theta_i + \theta_j > 0$ with $\theta_1+\theta_2+\theta_3 \geq \pi/2$. Then

$$\int_{B_r} \sigma_2 dx \leq \|Du\|_{L^\infty(\partial B_r)} \int_{\partial B_r} \Delta u dA.$$

Integrating the boundary integral from $r=6$ to $r=7$, we get

$$\begin{aligned}\int_{B_6} \sigma_2 dx &\leq \|Du\|_{L^\infty(B_7)} \min_{r\in[6,7]} \int_{\partial B_r} \Delta u dA \\ &\leq \|Du\|_{L^\infty(B_7)} \int_{B_7} \Delta u dx \\ &\leq C(3) \|Du\|^2_{L^\infty(B_7)}.\end{aligned}$$

It follows that for $\Theta \geq \pi/2$

$$\begin{aligned}\int_{B_6} V dx &= \frac{1}{|\cos\Theta|}\int_{B_6} (\sigma_2 - 1)\, dx \leq \frac{1}{|\cos\Theta|}\int_{B_6} \sigma_2 dx \\ &\leq \frac{C(3)}{|\cos\Theta|} \|Du\|^2_{L^\infty(B_7)}\end{aligned}$$

or

$$\int_{B_6} V dx \leq \frac{C(3)}{|\cos\Theta|\, \|Du\|_{L^\infty(B_8)}} \|Du\|^3_{L^\infty(B_8)}. \tag{1.59}$$

In order to get Θ-independent control on the volume, we estimate the volume in another way. By the Sobolev inequality on the minimal surface $\mathfrak{M}$ ([32], Theorem 2.1) or ([1], Theorem 7.3), we have

$$\int_{B_6} V dx = \int_{B_6} dv_g \leq \int_{B_7} \phi^6 dv_g \leq C(3) \left[\int_{B_7} |\nabla_g \phi|^2 dv_g\right]^3,$$

where the nonnegative cut-off function $\phi \in C_0^\infty(B_7)$ satisfies $\phi = 1$ on B_6 and $|D\phi| \leq 1.1$.

Observe Lemma 1.4.2 ("conformality") for large phase

$$\left(\frac{1}{1+\lambda_1^2}, \cdots, \frac{1}{1+\lambda_3^2}\right) V =$$
$$\left(\sin\Theta(\sigma_1 - \lambda_1) + \cos\Theta\left(1 - \frac{\sigma_3}{\lambda_1}\right), \cdots, \sin\Theta(\sigma_1-\lambda_3) + \cos\Theta\left(1-\frac{\sigma_3}{\lambda_3}\right)\right).$$

We then have

$$\int_{B_7} |\nabla_g \phi|^2 \, dv_g = \int_{B_7} \sum_{i=1}^{3} \frac{|\phi_i|^2}{1+\lambda_i^2} V dx$$
$$\leq 1.21 \int_{B_7} \left[2 \sin \Theta \sigma_1 + \cos \Theta \left(3 - \sigma_2\right)\right] dx$$
$$\leq C\left(3\right) \left[\left|\sin \Theta\right| \|Du\|_{L^\infty(B_8)} + \left|\cos \Theta\right| \|Du\|_{L^\infty(B_8)}^2\right]$$

for $\Theta \geq \pi/2$, where we used the argument leading to (1.59). Thus we get

$$\int_{B_6} V dx \leq C\left(3\right) \left[\left|\sin \Theta\right| \|Du\|_{L^\infty(B_8)} + \left|\cos \Theta\right| \|Du\|_{L^\infty(B_8)} \|Du\|_{L^\infty(B_8)}\right]^3 \qquad (1.60)$$

Now either $|\cos \Theta| \, \|Du\|_{L^\infty(B_8)} \leq 1$ or $-\cos \Theta \, \|Du\|_{L^\infty(B_8)} > 1$; combining (1.59) and (1.60), we have in either case

$$\int_{B_6} V dx \leq C\left(3\right) \|Du\|_{L^\infty(B_8)}^3 .$$

Finally from the above inequality and (1.58), we conclude

$$b(0) \leq C(3)\rho^4 \mu(\kappa, \delta) \, \|Du\|_{L^\infty(B_8)}^3$$
$$\leq C(3)\rho^4 \, \|Du\|_{L^\infty(B_8)}^3 \min\left\{1 + C(3) \exp\left[C(3)\kappa^3\right], 1 + \left(\cot\frac{\delta}{3}\right)^2\right\}.$$

Exponentiating, and recalling (1.51), (1.52) and (1.54), we have the Θ-independent bound

$$\left|D^2 u(0)\right| \leq C(3) \exp\left\{C(3) \exp\left[C(3) \, \|Du\|_{L^\infty(B_8)}^3\right]\right\}$$

and the Θ-dependent bound

$$\left|D^2 u(0)\right| \leq C(3) \exp\left\{C(3) \left[1 + \left(\cot\frac{\delta}{3}\right)^2\right] \left[1 + \|Du\|_{L^\infty(B_8)} \sin\frac{\delta}{3}\right]^4 \|Du\|_{L^\infty(B_8)}^3\right\}.$$

Simplifying the above expressions, we arrive at the conclusion of Theorem 2.

1.8 *Proof of Theorem 4: Estimates in Dimension Two*

In this section we prove uniform estimates for special Lagrangian equations in dimension two. When $n=2$, equation (1.2) reduces to the simpler

$$\cos\Theta \bigtriangleup u+\sin\Theta\left(\det(D^2u)-1\right)=0. \tag{1.61}$$

1.8.1 *Jacobi Inequality*

Lemma 1.8.1. *Let u be a smooth solution to (1.1), with phase $\Theta\geq 0$, and $n=2$. Set $b=\ln V=\ln\sqrt{\det\left(I+D^2uD^2u\right)}$. Then b satisfies*

$$\triangle_g b\ \geq \sin\Theta|\nabla_g b|^2 \tag{1.62}$$

and for $\Theta\geq\pi/2$,

$$\triangle_g b\ \geq |\nabla_g b|^2. \tag{1.63}$$

Proof. Assume that D^2u is diagonalized at a point p. The calculation

$$\triangle_g \ln\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}=\left[4+(\lambda_1+\lambda_2)^2\right]\left(h_{111}^2+h_{112}^2\right) \tag{1.64}$$

follows from Lemma 1.4.1, again using the notation $h_{ijk}=\sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$. Similarly,

$$\begin{aligned}
|\nabla_g b|^2&=\sum_{i=1}^{2}g^{ii}\left(\partial_i\ln\sqrt{\det g}\right)^2\\
&=\sum_{i=1}^{2}g^{ii}\left(\frac{1}{2}\sum_{a,b=1}^{2}g^{ab}\partial_i g_{ab}\right)^2=\sum_{i=1}^{2}g^{ii}\left(\sum_{j=1}^{2}g^{jj}\lambda_j u_{jji}\right)^2\\
&=g^{11}\left[g^{11}u_{111}\left(\lambda_1-\lambda_2\right)\right]^2+g^{22}\left[g^{11}u_{112}\left(\lambda_1-\lambda_2\right)\right]^2\\
&=\left(h_{111}^2+h_{112}^2\right)(\lambda_1-\lambda_2)^2,
\end{aligned} \tag{1.65}$$

where we used the minimal surface equation (1.21)

$$g^{11}u_{111}+g^{22}u_{221}=g^{11}u_{112}+g^{22}u_{222}=0.$$

With (1.61) in mind, we compute

$$\begin{aligned} 4+(\lambda_1+\lambda_2)^2-\sin\Theta(\lambda_1-\lambda_2)^2 &= 4+\sigma_1^2-\sin\Theta\left(\sigma_1^2-4\sigma_2\right) \\ &= 4+\sigma_1^2-\sin\Theta\left(\sigma_1^2-4+4\cot\Theta\sigma_1\right) \\ &= (1-\sin\Theta)\left[\sigma_1-2\frac{\cos\Theta}{(1-\sin\Theta)}\right]^2 \\ &\geq 0. \end{aligned}$$

Accordingly,

$$\triangle_g b-\sin\Theta|\nabla_g b|^2=\left[4+(\lambda_1+\lambda_2)^2-\sin\Theta(\lambda_1-\lambda_2)^2\right]\left(h_{111}^2+h_{112}^2\right)\geq 0.$$

The Jacobi inequality (1.62) is proved.

For large phase, $\Theta\geq\pi/2$, the equation (1.1) dictates that both eigenvalues are positive, and one can see easily from (1.64) and (1.65) that (1.63) holds. □

In two dimensions, we take advantage of a certain "super" isoperimetric inequality on the level sets of "subharmonic" functions. The resulting Poincaré type inequality will be used in the proof of Theorem 4. Note that this takes the place of the mean value inequality of Michael and Simon, as it was used in the proof of Theorem 1.

Proposition 1.8.1. *Let f be a smooth, positive function on $B_2(0)\subset\mathbb{R}^2$. Suppose that f satisfies the weak maximum principle: f attains its maximum on the boundary of any subdomain of B_2. Then*

$$||f||_{L^\infty(B_1)}\leq\int_{B_2}|Df|\,dx+\int_{B_2}f dx.$$

Proof. Set $\alpha=\int_{B_2}f dx$. We may assume $M\triangleq||f||_{L^\infty(B_1)}>\alpha$. By Sard's theorem, the level set $\{x|\ f(x)=t\}\cap B_2$ is C^1 for almost all t with $\alpha<t<||f||_{L^\infty(B_1)}$. For such (almost all) t, we show that $\{x|\ f(x)=t\}\cap B_2$ has length at least 1 in the following. The set $\{x|\ f(x)\leq t\}\cap B_1$ is nonempty and satisfies

$$|\{x|\ f(x)\leq t\}\cap B_1|>|B_1|-1, \tag{1.66}$$

otherwise we have a contradiction:

$$\alpha>\int_{B_1}f dx>t\ |\{x|\ f(x)>t\}\cap B_1|>\alpha.$$

If any component of $\{x|\ f(x)=t\}\cap B_2$ stretches from the interior B_1 to the boundary ∂B_2, then the length $|\{x|\ f(x)=t\}\cap B_2|>1$. Otherwise, each component of $\{x|\ f(x)=t\}\cap B_2$ which intersects B_1 must be a closed curve in B_2, as we are using the fact that t is not a critical value for f. From the maximum principle for f, it follows that $f\le t$ inside any such closed curve. By (1.66) and the usual isoperimetric inequality for each of these (finitely many) C^1 regions where $f\le t$, we have

$$|\{x|\ f(x)=t\}\cap B_2| \ge \sqrt{4\pi\,|\{x|\ f(x)\le t\}\cap B_1|} > 1. \tag{1.67}$$

Now we proceed as follows. For any $q\ge 1$,

$$\begin{aligned}
\left[\int_{B_1} |(f-\alpha)^+|^q\,dx\right]^{1/q} &= \left[\int_0^{M-\alpha} |\{x|\ f(x)-\alpha>t\}\cap B_1|\,dt^q\right]^{1/q} \\
&\le \int_0^{M-\alpha} |\{x|\ f(x)-\alpha>t\}\cap B_1|^{1/q}\,dt \\
&\le |B_1|^{1/q}\int_0^{M-\alpha} |\{x|\ f(x)-\alpha=t\}\cap B_2|\,dt \\
&\le |B_1|^{1/q}\int_{B_2} |D[f(x)-\alpha]|\,dx,
\end{aligned}$$

where the last inequality followed from the coarea formula; the second inequality followed from (1.67); and the first inequality followed from the Hardy-Littlewood-Polya inequality (just as in Step 2.2 in the proof of Theorem 2, see also ([4], p.258).

Letting q go to ∞, we have

$$\|(f-\alpha)^+\|_{L^\infty(B_1)} \le \int_{B_2} |D(f-\alpha)|\,dx.$$

Thus

$$\|f\|_{L^\infty(B_1)} \le \int_{B_2} |Df|\,dx + \int_{B_2} f dx.$$

□

1.8.2 Proof of Uniform Estimate

We combine two estimates to obtain a uniform Hessian estimate for any given height bound. The first estimate, which uses the Jacobi inequality, deteriorates as $\Theta\to 0$. The second estimate holds for small Θ with constrained height, and follows easily from a standard technique for harmonic functions, combined with a Lewy-Yuan rotation of coordinates. For simplicity, we assume that $R = 4$ and u is a solution on $B_4 \subset \mathbb{R}^2$. By scaling $u\left(\frac{R}{4}x\right) / \left(\frac{R}{4}\right)^2$, we still get the estimate in Theorem 4.

Case with Θ-dependence. By the symmetry of the equation (1.1), we assume $\Theta > 0$. From inequality (1.62) in Lemma 1.8.1, $b = \ln V$ is subharmonic with respect to the induced metric on B_2; hence b satisfies the weak maximum principle. We apply Proposition 1.8.1

$$\begin{aligned}\|b\|_{L^\infty(B_1)} &\leq \int_{B_2} |Db|\, dx + \int_{B_2} b\, dx \\ &\leq \int_{B_2} |\nabla_g b|\, dv_g + \int_{B_2} b\, dx \\ &\leq \left(\int_{B_2} |\nabla_g b|^2\, V dx\right)^{1/2} \left(\int_{B_2} V dx\right)^{1/2} + \int_{B_2} V dx. \end{aligned} \tag{1.68}$$

Multiplying both sides of the Jacobi equation (1.62) by a non-negative cut-off function $\psi \in C_0^\infty(B_3)$ with $\psi = 1$ on B_2 and $|D\psi| \leq 1.1$, then integrating, we obtain

$$\begin{aligned}\int_{B_3} \psi^2 |\nabla_g b|^2\, dv_g &\leq \frac{1}{\sin\Theta} \int_{B_3} \psi^2\, \Delta_g\, b\, dv_g \\ &= -\frac{1}{\sin\Theta} \int_{B_3} \langle 2\psi \nabla_g \varphi, \nabla_g b\rangle_g\, dv_g \\ &\leq \frac{1}{2} \int_{B_3} \psi^2 |\nabla_g b|^2\, dv_g + 2\left(\frac{1}{\sin\Theta}\right)^2 \int_{B_3} |\nabla_g \psi|^2\, dv_g. \end{aligned}$$

It follows that

$$\int_{B_2} |\nabla_g b|^2\, V dx \leq \int_{B_3} \psi^2 |\nabla_g b|^2\, dv_g \leq 4 \csc^2\Theta \int_{B_3} |\nabla_g \psi|^2\, dv_g. \tag{1.69}$$

Observe that by equation (1.1) or (1.61) the volume element takes a simple form

$$\begin{aligned} V &= \sqrt{\left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)} = \left|\left(1+i\lambda_1\right)\left(1+i\lambda_2\right)\right| = \left|1-\sigma_2+i\sigma_1\right| \\ &= \frac{\sigma_1}{\sin\Theta} = \csc\Theta\, \Delta\, u. \end{aligned}$$

Hence,

$$\int_{B_2} V dx \leq \frac{C(2)}{\sin\Theta} \|Du\|_{L^\infty(B_2)}$$

and

$$\begin{aligned} |\nabla_g \psi|^2\, V &\leq \left(\frac{|D\psi|^2}{1+\lambda_1^2} + \frac{|D\psi|^2}{1+\lambda_2^2}\right) V = |D\psi|^2 \left(\frac{2+\lambda_2^2+\lambda_1^2}{V}\right) \\ &= |D\psi|^2 \left[2(1-\sigma_2)+\sigma_1^2\right] \frac{\sin\Theta}{\sigma_1} = |D\psi|^2 \left(2\cos\Theta + \sigma_1 \sin\Theta\right), \end{aligned}$$

where we used the equation (1.61). We then have from (1.69)

$$\int_{B_2} |\nabla_g b|^2 \, V dx \leq C(2) \csc^2\Theta \int_{B_3} (2\cos\Theta + \sigma_1 \sin\Theta)\, dx$$
$$\leq C(2)\left(\csc^2\Theta + \csc\Theta \, \|Du\|_{L^\infty(B_3)}\right).$$

Thus from (1.68),

$$\|b\|_{L^\infty(B_1)} \leq C(2)\frac{\left(1+\sin\Theta\,\|Du\|_{L^\infty(B_3)}\right)^{\frac{1}{2}}}{\sin\Theta}\left[\frac{\|Du\|_{L^\infty(B_2)}}{\sin\Theta}\right]^{\frac{1}{2}} + C\,(2)\,\frac{\|Du\|_{L^\infty(B_2)}}{\sin\Theta};$$

that is

$$\|b\|_{L^\infty(B_1)} \leq C(2)\frac{\|Du\|_{L^\infty(B_3)}}{\sin\Theta}\left(1+\frac{1}{\sin^{1/2}\Theta\,\|Du\|^{1/2}_{L^\infty(B_3)}}\right). \tag{1.70}$$

The estimate (1.12) follows by exponentiating (1.70).

Next, for very large phase, $\Theta > 3\pi/4$, we adapt the proof of (1.70) to obtain a bound that does not deteriorate as $\Theta \to \pi$. First we note that from the Jacobi inequality (1.63) the Θ-dependence in (1.69) is no longer needed, and we have

$$\int_{B_2} |\nabla_g b| dv_g \leq 4\left(\int_{B_3} |\nabla_g \psi|^2 \, dv_g\right)^{1/2}\left(\int_{B_2(0)} V dx\right)^{1/2} \leq C(2)\int_{B_3(0)} V dx. \tag{1.71}$$

Using another expression for the volume form

$$V = |1-\sigma_2 + i\sigma_1| = |\sec\Theta|\,(\sigma_2 - 1)$$

for $\Theta > \pi/2$,

$$\int_{B_r(0)} V dx = \int_{B_r(0)} |\sec\Theta|\,(\sigma_2-1)\,dx \leq |\sec\Theta| \int_{B_r(0)} \sigma_2 dx$$
$$= |\sec\Theta| \int_{B_r(0)} \operatorname{div}(u_1 u_{22}, -u_1 u_{21}) dx$$
$$\leq |\sec\Theta|\,\|Du\|_{L^\infty(B_r)} \int_{\partial B_r(0)} |D^2 u| ds.$$

By the convexity of u for large phase $\Theta > \pi/2$, we know $\Delta u \geq \left|D^2 u\right|$, so

$$\int_{B_r(0)} V dx \leq |\sec\Theta|\,\|Du\|_{L^\infty(B_r)} \int_{\partial B_r(0)} \Delta u ds.$$

Integrating the right hand side from $r=3$ to $r=4$, we deduce

$$\begin{aligned}\int_{B_3(0)} V dx &\leq |\sec\Theta| \, \|Du\|_{L^\infty(B_4)} \min_{r\in[3,4]} \int_{\partial B_r(0)} \triangle u ds \\ &\leq |\sec\Theta| \, \|Du\|_{L^\infty(B_4)} \int_{B_4(0)} \triangle u dx \leq |\sec\Theta| \, \|Du\|^2_{L^\infty(B_4)} .\end{aligned}$$

In light of (1.68) and (1.71), we then have for $\Theta > \pi/2$,

$$\|b\|_{L^\infty(B_1)} \leq C(2) \, |\sec\Theta| \, \|Du\|^2_{L^\infty(B_4)},$$

and finally

$$\left\|D^2 u\right\|_{L^\infty(B_1)} \leq \exp\left[C\,(2) \, |\sec\Theta| \, \|Du\|^2_{L^\infty(B_4)}\right]. \tag{1.72}$$

This finishes the estimates with Θ-dependence in Theorem 4.

Case without Θ-dependence. In order to prove the Hessian bound (1.11) that does not deteriorate for small Θ, we need the following.

Proposition 1.8.2. *Let u be a smooth solution to (1.1) with $n=2$ and $\Theta \in [0, \pi/4]$ on $B_1(0) \subset \mathbb{R}^2$. Suppose that*

$$\|Du\|_{L^\infty(B_1)} \leq \frac{1}{8\sin\Theta}.$$

Then we have

$$\left|D^2 u(0)\right| \leq C(2) \left(\|Du\|_{L^\infty(B_1)} + 1\right).$$

Proof. We first find a harmonic representation of $\mathfrak{M} = (x, Du)$ via Lewy-Yuan rotation as in Proposition 1.5.1:

$$\begin{cases} \bar{x} = x\cos\frac{\Theta}{2} + Du\,(x)\sin\frac{\Theta}{2} \\ \bar{y} = -x\sin\frac{\Theta}{2} + Du\,(x)\cos\frac{\Theta}{2} \end{cases}. \tag{1.73}$$

We see that $\mathfrak{M}$ is a (special Lagrangian) graph over $\bar{x}$ space: $\mathfrak{M} = (\bar{x}, D\bar{u}\,(\bar{x}))$, where $\bar{u}$ is a smooth function. Let $\bar{\lambda}_i$ be the eigenvalues of the Hessian $D^2\bar{u}$. From (1.1) we have that $\arctan\lambda_i > \Theta - \pi/2$. (When $n=2$, all phases are considered large.) Then

$$\arctan\bar{\lambda}_i = \arctan\lambda_i - \frac{\Theta}{2} \in \left(-\frac{\pi}{2} + \frac{\Theta}{2}, \frac{\pi}{2} - \frac{\Theta}{2}\right). \tag{1.74}$$

It follows that

$$\arctan\bar{\lambda}_1 + \arctan\bar{\lambda}_2 = 0 \quad \text{or}$$

$$\Delta\bar{u} = 0.$$

Moreover, the domain of $\bar{u}$, $\bar{x}(B_1)$ contains a ball in $\bar{x}$ space with radius $\bar{R}$ at least

$$\bar{R} \geq \frac{1}{2\cos\Theta/2}$$

around $\bar{x}(0)$.

For the harmonic function $\bar{u}_{\bar{e}\bar{e}}$ with $\bar{e}$ being an arbitrary unit vector in $\bar{x}$ space, the mean value formula implies

$$\bar{u}_{\bar{e}\bar{e}}(\bar{0}) = \frac{1}{\pi\bar{R}^2}\int_{\bar{B}_{\bar{R}}} \bar{u}_{\bar{e}\bar{e}}\, d\bar{x} \leq \frac{2\pi\bar{R}}{\pi\bar{R}^2}\|D\bar{u}\|_{L^\infty(\bar{B}_{\bar{R}})} \leq 4\cos\frac{\Theta}{2}\|D\bar{u}\|_{L^\infty(\bar{B}_{\bar{R}})}.$$

From the above harmonic parametrization (1.73) of $\mathfrak{M}$, we know

$$\|D\bar{u}\|_{L^\infty(\bar{B}_{\bar{R}})} \leq \sin\frac{\Theta}{2} + \cos\frac{\Theta}{2}\|Du\|_{L^\infty(B_1)}.$$

Thus we get

$$\bar{\lambda}_i(\bar{0}) \leq 4\cos\frac{\Theta}{2}\left[\sin\frac{\Theta}{2} + \cos\frac{\Theta}{2}\|Du\|_{L^\infty(B_1)}\right].$$

From (1.74), we see that

$$\lambda_i(0) = \frac{\bar{\lambda}_i(\bar{0}) + \tan\frac{\Theta}{2}}{1 - \bar{\lambda}_i(\bar{0})\tan\frac{\Theta}{2}}.$$

Note that $\lambda_{\max}(0) \geq |\lambda_i(0)|$ for $\Theta \geq 0$. It follows that

$$\begin{aligned}
\left|D^2u(0)\right| &\leq \lambda_{\max}(0) \\
&\leq \frac{\tan\frac{\Theta}{2} + 4\cos\frac{\Theta}{2}\left[\sin\frac{\Theta}{2} + \cos\frac{\Theta}{2}\|Du\|_{L^\infty(B_1)}\right]}{2\cos\Theta - 1 - 2\sin\Theta\|Du\|_{L^\infty(B_1)}} \\
&\leq C(2)\left[\|Du\|_{L^\infty(B_1)} + 1\right],
\end{aligned}$$

provided that, say, $\left[2\cos\Theta - 1 - 2\sin\Theta\|Du\|_{L^\infty(B_1)}\right] > 0.15$, which is available under our assumption. □

We finish the proof of Theorem 4 without Θ-dependence. By symmetry, we only consider the cases $0 \leq \Theta < \pi$.

We first consider the small phase $0 \leq \Theta \leq \pi/4$.

If $\|Du\|_{L^\infty(B_3)} \geq \frac{1}{8\sin\Theta}$, then from the first estimate (1.70) we have

$$\begin{aligned}
\left\|D^2u\right\|_{L^\infty(B_1)} &\leq C(2)\exp\left\{C(2)\frac{\|Du\|_{L^\infty(B_3)}}{\sin\Theta}\left[1 + \frac{1}{\sin^{1/2}\Theta\|Du\|_{L^\infty(B_3)}^{1/2}}\right]\right\} \\
&\leq C(2)\exp\left[C(2)\|Du\|_{L^\infty(B_4)}^2\right].
\end{aligned}$$

If $\|Du\|_{L^\infty(B_2)} < \frac{1}{8\sin\Theta}$, then Proposition 1.8.2 (applied at any point in B_1) implies

$$\begin{aligned}\left\|D^2u\right\|_{L^\infty(B_1)} &\leq C\,(2)\left[\|Du\|_{L^\infty(B_2)}+1\right]\\ &\leq C(2)\exp\left[C(2)\,\|Du\|^2_{L^\infty(B_4)}\right].\end{aligned}$$

For phases $\pi/4 \leq \Theta \leq 3\pi/4$, $\sin\Theta$ is bounded away from 0 and (1.70) gives

$$\begin{aligned}\left\|D^2u\right\|_{L^\infty(B_1)} &\leq C(2)\exp\left(C(2)\,\|Du\|_{L^\infty(B_3)}\right)\\ &\leq C(2)\exp\left(C(2)\,\|Du\|^2_{L^\infty(B_4)}\right).\end{aligned}$$

For large phase $\Theta \geq 3\pi/4$, $\sec\Theta$ is bounded and we have from (1.72)

$$\left\|D^2u\right\|_{L^\infty(B_1)} \leq C(2)\exp\left(C(2)\,\|Du\|^2_{L^\infty(B_4)}\right).$$

The proof of estimate (1.11) without Θ-dependence in Theorem 4 is complete after a scaling.

1.9 Proof of Theorem 5: Estimates for Convex Solutions

In this section we prove estimates for convex solutions of (1.1).

1.9.1 A Jacobi Inequality for the Volume Element

We now derive a Jacobi inequality for the volume element

$$V = \sqrt{\det g} = \prod_{i=1}^{n}(1+\lambda_i^2)^{\frac{1}{2}}.$$

Proposition 1.9.1. *Suppose that u is a smooth convex solution to (1.1) in $B_R \subset \mathbb{R}^n$. Then*

$$\triangle_g \ln V \geq \frac{1}{n}|\nabla_g \ln V|^2.$$

Consequently

$$\int_{B_r} |\nabla_g \ln V|^2 \, dv_g \leq \frac{C(n)}{R-r}\int_{B_R} dv_g.$$

Proof. At any fixed point, we assume that D^2u is diagonalized, then using (1.4.1)

$$\triangle_g \ln V = \sum_{i,j,k=1}^{n}(1+\lambda_i\lambda_j)h_{ijk}^2,$$

where $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$ are the components of the second fundamental form of the graph. Gathering all terms containing $h_{ijj}^2 = h_{jij}^2 = h_{jji}^2$ for a fixed i, we have

$$\begin{aligned}&(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(1+\lambda_j^2)h_{jji}^2 + \sum_{j\neq i}(1+\lambda_i\lambda_j)h_{ijj}^2 + \sum_{j\neq i}(1+\lambda_j\lambda_i)h_{jij}^2\\ &= (1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2.\end{aligned}$$

Thus

$$\begin{aligned}\triangle_g \ln V &= \sum_{i}^{n}\left[(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2\right]\\ &\quad + 2\sum_{i<j<k}(3+\lambda_i\lambda_j+\lambda_j\lambda_k+\lambda_k\lambda_i)h_{ijk}^2.\end{aligned}$$

To bound the gradient, we compute (still at the same fixed point with D^2u diagonalized)

$$\partial_i \ln V = \sum_{j=1}^{n} g^{jj}\lambda_j u_{jji},$$

then

$$|\nabla_g \ln V|_g^2 = \sum_{i=1}^{n} g^{ii} \left(\sum_{j=1}^{n} g^{jj} \lambda_j u_{jji} \right)^2$$
$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_j h_{jji} \right)^2$$
$$\leq n \sum_{i,j=1}^{n} {\lambda_j}^2 h_{jji}^2.$$

From convexity of u, we have

$$\triangle_g \ln V - \frac{1}{n} |\nabla_g \ln V|^2$$
$$\geq \sum_{i=1}^{n} h_{iii}^2 + \sum_{j \neq i} [3 + 2\lambda_i \lambda_j] h_{jji}^2 \geq 0.$$

For any smooth cut-off function $\varphi \in C_0^\infty (B_R)$,

$$\int_{B_R} \varphi^2 |\nabla_g \ln V|^2 dv_g \leq n \int_{B_R} \varphi^2 \triangle_g \ln V dv_g$$
$$= -n \int_{B_R} \langle 2\varphi \nabla_g \varphi, \nabla_g \ln V \rangle_g dv_g$$
$$\leq \frac{1}{2} \int_{B_R} \varphi^2 |\nabla_g \ln V|^2 dv_g + 2n^2 \int_{B_R} |\nabla_g \varphi|^2 dv_g.$$

In particular, choosing φ to be 1 on B_r and have gradient bounded by $2/(R-r)$ in B_R, we have

$$\int_{B_r} |\nabla_g \ln V|^2 dv_g \leq \int_{B_R} \varphi^2 |\nabla_g \ln V|^2 dv_g$$
$$\leq 4n^2 \int_{B_R} |\nabla_g \varphi|^2 dv_g$$
$$\leq \frac{C(n)}{R-r} \int_{B_R} dv_g.$$

This completes the proof of Proposition 1.9.1. □

1.9.2 Proof of Theorem 5

We assume that $R = 7$ and u is a solution on $B_7 \subset \mathbb{R}^n$ for simplicity of notation. By scaling $v(x) = u\left(\frac{R}{7}x\right) / \left(\frac{R}{7}\right)^2$, we still get the estimate in Theorem 5. The proof of Theorem 5 consists of five steps.

Step 1. We take a new representation $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ for the original (special) Lagrangian graph $\mathfrak{M} = (x, Du(x))$ in new coordinate system of $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{C}^n$, $\bar{z} = e^{-\sqrt{-1}\pi/4}z$ with $z = x + \sqrt{-1}y$ and $\bar{z} = \bar{x} + \sqrt{-1}\bar{y}$:

$$\begin{cases} \bar{x} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}Du(x) \\ \bar{y} = D\bar{u} = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}Du(x) \end{cases} \tag{1.75}$$

The following instance of Proposition 1.5.1 is contained in [56], p. 122.

Proposition 1.9.2. *1.5.1 Let u be a smooth convex function $B_R(0) \subset \mathbb{R}^n$. Then the Lagrangian submanifold $\mathfrak{M} = (x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ can be represented as a gradient graph $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ of the new potential $\bar{u}$ in a domain containing a ball of radius*

$$\bar{R} \geq \frac{\sqrt{2}R}{2} \tag{1.76}$$

such that in these coordinates the new Hessian satisfies

$$-I \ \leq D^2\bar{u} \leq I. \tag{1.77}$$

Define

$$\bar{\Omega}_r = \bar{x}(B_r(0)).$$

Then we have from (1.76)

$$\operatorname{dist}(\bar{\Omega}_1, \partial\bar{\Omega}_5) \geq \frac{4}{\sqrt{2}} > 2. \tag{1.78}$$

We see from (1.75) that $|\bar{x}| \leq \rho$ for $\bar{x} \in \bar{\Omega}_6$ with

$$\rho = 6\frac{\sqrt{2}}{2} + \|Du\|_{L^\infty(B_6)}\frac{\sqrt{2}}{2}. \tag{1.79}$$

From (1.77), it follows that the induced metric on $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in $\bar{x}$ coordinates is bounded by

$$d\bar{x}^2 \leq g(\bar{x}) \leq 2d\bar{x}^2, \tag{1.80}$$

Step 2. As in the proof of Theorem 2, we need a Sobolev type inequality. This following Sobolev inequality is proved similar to the one in the proof of Theorem 2.

Proposition 1.9.3. *Let u be a smooth convex function on $B_5(0) \subset \mathbb{R}^n$. Let f be a smooth positive function on the Lagrangian surface $\mathfrak{M} = (x, Du(x))$. Then*

$$\left(\int_{B_1} \left|(f-\iota)^+\right|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}} \leq C(n)\rho^{2(n-1)} \int_{B_5} \left|\nabla_g (f-\iota)^+\right| dv_g,$$

where ρ was defined in (1.79) and $\iota = \frac{2}{|B_1|}\int_{B_5(0)} f dx$

Proof. Let $M = \|f\|_{L^\infty(B_1)}$. We may assume $\iota < M$ since otherwise the desired result holds trivially. By Sard's theorem, the level set $\{x|\ f(x) = t\}$ is C^1 for almost all t. We first show that for all such $t \in [\iota, M]$,

$$|\{x|\ f(x) > t\} \cap B_1|_g \leq C(n)\rho^{2n} \left|\{x|\ f(x) = t\} \cap B_5\right|_g^{\frac{n}{n-1}}. \tag{1.81}$$

Here $|\ |_g$ denotes the area or volume with respect to the induced metric; $|\ |$ denotes the same with respect to the Euclidean metric.

From $t > \frac{2}{|B_1|}\int_{B_5(0)} f dx$, it follows that $|\{x|\ f(x) > t\} \cap B_1| < |B_1|/2$ and consequently

$$|\{x|\ f(x) \leq t\} \cap B_1| > \frac{|B_1|}{2}. \tag{1.82}$$

Now we use instead the coordinates for $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ given by the Lewy rotation (1.75). Let

$$A_t = \{\bar{x}|\ f(\bar{x}) > t\} \cap \bar{\Omega}_5,$$

where we are treating f as a function on the special Lagrangian surface $\mathfrak{M}$. From (1.82) we see that $\left|A_t^c \cap \bar{\Omega}_1\right| > c(n) > 0$, and thus

$$|A_t| \leq \left(1 - c(n)\rho^{-n}\right)\left|\bar{\Omega}_1\right|.$$

Applying Proposition 1.5.2 (see section 1.5) with (1.78) and (1.79), we see that

$$\left|A_t \cap \bar{\Omega}_1\right| \leq C(n)\rho^{2n}\left|\partial A \cap \partial A^c \cap \bar{\Omega}_5\right|^{\frac{n}{n-1}}.$$

Then we have from (1.80)

$$\begin{aligned}\left|A_t \cap \bar{\Omega}_1\right|_{\bar{g}} &\leq 2^{\frac{n}{2}}\left|A_t \cap \bar{\Omega}_1\right| \\ &\leq C(n)\rho^{2n}\left|\partial A_t \cap \partial A_t^c\right|^{\frac{n}{n-1}} \\ &\leq C(n)\rho^{2n}\left|\partial A_t \cap \partial A_t^c\right|_{g(\bar{x})}^{\frac{n}{n-1}}.\end{aligned}$$

We have the desired isoperimetric inequality (now given in the new coordinates for $\mathfrak{M}$) which holds for $\iota < t < M$

$$\left|A_t \cap \bar{\Omega}_1\right|_{g(\bar{x})} \leq C(n)\rho^{2n} \left|\partial A_t \cap \partial A_t^c\right|_{g(\bar{x})}^{\frac{n}{n-1}},$$

or equivalently (1.81) in the original coordinates.

We then proceed to prove the Sobolev inequality via the Federer-Fleming argument, as before. We have

$$\begin{aligned}
\left(\int_{B_1} \left|(f-\iota)^+\right|^{\frac{n}{n-1}} dv_g\right)^{\frac{n-1}{n}} &= \left(\int_0^{M-\iota} \left|\{x|\ f(x)-\iota > t\} \cap B_1\right|_g dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \\
&\leq \int_0^{M-\iota} \left|\{x|\ f(x)-\iota > t\} \cap B_1\}\right|_g^{\frac{n-1}{n}} dt \\
&\leq C(n)\rho^{2(n-1)} \int_\iota^M \left|\{x|\ f(x) = t\} \cap B_5\right|_g dt \\
&\leq C(n)\rho^{2(n-1)} \int_{B_5} \left|\nabla_g (f-\iota)^+\right| dv_g,
\end{aligned}$$

by (1.57), (1.81) and the coarea formula for the above three inequalities, respectively. The proposition is thus proved. □

Step 3. We consider the function $f = \ln V$ on the special Lagrangian manifold $\mathfrak{M}$, where V is the volume element with respect to the original x-coordinate system. Observe that $\ln V\,(\bar{x})$ in the rotated $\bar{x}$-coordinate system satisfies

$$\sum_{i,j=1}^{n} g^{ij}\,(\bar{x})\, \frac{\partial^2 \ln V\,(\bar{x})}{\partial \bar{x}_i \partial \bar{x}_j} = \Delta_{g(\bar{x})} \ln V\,(\bar{x}) \geq 0$$

because of (1.20) and Proposition 1.9.1. Note that the above non-divergence and divergence elliptic operator are both uniformly elliptic by (1.80).

Via the De Giorgi-Moser iteration (cf. Theorem 8.17 in [16]), we have

$$
\begin{aligned}
(\ln V - \iota)^+(0) &= (\ln V - \iota)^+(\bar{0}) \\
&\le C(n)\left(\int_{B_{1/\sqrt{2}}(\bar{0})} \left|(\ln V - \iota)^+(\bar{x})\right|^{\frac{n}{n-1}} d\bar{x}\right)^{\frac{n-1}{n}} \\
&\le C(n)\left(\int_{B_{1/\sqrt{2}}(\bar{0})} \left|(\ln V - \iota)^+(\bar{x})\right|^{\frac{n}{n-1}} dv_{g(\bar{x})}\right)^{\frac{n-1}{n}} \\
&\le C(n)\left(\int_{B_1(0)} \left|(\ln V - \iota)^+(x)\right|^{\frac{n}{n-1}} dv_{g(x)}\right)^{\frac{n-1}{n}}
\end{aligned}
$$

where

$$
\iota = \frac{2}{|B_1(0)|}\int_{B_5(0)} \ln V dx.
$$

Using the non-divergence structure of the Laplace-Beltrami operator, one may also get the above mean value inequality by a local maximum principle (Theorem 9.20 in [16]).

Thus by Proposition 1.9.3 and Proposition 1.9.1, we obtain

$$
\begin{aligned}
\ln V(0) &\le C(n)\rho^{2(n-1)}\int_{B_5} \left|\nabla_g(\ln V - \iota)^+\right| dv_g + C(n)\int_{B_5} \ln V dx \\
&\le C(n)\rho^{2(n-1)}\left(\int_{B_5} |\nabla_g \ln V|^2 dv_g\right)^{\frac{1}{2}}\left(\int_{B_5} V dx\right)^{\frac{1}{2}} + C(n)\int_{B_5} V dx \\
&\le C(n)\rho^{2(n-1)}\int_{B_6} V dx. \qquad (1.83)
\end{aligned}
$$

Step 4. We finish the proof of Theorem 5 by bounding $\int_{B_6} V dx$. Returning to our rotated coordinates, from (1.80) we see

$$
\bar{V} \le 2^{\frac{n}{2}}.
$$

Since $\bar{\Omega}_6 = \bar{x}(B_6(0))$,

$$
\int_{B_6} V dx = \int_{\bar{\Omega}_6} \bar{V} d\bar{x} \le 2^n \int_{\bar{\Omega}_6} d\bar{x} \le C(n)\rho^n,
$$

then (1.79)

$$
\rho = 6\cos\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\|Du\|_{L^\infty(B_6)}
$$

leads to

$$
\int_{B_6} V dx \le C(n)(1 + \|Du\|^n_{L^\infty(B_6)}).
$$

Finally, from the above and (1.83) we conclude

$$\ln V(0) \leq C(n)\rho^{2(n-1)}\left(1+\|Du\|^{n}_{L^\infty(B_6)}\right)$$

or

$$\left|D^2u(0)\right| \leq C(n)\exp\left\{C(n)\,\|Du\|^{3n-2}_{L^\infty(B_6)}\right\}.$$

Step 5. Now the gradient of the convex function u is bounded by its oscillation

$$\|Du\|_{L^\infty(B_6)} \leq \operatorname*{osc}_{B_7} u.$$

By scaling, we arrive at the conclusion of Theorem 5.

1.9.3 Interior Regularity for Special Lagrangian Equations with Very Large Phase

In this section, we apply our a priori interior estimates of Corollary 1 to show Theorem 6 which asserts that any C^0 viscosity solution to special Lagrangian equation (1.1) with very large phase $|\Theta| \geq (n-1)\,\pi/2$ is regular, that is, analytic inside the domain of the C^0 viscosity solution.

Proof. By symmetry, we assume $\Theta \geq (n-1)\,\pi/2$. By scaling, we assume that the C^0 viscosity solution u satisfies (1.1) in the unit ball around any interior point of the domain of u. We approximate the C^0 boundary data $u|_{\partial B_1} = \varphi$ by smooth φ_ε. Notice that the level set of $F(D^2u) = \arctan\lambda_1 + \cdots + \arctan\lambda_n$ in the λ-space, corresponding to the elliptic equation (1.1),

$$\Sigma = \{(\lambda_1,\cdots,\lambda_n)\,|\,\ \arctan\lambda_1+\cdots+\arctan\lambda_n = \Theta\} \tag{1.84}$$

is convex for $\Theta \geq (n-1)\,\pi/2$. By Theorem 4 of [7] (see also a simplification [42]), we obtain a smooth solution u_ε to (1.1) with smooth boundary data φ_ε in B_1.

Applying the (easy) comparison theorem to the C^0 viscosity solution u and the C^2 solution u_ε,

$$\|u-u_\varepsilon\|_{L^\infty(B_1)} \leq \|u-u_\varepsilon\|_{L^\infty(\partial B_1)} \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{1.85}$$

In fact, one can deduce (1.85) from the definition of viscosity solution as follows. Suppose

$$M \overset{\text{def}}{=} \max_{B_1}(u-u_\varepsilon) > \max_{\partial B_1}(u-u_\varepsilon) \overset{\text{def}}{=} M_\partial,$$

then the function $u - \left(u_\varepsilon - \frac{(M-M_\partial)}{2}|x|^2\right)$ achieves its maximum at an interior point x_0. By the definition of viscosity solution,

$$F\left(D^2\left(u_\varepsilon - \frac{(M-M_\partial)}{2}|x|^2\right)\right) \geq \Theta$$

at x_0. But $F(D^2u)$ is strictly elliptic,

$$\Theta = F\left(D^2u_\varepsilon\right) > F\left(D^2u\left(x_0\right) - \left(M - M_\partial\right)I\right) \geq \Theta.$$

This contradiction shows $\max_{B_1}(u-u_\varepsilon) \leq \max_{\partial B_1}\left(u - u_\varepsilon\right)$. Similarly, one proves $\min_{B_1}(u-u_e) \geq \min_{\partial B_1}(u - u_e)$. Therefore, (1.85) holds.

From our Corollary 1 combined with the Evans-Krylov-Safonov theory (cf. Theorem 17.15 in [16]), we have

$$\|u_\varepsilon\|_{C^{2,\alpha}(\Omega)} \leq C\left(\Omega, \|u\|_{L^\infty(B_1)}\right)$$

for any subdomain $\Omega \subset \mathring{B}_1$. Thus u is $C^{2,\alpha}$ inside B_1. Now by the classical elliptic theory (cf. Theorem 17.16 in [16] and [33] p. 203), u is smooth, even analytic inside B_1. □

In closing, we make some remarks on the existence and uniqueness of C^0 viscosity solution for the Dirichlet problem of special Lagrangian equation (1.1). As we see in the above, the approximation and the existence of smooth solutions with smooth boundary data to (1.1) with $|\Theta| \geq (n-1)\pi/2$ in any (strongly) convex domain already lead to the existence and uniqueness of C^0 viscosity solution to the Dirichlet problem with C^0 boundary data.

Notice that (1.1) is strictly elliptic in the sense

$$F_{u_{ij}}\left(D^2u\right) > 0.$$

The existence and uniqueness of C^0 viscosity solution for the Dirichlet problem of (1.1) then follow from those for general strictly elliptic equation. This general result is known. It follows from the comparison principle for C^0 viscosity solutions to strictly elliptic equations. In fact, strict ellipticity (rather than uniform ellipticity) is enough for the proof of the uniqueness of solution (also comparison principle) for fully nonlinear elliptic equation presented in [8], p.43–46. For details, see for example [57].

Recently, there is a new approach toward the existence and uniqueness of C^0 viscosity solution to the special Lagrangian equation (1.1), as stated in [18], p.46. Actually, the result on the Dirichlet problem for general fully nonlinear elliptic equations with starlike/convex level set (Theorem 6.2 in [18]) already applies to (1.1) with $|\Theta| \geq (n-1)\,\pi/2$, even $|\Theta| \geq (n-2)\,\pi/2$. This is because the level set (1.84) Σ is convex for $|\Theta| \geq (n-2)\,\pi/2$; see Lemma 2.1 in [58].

We point out that the above two conditions on the fully nonlinear elliptic equations, strict ellipticity and starlike/convex level set are independent.

1.9.4 Convex Solutions with Variable Phase

We note that in the proof of Theorem 5 we do not use any special structure coming from the minimality of the Lagrangian surface, we use only the equation itself with the convexity of u. In particular, we may easily adapt the proof to when $\theta(x)$ varies. In principle, estimates for variable right hand side nonlinear elliptic equations can be deduced from those with constant right hand side, by the work of Caffarelli and Safanov. However, it is not a simple matter to make this deduction explicit. We need not rely on any other arguments to obtain the following.

Consider

$$F\left(D^2u\right) = \sum_{i=1}^{n} \arctan \lambda_i = \Theta(x) \tag{1.86}$$

where λ_i are the eigenvalues of the Hessian D^2u .

Theorem 8. *Let u be a convex solution to (1.86) on $B_6(0)$. Then*

$$\left|D^2u(0)\right| \leq C(n, D\theta, D^2\theta) \exp\left\{C(n) \|Du\|^{3n-2}_{L^\infty(B_6)}\right\}.$$

It is well known [19] that

$$\begin{aligned}\triangle_g(x, Du(x)) &= H = J\nabla\theta \\ &= \sum_{i,j} g^{ij}\partial_j\theta(-\sum_k u_{ki}e_k + Je_i).\end{aligned}$$

Using this fact, a long and straightforward computation, which we omit, will show the following Jacobi inequality for $\ln V$.

Proposition 1.9.4. *Suppose that u is a smooth convex solution to (1.86) in $B_R \subset \mathbb{R}^n$. Then*

$$\triangle_g \ln V(x) \geq \frac{1}{2n}|\nabla_g \ln V|^2(x) - C(|D\theta(x)|^2 + |D^2\theta(x)|).$$

Consequently

$$\int_{B_r} |\nabla_g \ln V|^2 \, dv_g \leq \left(\frac{C(n)}{R-r} + C(D\theta, D^2\theta)\right) \int_{B_R} dv_g.$$

We can proceed as in the proof of the estimate for convex functions. Taking a Lewy rotation, applying the Sobolev inequality (Proposition 1.9.3) and the DeGiorgi-Moser estimate in Step 3, we have

$$
\begin{aligned}
(\ln V - \iota)^{+}(0) &= (\ln V - \iota)^{+}(\bar{0}) \\
&\leq C(n)\left(\left(\int_{B_{1/\sqrt{2}}(\bar{0})} \left|(\ln V - \iota)^{+}(\bar{x})\right|^{\frac{n}{n-1}} d\bar{x}\right)^{\frac{n-1}{n}} + \left|C(n, D\theta, D^{2}\theta)\right|_{L^{n}(B_{1/\sqrt{2}}(\bar{0}))}\right).
\end{aligned}
$$

The same considerations as in the proof of Theorem 5 then lead to Theorem 8.

1.10 Proof of Theorem 3: Gradient Estimates

We assume that $R = 1$ by scaling $u(Rx)/R^2$, and $\Theta \geq (n-2)\pi/2$ by symmetry.

Case $\Theta = (n-2)\pi/2$. Set $M = \operatorname{osc}_{B_1} u$. We may assume $M > 0$. By replacing u with $u - \min_{B_1} u + M$, we have $M \leq u \leq 2M$ in B_1. Let

$$w = \eta |Du| + Au^2$$

with $\eta = 1 - |x|^2$ and $A = n/M$. We assume that w attains its maximum at an interior point $x^* \in B_1$, otherwise w would take its maximum on the boundary ∂B_1 and the conclusion would be straightforward. Choose a coordinate system so that D^2u is diagonalized at x^*. We assume, say $u_n \geq \frac{|Du|}{\sqrt{n}}\ (> 0)$ at x^*. For all $i = 1, \cdots, n$, we have at x^*

$$0 = w_i = \eta |Du|_i + \eta_i |Du| + 2Auu_i,$$

then

$$\frac{u_i u_{ii}}{|Du|} = |Du|_i = -\frac{\eta_i |Du| + 2Auu_i}{\eta}. \tag{1.87}$$

In particular, we have $u_{nn} < 0$ by the choice of A. Since the phase $\Theta \geq (n-2)\pi/2$, it follows that $\lambda_n = \lambda_{\min}$, $|\lambda_n| \leq \lambda_k$, and

$$g^{nn} = \frac{1}{1+\lambda_n^2} \geq \frac{1}{1+\lambda_k^2} = g^{kk} \tag{1.88}$$

for $k = 1, \cdots, n-1$ at x^*.

Next, we show

$$\triangle_g u \geq 0.$$

When D^2u is diagonalized,

$$\triangle_g u = \sum_{i=1}^n g^{ii} u_{ii} = \sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i^2} = \frac{1}{2}\sum_{i=1}^n \sin(2\theta_i).$$

Let $S \subset \mathbb{R}^n$ be the hypersurface (with boundary) given by

$$S = \left\{ \theta \mid \theta_1 + \theta_2 + \cdots + \theta_n = \frac{\pi}{2}(n-2), |\theta_i| \leq \frac{\pi}{2} \right\},$$

where $\theta = (\theta_1, \cdots, \theta_n)$. Set $\Gamma(\theta) = \frac{1}{2}\sum_{i=1}^n \sin(2\theta_i)$. Suppose that Γ obtains a negative minimum on the interior of S at θ^*. At this point $D\Gamma$ vanishes on $T_{\theta^*}S$, thus we have

$$\cos(2\theta_i) = \cos(2\theta_j), \text{ then } \theta_i = \pm\theta_j.$$

The only two possible configurations for θ are

$$\theta_1 = \cdots = \theta_n = \frac{(n-2)\,\pi}{2n} \quad \text{or}$$
$$\theta_1 = \cdots = \theta_{n-2} = \frac{\pi}{2},\ \theta_{n-1} = -\theta_n.$$

In either case, Γ is nonnegative. This contradiction allows us to verify the nonnegativity of Γ along the boundary ∂S. It follows easily that $\Gamma \geq 0$ there by induction on dimension n, as

$$\partial S = \bigcup_{k=1}^{n} \left\{ \theta \mid \theta_1 + \cdots + \widehat{\theta_k} + \cdots + \theta_n = \frac{\pi}{2}(n-3),\ |\theta_i| \leq \frac{\pi}{2} \right\}$$

and $\Gamma\left(\theta_1, \theta_2\right) = 0$ for $\theta_1 + \theta_2 = 0$.

Further, we show

$$\triangle_g |Du| \geq 0.$$

We calculate

$$\begin{aligned}
\triangle_g |Du| &= \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \partial_{\alpha\beta} |Du| = \sum_{\alpha,\beta,i=1}^{n} g^{\alpha\beta} \left(\frac{u_i u_{i\beta\alpha}}{|Du|} + \frac{u_{i\alpha} u_{i\beta}}{|Du|} - \sum_{j=1}^{n} \frac{u_i u_{i\beta} u_j u_{j\alpha}}{|Du|^3} \right) \\
&= \sum_{\alpha,\beta,i=1}^{n} g^{\alpha\beta} \left(\frac{u_{i\alpha} u_{i\beta}}{|Du|} - \sum_{j=1}^{n} \frac{u_i u_{i\beta} u_j u_{j\alpha}}{|Du|^3} \right) \\
&\overset{D^2 u \text{ is diagonal}}{=} \sum_{\alpha=1}^{n} g^{\alpha\alpha} \frac{\left(|Du|^2 - u_\alpha^2 \right) \lambda_\alpha^2}{|Du|^3} \geq 0,
\end{aligned}$$

where we used the minimality equation (1.21).

Combining the subharmonicity of u and $|Du|$ with (1.88) and (1.87), we have at x^*

$$\begin{aligned}
0 \geq \triangle_g w &= |Du| \triangle_g \eta + 2 \sum_{\alpha=1}^{n} g^{\alpha\alpha} \eta_\alpha |Du|_\alpha + \underbrace{\eta \triangle_g |Du| + 2Au \triangle_g u}_{\geq 0} + 2A \sum_{\alpha=1}^{n} g^{\alpha\alpha} u_\alpha^2 \\
&\geq |Du| \triangle_g \eta + 2 \sum_{\alpha=1}^{n} g^{\alpha\alpha} \eta_\alpha \underline{|Du|_\alpha} + 2A \sum_{\alpha=1}^{n} g^{\alpha\alpha} u_\alpha^2 \\
&\geq -2n g^{nn} |Du| - 2 \sum_{\alpha=1}^{n} g^{\alpha\alpha} \eta_\alpha \left(\frac{\eta_\alpha |Du| + 2Auu_\alpha}{\eta} \right) + \frac{2A}{n} g^{nn} |Du|^2 \\
&\geq -2n g^{nn} |Du| - 4 g^{nn} \frac{|Du|}{\eta} - 8 g^{nn} Au \frac{|Du|}{\eta} + \frac{2A}{n} g^{nn} |Du|^2;
\end{aligned}$$

It follows that

$$0 \geq -2n\eta - 4 - 8Au + \frac{2A}{n}\eta\,|Du|\,.$$

Then by the assumption $M \leq u \leq 2M$ and $A = n/M$

$$\eta\,|Du|\,(x^*) \leq (n + 2 + 8n)\,M.$$

So we obtain

$$|Du\,(0)| \leq w\,(x^*) \leq 15nM. \tag{1.89}$$

Case $\Theta > (n-2)\,\pi/2$. Let $\Theta = \delta + (n-2)\,\pi/2$. From our special Lagrangian equation (1.1), we know

$$\theta_i + (n-1)\,\frac{\pi}{2} > (n-2)\,\frac{\pi}{2} + \delta \;\text{ or }\; \theta_i > -\frac{\pi}{2} + \delta.$$

We can control the gradient of the convex function $u\,(x) + \frac{1}{2}\max\{\cot\delta, 0\}\,|x|^2$ by its oscillation, thus

$$|Du\,(0)| \leq \operatorname*{osc}_{B_1} + \frac{1}{2}\max\{\cot\delta, 0\}\,. \tag{1.90}$$

In order to get rid of the δ-dependence in the gradient estimate, we need the following.

Proposition 1.10.1. *Let smooth u satisfy (1.1) with $\Theta - (n-1)\,\frac{\pi}{2} = \delta \in (0, \pi/4)$ on $B_2(0)$. Suppose that*

$$\operatorname*{osc}_{B_2} u \leq \frac{1}{2\sin\delta}. \tag{1.91}$$

Then

$$|Du(0)| \leq C(n)\left(\operatorname*{osc}_{B_2} u + 1\right).$$

Proof. We take the Lewy-Yuan rotation in the proof of Proposition 1.5.1, to obtain a "critical" representation $\mathfrak{M} = (\bar{x}, D\bar{u}\,(\bar{x}))$ for the original special Lagrangian graph $\mathfrak{M} = (x, Du(x))$ with $x \in B_2$. Recentering the new coordinates, we take

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{n} + Du\,(x)\sin\frac{\delta}{n} - Du\,(0)\sin\frac{\delta}{n} \\ D\bar{u}\,(\bar{x}) = -x\sin\frac{\delta}{n} + Du\,(x)\cos\frac{\delta}{n} \end{cases}. \tag{1.92}$$

By (1.23) we see that the potential $\bar{u}$ is defined on a ball in $\bar{x}$-space around the origin of radius

$$\bar{R} = \frac{2}{2\cos(\frac{\delta}{n})} > 1.$$

From (1.92) and the estimate (1.89) for the critical potential, we have

$$|Du(0)| = \frac{|D\bar{u}(\bar{0})|}{\cos(\delta/n)} \leq C\,(n) \operatorname*{osc}_{\bar{B}_1} \bar{u}.$$

Next, we estimate the oscillation of $\bar{u}$ in terms of u. We may assume that $\bar{u}(\bar{0}) = 0$. Without loss of generality we assume the maximum of $|\bar{u}|$ on $\bar{B}_1(\bar{0})$ happens along the positive $\bar{x}_1$-axis, and even on the boundary $\partial\bar{B}_1$. Thus we have

$$\operatorname*{osc}_{\bar{B}_1} \bar{u} \leq 2 \left| \int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} d\bar{x}_1 \right|.$$

In the following, we convert the integral of $\bar{u}_{\bar{x}_1}$ to one in terms of u_{x_1}, then recover the oscillation of $\bar{u}$ from that of u.

We work on the x_1-y_1 plane in the remaining of the proof. Under our above assumption, the $\bar{x}_1$-axis is given by the line

$$y_1 = \tan\left(\frac{\delta}{n}\right) x_1$$

and the curve $\gamma : (x_1, u_1(x_1))$ with $|x_1| < 2$ forms a graph over the $\bar{x}_1$-axis. Let l_0 be the line perpendicular to the $\bar{x}_1$-axis and intersecting the curve γ at $(0, u_1(0))$ along the y_1-axis. The intersection of l_0 and the $\bar{x}_1$-axis (which is also the origin of the recentered the $\bar{x}_1$-$\bar{y}_1$ plane) has distance to the origin of the x_1-y_1 plane given by

$$|u_1(0)| \sin\left(\frac{\delta}{n}\right) \leq \left(\operatorname*{osc}_{B_1} u + \frac{1}{2}\cot\delta\right) \sin\left(\frac{\delta}{n}\right) \leq 1 \tag{1.93}$$

by the rough bound (1.90) and the condition (1.91). Now let l_1 be the line parallel to l_0 passing through the point $\bar{x}_1 = 1$ along the $\bar{x}_1$-axis.

The integral

$$\int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} d\bar{x}_1$$

is the signed area between the $\bar{x}_1$-axis and the curve γ, and lying between the lines l_0 and l_1. We convert this to an integral over x_1,

$$\int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} d\bar{x}_1 = \int_{P(l_0\cap\bar{x}_1\text{-axis})}^{P(l_1\cap\bar{x}_1\text{-axis})} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right) x_1\right] dx_1 + K_0 + K_1,$$

where P denotes projection to the x_1-axis, and K_0 as well as K_1 denotes the signed areas to the left or right of the desired region, forming the difference.

It is important to note the following for $j = 1, 2$:

(i) $P(l_j \cap \bar{x}_1\text{-axis})$ is in the x_1-domain of u_1 by (1.93),

$$|P(l_0 \cap \bar{x}_1\text{-axis})| \leq 1 \cdot \cos\left(\frac{\delta}{n}\right) < 1,$$

$$|P(l_1 \cap \bar{x}_1\text{-axis})| \leq (1+1) \cdot \cos\left(\frac{\delta}{n}\right) < 2;$$

(ii) $P(l_j \cap \gamma)$ is also in the x_1-domain of u_1 as the whole Lagrangian surface $\mathfrak{M}$ is a graph over B_2,

$$|P(l_j \cap \gamma)| \leq 2;$$

(iii) the region K_j is bounded by the line l_j, the vertical line $x_1 = P(l_j \cap \bar{x}_1\text{-axis})$, and the curve γ, also each region K_j is on one side of the $\bar{x}_1$-axis.

Thus from (i)

$$\left| \int_{P(l_0 \cap \bar{x}_1\text{-axis})}^{P(l_1 \cap \bar{x}_1\text{-axis})} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right) x_1 \right] dx_1 \right| \leq \operatorname*{osc}_{B_2} u + C(n)$$

and from (ii) (iii)

$$|K_j| \leq \left| \int_{P(l_j \cap \bar{x}_1\text{-axis})}^{P[l_j \cap \gamma]} \left[u_1(x_1) - \tan(\frac{\delta}{n}) x_1 \right] dx_1 \right| \leq \operatorname*{osc}_{B_2} u + C(n).$$

It follows that we have the conclusion of Proposition 1.10.1

$$|Du(0)| \leq C(n) \operatorname*{osc}_{\bar{B}_1} \bar{u} \leq C(n) \left(\operatorname*{osc}_{B_2} u + 1 \right).$$

$\square$

We finish the proof of Theorem 3. For $\delta \geq \pi/4$, the bound (1.90) gives

$$|Du(0)| \leq \operatorname*{osc}_{B_1} u + \frac{1}{2} \leq C(n) \left[\operatorname*{osc}_{B_2} u + 1 \right].$$

For $\delta \leq \pi/4$, if $\operatorname{osc}_{B_2} u \leq 1/(2 \sin\delta)$, then Proposition 1.10.1 gives

$$|Du(0)| \leq C(n) \left[\operatorname*{osc}_{B_2} u + 1 \right].$$

Otherwise, $\operatorname{osc}_{B_2} u > 1/(2 \sin\delta)$, and from (1.90)

$$|Du(0)| \leq \operatorname*{osc}_{B_1} u + \operatorname*{osc}_{B_2} u \leq C(n) \left[\operatorname*{osc}_{B_2} u + 1 \right].$$

Applying this estimate on $B_2(x)$ for any $x \in B_1(0)$, we arrive at the conclusion of Theorem 3.

1.11 A Liouville Type Theorem

In this section, we prove Theorem 7.

1.11.1 Proof of Liouville Theorem

We begin by demonstrating a Jacobi inequality for the volume element

$$V = \sqrt{\det g} = \prod_{i=1}^{n}(1+\lambda_i^2)^{\frac{1}{2}}.$$

Lemma 1.11.1. *Suppose that u is a smooth solution to (1.1) satisfying (1.14). Then*

$$\triangle_g \ln V \ \geq \frac{\varepsilon}{n}|\nabla_g \ln V|_g^2$$

or equivalently

$$\triangle_g V^{\frac{\varepsilon}{n}} \geq 2\frac{|\nabla_g V^{\frac{\varepsilon}{n}}|_g^2}{V^{\frac{\varepsilon}{n}}}. \tag{1.94}$$

Proof. At any fixed point, we assume that D^2u is diagonalized, then using (1.4.1)

$$\triangle_g \ln V = \sum_{i,j,k=1}^{n}(1+\lambda_i\lambda_j)h_{ijk}^2,$$

where $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$. Gathering all terms containing $h_{ijj}^2 = h_{jij}^2 = h_{jji}^2$ for a fixed i, we have

$$\begin{aligned}
&(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(1+\lambda_j^2)h_{jji}^2 + \sum_{j\neq i}(1+\lambda_i\lambda_j)h_{ijj}^2 + \sum_{j\neq i}(1+\lambda_j\lambda_i)h_{jij}^2 \\
&= (1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\triangle_g \ln V \ &= \sum_{i}^{n}\left[(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2\right] \\
&\quad + 2\sum_{i<j<k}(3+\lambda_i\lambda_j+\lambda_j\lambda_k+\lambda_k\lambda_i)h_{ijk}^2.
\end{aligned} \tag{1.95}$$

Condition (1.14) gives that

$$\underline{3} + (1-\varepsilon)\,\lambda_i^2 + \lambda_i\lambda_j + \underline{\lambda_i\lambda_j + \lambda_k(\lambda_j+\lambda_i)} - \lambda_k(\lambda_j+\lambda_i) \geq 0$$

that is

$$S_{ijk} = 3 + \lambda_i\lambda_j + \lambda_j\lambda_k + \lambda_k\lambda_i \geq (\lambda_k - \lambda_i)(\lambda_i + \lambda_j) + \varepsilon\lambda_i^2.$$

Switching λ_i and λ_j, we also have

$$S_{ijk} = S_{jik} \geq (\lambda_k - \lambda_j)(\lambda_j + \lambda_i) + \varepsilon\lambda_j^2.$$

By symmetry of S_{ijk}, we may assume

$$\lambda_i \geq \lambda_k \geq \lambda_j, \tag{1.96}$$

then either $(\lambda_k - \lambda_i)(\lambda_i + \lambda_j)$ or $(\lambda_k - \lambda_j)(\lambda_j + \lambda_i)$ has to be non-negative, thus

$$S_{ijk} \geq \varepsilon \min\left\{\lambda_i^2, \lambda_j^2\right\}. \tag{1.97}$$

We conclude that

$$\triangle_g \ln V \geq \sum_i^n \left[(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2\right]. \tag{1.98}$$

To bound the gradient, we compute, (still at the same fixed point with D^2u diagonalized,)

$$\partial_i \ln V = \sum_{j=1}^n g^{jj}\lambda_j u_{jji},$$

then

$$\begin{aligned}
|\nabla_g \ln V|_g^2 &= \sum_{i=1}^n g^{ii}\left(\sum_{j=1}^n g^{jj}\lambda_j u_{jji}\right)^2 \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n \lambda_j h_{jji}\right)^2 \leq n\sum_{i,j=1}^n {\lambda_j}^2 h_{jji}^2.
\end{aligned} \tag{1.99}$$

Combining (1.14) with (1.98) and (1.99) we have

$$\begin{aligned}
&\triangle_g \ln V - \frac{\varepsilon}{n}|\nabla_g \ln V|^2 \\
&\geq \sum_{i=1}^n(\left[1+(1-\varepsilon)\lambda_i^2\right]h_{iii}^2 + \sum_{j\neq i}(\left[3+(1-\varepsilon)\lambda_j^2+2\lambda_i\lambda_j\right]h_{jji}^2 \geq 0.
\end{aligned} \tag{1.100}$$

The proof of Lemma 1.11.1 is complete. $\square$

Lemma 1.11.2. *Suppose that u is a smooth solution to (1.1) on $B_1(0)$ satisfying condition (1.14) and*

$$|\nabla u| \le \delta < \frac{1}{\sqrt{n-1}}.$$

Then

$$|D^2 u(0)| \le C(n, \delta, \varepsilon).$$

Proof. Set

$$v = u + \alpha \frac{1}{|\nabla u(0)|} \langle \nabla u(0), x \rangle \quad \text{or} \quad u + \alpha x_1 \text{ if } \nabla u(0) = 0,$$

where $\alpha = \left(\frac{1}{\sqrt{n-1}} - \delta\right)/2$. Now v satisfies in B_1 the following

$$D^2 v = D^2 u, \quad |\nabla v(0)| \ge \alpha, \quad \text{and} \quad |\nabla v| \le \alpha + \delta < \frac{1}{\sqrt{n-1}}.$$

Set $b = V^{\frac{\varepsilon}{n}}$, and consider the function

$$w = \eta b = \left[|\nabla u|^2 - (\alpha+\delta)^2 |x|^2\right]^+ b \ge 0.$$

A positive maximum for w will be attained at a point p on the interior, since $w(0) > 0$ and $w(x)$ vanishes on the boundary ∂B_1. At this point p,

$$\nabla_g (\eta b) = 0 \quad \text{or} \quad \nabla_g \eta = -\frac{\eta}{b} \nabla_g b,$$

$$\begin{aligned} 0 \ge \triangle_g (\eta b) &= \eta \triangle_g b + 2 \langle \nabla_g \eta, \nabla_g b \rangle_g + b \triangle_g \eta \\ &= \eta \left(\triangle_g b - 2 \frac{|\nabla_g b|_g^2}{b} \right) + b \triangle_g \eta \\ &\ge b \triangle_g \eta, \end{aligned}$$

by the inequality (1.94) in Lemma 1.11.1. This last inequality implies a bound on $|D^2 v(p)|$ as the following. We have

$$\begin{aligned} 0 \ge \triangle_g \eta &= \triangle_g \left[|\nabla v|^2 - (\alpha+\delta)^2 |x|^2\right] \\ &= \sum_{i,j=1}^n g^{ij} \left[2 \sum_{k=1}^n (v_{ki} v_{kj} + v_k \partial_{ij} v_k) - (\alpha+\delta)^2 \partial_{ij} |x|^2 \right] \\ &= 2 \sum_{i=1}^n \frac{\lambda_i^2 - (\alpha+\delta)^2}{1+\lambda_i^2} \ge 2 \left[\frac{\lambda_1^2 - (\alpha+\delta)^2}{1+\lambda_1^2} - (n-1)(\alpha+\delta)^2 \right], \end{aligned}$$

using the minimal surface equation (1.21) and assuming $|\lambda_1| \geq |\lambda_i|$ for all i. It follows that

$$1+\lambda_1^2(p) \leq \frac{1+(\alpha+\delta)^2}{1-(n-1)(\alpha+\delta)^2}.$$

We get

$$\alpha^2 b(0) \leq |\nabla v(0)|^2 b(0) \leq \eta(p) b(p) \leq (\alpha+\delta)^2 \left[\frac{1+(\alpha+\delta)^2}{1-(n-1)(\alpha+\delta)^2}\right]^{\frac{\varepsilon}{2}},$$

then

$$1+\lambda_i^2(0) \leq \left(1+\frac{\delta}{\alpha}\right)^{\frac{4n}{\varepsilon}} \left[\frac{1+(\alpha+\delta)^2}{1-(n-1)(\alpha+\delta)^2}\right]^n. \tag{1.101}$$

Therefore, we conclude the estimate $|D^2u(0)| \leq C(n,\delta,\varepsilon)$ in Lemma 1.11.2. □

Lemma 1.11.3. *Let $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a solution to the special Lagrangian equation (1.1) and homogeneous of order 2; that is, $u(x) = |x|^2 u(x/|x|)$. Suppose that the eigenvalues λ_i of the Hessian $D^2u(x)$ satisfy (1.14). Then u must be quadratic.*

Proof. Lemma 1.11.3 follows from Proposition 1.11.1; nonetheless we give a direct proof in the following. Considering (1.14), (1.96), and (1.97), we observe that the coefficients of h_{ijk}^2 in (1.95) are strictly positive. Accordingly,

$$\triangle_g \ln V \geq c(\lambda) \sum_{i,j,k=1}^{n} h_{ijk}^2 \tag{1.102}$$

with $c(\lambda) > 0$.

Since u is homogeneous of order 2, the homogeneous order 0 function $\ln V$ attains its maximum along a ray. We infer from the strong maximum principle that $\ln V \equiv$ *const*. It follows from (1.102) that $D^3u \equiv 0$. Therefore, u must be quadratic, as claimed in Lemma 1.11.3. □

Proof of Theorem 7. Now the Hessian bound is available by Lemma 1.11.2. We run the "routine" blow-down procedure "in detail" to finish the proof of Theorem 7, as in [56].

Step 1. From the assumption that $|\nabla u(x)| \leq \delta|x|$ for large x, we have on the ball $B_R(p)$ with any fixed $p \in \mathbb{R}^n$

$$|\nabla u(x)| \leq \delta(|p|+R) = \left(\delta + \frac{\delta|p|}{R}\right) R.$$

A rescaled version of Lemma 1.11.2 with R going to ∞ then leads to a Hessian bound, $|D^2u(p)| \leq C(n, \delta, \varepsilon) \triangleq K$, which must hold at each point $p \in \mathbb{R}^n$.

Step 2. Repeating verbatim the argument in ([55], p.263–264), we show that we can find a tangent cone of the special Lagrangian graph $(x, \nabla u(x))$ at ∞ whose potential function is $C^{1,1}$, homogenous order 2, and still satisfies the "convexity" condition (1.14).

Without loss of generality, we assume $u(0) = 0$, $\nabla u(0) = 0$. We "blow down" u at ∞.

Set

$$u_k(x) = \frac{u(kx)}{k^2}, \ k = 1, 2, 3, \cdots .$$

We see that

$$\|u_k\|_{C^{1,1}(B_R)} \leq C(K, R),$$

so there exists a subsequence, still denoted by $\{u_k\}$ and a function $u_R \in C^{1,1}(B_R)$ such that $u_k \to u_R$ in $C^{1,\alpha}(B_R)$ as $k \to \infty$, and $|D^2u_R| \leq K$. By the fact that the family of viscosity solution is closed under C^0 uniform limit, we know that u_R is also a viscosity solution of

$$F\left(D^2u\right) = \sum_{i=1}^{n} \arctan \lambda_i = c \ \text{ on } \ B_R.$$

Applying the $W^{2,\delta}$ estimate (cf. [8], Proposition 7.4) to the difference $u_k - u_R$, we have

$$\left\|D^2u_k - D^2u_R\right\|_{L^\delta(B_{R/2})} \leq C(K, R)\, \|u_k - u_R\|_{L^\infty(B_R)} \to 0 \text{ as } k \to \infty.$$

Note that $\left|D^2u_k\right|, \left|D^2u_R\right| \leq K$, so also

$$\left\|D^2u_k - D^2u_R\right\|_{L^n(B_{R/2})} \to 0 \text{ as } k \to \infty.$$

By a standard fact from real analysis, there exists another subsequence and $C^{1,1}$ function on B_R, still denoted by $\{u_k\}$ and $u_{R/2}$ such that $D^2u_k \to D^2u_{R/2}$ almost everywhere as $k \to \infty$. So D^2u_R still satisfies (1.14) almost everywhere on $B_{R/2}$.

The diagonalizing process yields yet another subsequence, again denoted by $\{u_k\}$ and $v \in C^{1,1}(\mathbb{R}^n)$ such that $u_k \to v$ in $W^{2,n}_{loc}(\mathbb{R}^n)$ as $k \to \infty$; v is a viscosity solution of (1.1) on $\mathbb{R}^n$; $\left|D^2v\right| \leq K$; and D^2v still satisfies (1.14) almost everywhere on $\mathbb{R}^n$.

The surfaces $(x, \nabla u_k(x))$ are minimal in $\mathbb{R}^n \times \mathbb{R}^n$ and their potentials u_k converge to v in $W^{2,n}_{loc}(\mathbb{R}^n)$, so by the monotonicity formula (cf. [39], p.84, Theorem 19.3), we conclude that $M_v = (x, \nabla v(x))$ is a cone.

Step 3. We claim that M_v is smooth away from the vertex. Suppose M_v is singular at P away from the vertex. We blow up M_v at P to get a tangent cone, which is a lower dimensional special Lagrangian cone crossing a line; repeat the procedure if the resulting cone is still singular away from the vertex. Finally we get a special Lagrangian cone which is smooth away from the vertex, and the bounded eigenvalues of the Hessian of the potential function satisfies (1.14), by a similar $W^{2,\delta}$ argument as in Step 2. By Lemma 1.11.3, the cone is flat. This is a contradiction to Allard's regularity result (cf. [39], Theorem 24.2).

Applying Lemma 1.11.3 to M_v, we see that M_v is flat.

Step 4. Now with the flatness of M_v, a final application of the monotonicity formula yields that the original gradient graph $(x, \nabla u(x))$ is also a plane (cf. [56], p.123). Therefore, u is a quadratic polynomial. □

1.11.2 More on Cones

We include here a uniqueness result for global solutions to the special Lagrangian equation (1.1) with bounded Hessian satisfying certain "convexity" constraints (1.103). The constraints are only needed for $n \geq 4$.

Theorem 9. *Let u be a smooth solution to the special Lagrangian equation (1.1). Suppose that the eigenvalues λ_i of the Hessian $D^2u(x)$ are bounded $|\lambda_i(x)| \leq K$ and satisfy*

$$3 + \lambda_i^2(x) + 2\lambda_i(x)\lambda_j(x) \geq 0 \tag{1.103}$$

for all i, j, and x. Then u must be a quadratic polynomial.

Proof. The proof is identical to the one of Theorem 7 with Lemma 1.11.3 replaced by the following proposition. □

Proposition 1.11.1. *Let $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a solution to the special Lagrangian equation (1.1) and homogeneous of order 2, that is $u(x) = |x|^2 u(x/|x|)$. Suppose that the eigenvalues λ_i of the Hessian $D^2u(x)$ satisfy (1.103) for all i, j, and $x \neq 0$. Then u must be quadratic.*

Proof. By (1.103), we certainly have (1.100) with $\varepsilon = 0$ in Lemma 1.11.1, that is

$$\begin{aligned} \triangle_g \ln V &\geq \sum_{i=1}^{n} \left(1+\lambda_i^2\right) h_{iii}^2 + \sum_{j\neq i} \left(3+\lambda_j^2+2\lambda_i\lambda_j\right) h_{jji}^2 \\ &= \sum_{i=1}^{n} \frac{1}{\left(1+\lambda_i^2\right)^2} u_{iii}^2 + \sum_{j\neq i} \frac{\left(3+\lambda_j^2+2\lambda_i\lambda_j\right)}{\left(1+\lambda_j^2\right)^2\left(1+\lambda_i^2\right)} u_{jji}^2 \geq 0. \end{aligned} \tag{1.104}$$

Since u is homogeneous of order 2, the Hessian $D^2u\,(x)$ is homogeneous of order 0, hence $\ln V$ must attain its maximum along a ray. The strong maximum principle yields that $\ln V$ is constant, so in fact

$$0 = \triangle_g \ln V. \tag{1.105}$$

We claim now that

$$\triangle u \;= const \tag{1.106}$$

on $\mathbb{R}^n \setminus \{0\}$. At any point p compute the derivative

$$\partial_i\left(\triangle u\right) = \sum_j u_{jji} \tag{1.107}$$

for all i. Still assuming that D^2u is diagonalized at p, an inspection of (1.104), together with (1.105) shows that for all j with $u_{jji} \neq 0$,

$$3+\lambda_j^2+2\lambda_i\lambda_j \;= 0. \tag{1.108}$$

From $3+\lambda_i^2+2\lambda_i\lambda_j \geq 0$, we see that $\lambda_i^2 \geq \lambda_j^2$. Solving (1.108) for λ_j we get

$$\lambda_j = -\lambda_i - \sqrt{\lambda_i^2-3}, \text{ if } \lambda_i < 0$$

$$\lambda_j = -\lambda_i + \sqrt{\lambda_i^2-3}, \text{ if } \lambda_i > 0\ .$$

The minimal surface equation (1.21) at p then reads

$$0 = \triangle_g u_i \overset{p}{=} \sum_j \frac{1}{1+\lambda_j^2} u_{jji} = \frac{1}{1+\left(-\lambda_i \pm \sqrt{\lambda_i^2-3}\right)^2} \sum_j u_{jji}.$$

Hence $\partial_i(\triangle u) = 0$ and $\triangle u$ is constant.

Differentiating (1.106), we see that each u_{ij} satisfies

$$\triangle u_{ij} = 0.$$

Applying the strong maximum principle once again to each (homogeneous order 0) function u_{ij}, we have immediately

$$u_{ij} = const;$$

that is, u is quadratic. $\square$

Chapter 2

THE MONGE-AMPÈRE EQUATION

In this chapter we turn to calibrations on $\mathbb{R}^n \times \mathbb{R}^n$, with respect to the pseudo-Euclidean metric $dxdy$. The metric $dxdy$ can be expressed explicitly as the indefinite form

$$dxdy = \frac{1}{2}\sum_i (dx_i \otimes dy_i + dy_i \otimes dx_i).$$

Hitchin showed how the metric arises naturally in the study of the moduli space of calibrated submanifolds [23]. This metric also has been of interest recently to physicists who study what are called "para-complex" structures. Essentially, by replacing the complex structure J with an involution I, keeping the same Kähler form, while requiring that the Kähler form satisfy

$$\omega(X,Y) = g(IX,Y)$$

we arrive at the metric $g = dxdy$. For further discussion of para-complex manifolds and their relevance in string theory, see [12]. Our main result is the following.

Theorem 10. *Let $\Gamma = (x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ be the gradient graph of a convex function $u \in C^2(\Omega)$, for Ω a bounded, simply connected region with C^1 connected boundary. If u satisfies the Monge-Ampère equation*

$$\det(D^2u) = c, \tag{2.1}$$

or equivalently,

$$\sum \ln \lambda_i = \ln c,$$

where λ_i are the eigenvalues of D^2u, then Γ is volume maximizing in the pseudo-Euclidean space $(\mathbb{R}^n_x \times \mathbb{R}^n_y, dxdy)$. Precisely, if Σ is any C^1, space-like, oriented n-surface, with $\partial\Sigma = \partial\Gamma = \Gamma \cap \{x \in \partial\Omega\}$, homologous to Γ in $(\mathbb{R}^n \times \mathbb{R}^n, dxdy)$, then

$$Vol(\Sigma) \le Vol(\Gamma),$$

with equality only if $\Sigma = \Gamma$.

A Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ is one that can be described locally as a gradient graph, $(x, \nabla u(x))$ (see section 1.2). Hitchin ([23], §5) introduced a definition of "special Lagrangian" for Lagrangian submanifolds of $(\mathbb{R}^n \times \mathbb{R}^n, dxdy)$, and demonstrated that a gradient graph $(x, \nabla u(x))$ is special precisely when the potential $u(x)$ satisfies (2.1). Jost and Xin ([25], §4) then showed that such a submanifold has mean curvature $H \equiv 0$. Yu Yuan suggested that it is possible to find a calibration associated to (2.1), and that is what we do now. Specifically, we show that if $u(x)$ is a solution to (2.1), then the gradient graph $(x, \nabla u(x))$ is a calibrated submanifold of $(\mathbb{R}^n \times \mathbb{R}^n, dxdy)$, and consequently volume maximizing. In fact, any submanifold which is locally described by gradient graphs of functions satisfying (2.1) is calibrated, and therefore volume maximizing. Theorem 10 can be stated to allow a slightly larger class of n-surfaces, as we shall see in section 3.

The study of calibrated Lagrangian submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$ began with the work of Harvey and Lawson ([19], §3), who studied Lagrangian submanifolds of $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ with the Euclidean metric δ_0, and showed that a Lagrangian submanifold is calibrated, and therefore volume minimizing, if and only if the potential $u(x)$ is a solution to the special Lagrangian equation

$$\sum_i \arctan \lambda_i = c. \tag{2.2}$$

McLean ([31], Theorem 3.6, 3.10) showed that given $\Sigma \subset M$ a compact special Lagrangian submanifold, the moduli space X of special Lagrangian manifolds near Σ in M is itself a manifold which carries a natural Riemannian metric. Hitchin ([23], Proposition 2), showed that this metric can be obtained locally by embedding the moduli space X into $(V \oplus V^*, g_0)$, where $V = H^1(\Sigma, \mathbb{R})$. Hitchin then showed ([23], Proposition 3) that X is a special Lagrangian submanifold of $(V \oplus V^*, g_0)$, and that special Lagrangian submanifolds arise as solutions to the Monge-Ampère equation, $\det(D^2 u) = c$.

2.1 Preliminaries

For Γ the graph of F over a bounded region $\Omega \subset \mathbb{R}^n$, the volume of Γ is given by the functional

$$Vol(\Gamma) = \int_\Omega \sqrt{\det(I + (DF)(DF)^T)}dx$$

when the metric on $\mathbb{R}^n \times \mathbb{R}^n$ is the Euclidean metric δ_0. When the ambient metric is the pseudo-Euclidean metric g_0, the volume functional is given by

$$Vol(\Gamma) = \int_\Omega \sqrt{\det(DF + (DF)^T)}dx\,.$$

When $F = \nabla u$, then the Euler-Lagrange equations for the above functionals at F become (1.1) and (2.1), respectively.

We may use the idea of a calibration to detect maximal surface in the pseudo-Riemannian setting, provided we pay careful attention to orientation as it arises. Given an oriented k-plane ξ and a k-form φ, we will say that ξ is *oriented with respect to* φ whenever $\varphi(\xi) > 0$. Let M be a pseudo-Riemannian manifold with index $n-k$, and suppose φ is a closed exterior k-form on M with the property that on all oriented space-like k-planes, which are oriented with respect to φ,

$$\varphi(\xi) \geq Vol(\xi).$$

Let Σ be any compact, oriented, k-dimensional, space-like submanifold with the property that $\varphi|_\Sigma(\xi) = Vol|_\Sigma(\xi)$, for all oriented (with respect to the orientation on Σ) tangent k-planes ξ. If Σ is homologous to Σ', then Σ' has a prescribed orientation, and it is with respect to this orientation that we apply Stokes's Theorem and obtain

$$\int_\Sigma \varphi = \int_{\Sigma'} \varphi.$$

It is possible, however, for the orientation of Σ' to produce tangent planes which are not oriented with respect to φ. If this disagreement occurs, the calibrating inequality is reversed, and we are unable to make the volume comparison. This can happen if Σ' is disconnected, or is singular along a significant subset. The conditions in Theorem 10 are sufficient to preclude any such pathology, as we will see in section 3.

For the Euclidean case, Harvey and Lawson ([19], §3) define the S^1-family of forms

$$\alpha_\theta = Re(e^{-\sqrt{-1}\theta} dz),$$

where $dz = dz_1 \wedge ... \wedge dz_n$, and show that these are calibrations on $\mathbb{C}^n$. We briefly recall the idea.

For any real n-plane ξ in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$, we may choose an oriented orthonormal basis $\xi_1, ..., \xi_n$, with $\xi_i \in \mathbb{C}^n$. Define a complex-linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ by

$$A(e_{x_i}) = \xi_i,$$

$$A(e_{y_i}) = J\xi_i,$$

which is represented by a complex-valued matrix $A \in M(n, \mathbb{C})$, namely the complex $n \times n$ matrix with columns $\xi_1, ..., \xi_n$. This also defines a real-linear map $\hat{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, which is represented by a real-valued matrix $\hat{A} \in M(2n, \mathbb{R})$. Now

$$|\xi \wedge J\xi| = \det_{\mathbb{R}} \hat{A} = |\det_{\mathbb{C}} A|^2 = |dz(\xi)|^2 = \alpha^2(\xi) + \beta^2(\xi),$$

where $\alpha(\xi) = Re(dz(\xi))$, and $\beta(\xi) = Im(dz(\xi))$. By Hadamard's Inequality, $|\xi|^2 \geq |\xi \wedge J\xi|$, with equality if and only if ξ is Lagrangian. So now we have

$$|\xi|^2 \geq |\xi \wedge J\xi| = \alpha^2(\xi) + \beta^2(\xi) \geq \alpha^2(\xi),$$

with equality if and only if ξ is Lagrangian and $\beta(\xi) = 0$.

If ξ is Lagrangian, then $\xi_1, ..., \xi_n$ is an orthonormal basis for $\mathbb{C}^n$, so $A \in U(n)$. With the action of $SO(n)$ on $U(n)$ given by $n \times n$ matrix multiplication, choosing an orthonormal basis associates to each $\xi \in L$ an $A \in U(n)$ which is unique up to a factor of $SO(n)$. With this association, we have a transitive action of $U(n)$ on L. The isotropy group of $\xi_0 = \mathbb{R}^n_x$ is $SO(n)$, so L is a homogeneous space and the complex determinant descends to a map

$$\Theta : L \approx U(n)/SO(n) \overset{\det_{\mathbb{C}}}{\to} S^1.$$

For any $\xi \in L$, $dz(\xi) = e^{\sqrt{-1}\Theta(\xi)}$. Taking

$$\alpha_\theta = Re(e^{-\sqrt{-1}\theta} dz),$$

we see that $\alpha_\theta(\xi) = 1$ if and only if $\Theta(\xi) = \theta$, thus the $\alpha_\theta' s$ are calibrations for $\mathbb{C}^n$.

The special Lagrangian equations may then be deduced from the condition

$$\arg(\det(I + \sqrt{-1}D^2u)) = \arg((1 + \sqrt{-1}\lambda_1)...(1 + \sqrt{-1}\lambda_n)) = c$$

for some fixed c. This equation is satisfied precisely when Θ is constant along the graph of ∇u.

2.2 Calibrations for Pseudo-Euclidean Space

In the pseudo-Euclidean setting, a Lagrangian submanifold Σ is called *special* ([23], p. 510) if a linear combination of the volume forms $dx_1 \wedge ... \wedge dx_n$ and $dy_1 \wedge ... \wedge dy_n$ vanish along Σ.

Proposition 2.2.1. *For $c > 0$,*

$$\Phi_c = \frac{1}{2}[c\, dx_1 \wedge ... \wedge dx_n + \frac{1}{c} dy_1 \wedge ... \wedge dy_n]$$

is a calibration for $(\mathbb{R}^n \times \mathbb{R}^n, dxdy)$. Suppose ξ is an oriented space-like n-plane in $\mathbb{R}^n \times \mathbb{R}^n$, with $\Phi_c(\xi) > 0$. Then

$$\Phi_c(\xi) \geq Vol(\xi)$$

with equality if and only ξ is special Lagrangian, that is, if

$$dy_1 \wedge ... \wedge dy_n(\xi) = c^2 dx_1 \wedge ... \wedge dx_n(\xi).$$

In the development of the special Lagrangian calibrations, Harvey and Lawson used Hadamard's Inequality to compare Lagrangian planes to non-Lagrangian planes. In order to prove the Proposition 2.2.1, we need a result which serves this purpose in the pseudo-Euclidean case. The following lemma is from linear algebra.

Lemma 2.2.1. *Suppose $Q \in GL(n, \mathbb{R})$ satisfies $Q_{ij}x^i x^j > 0$, for all $0 \neq x \in \mathbb{R}^n$. Then*

$$\det(Q) \geq \det(\frac{Q + Q^T}{2})$$

with equality if and only if $Q = Q^T$.

Proof of Lemma. The matrix Q can be written $Q = S + A$, with S symmetric, and A antisymmetric. Choose a basis so that S is a diagonal matrix. With respect to this basis,

$$Q = \begin{pmatrix} \lambda_1 & a_{12} & \dots & a_{1n} \\ -a_{12} & \lambda_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & \dots & \lambda_n \end{pmatrix}.$$

Expand the determinant of Q, and group the terms according to the number of λ_i's each terms contains. For all $k \leq n$ define

$$P_k = \text{sum of all terms containing exactly } k \ \lambda_i's\,.$$

We see that P_n consists of one term, namely $\lambda_1 \dots \lambda_n = \sigma_n(S)$, and that there are no terms with $(n-1)$ λ_i's, so $P_{n-1} = 0$. For P_{n-k}, $k \geq 2$, we fix $i_1, \dots, i_{n-k}$, and look at the terms containing $\lambda_{i_1} \dots \lambda_{i_{n-k}}$. These occur as the determinant of a matrix, which after an orthogonal change of basis, looks like

$$\begin{pmatrix} \lambda_{i_1} & & & & & \\ & \dots & & & & \\ & & \lambda_{i_{n-k}} & & & \\ & & & 0 & a_{j_1 j_2} & \dots \\ & & & -a_{j_1 j_2} & 0 & \\ & & & \dots & & \end{pmatrix}$$

with $j_1, \dots, j_k \notin \{i_1, \dots, i_{n-k}\}$. This determinant is the product of the determinants of a positive diagonal matrix and an antisymmetric matrix. It follows that $P_{n-k} \geq 0$. We also see that

$$P_{n-2} = \sum_{i<j} a_{ij}^2 \ \lambda_1 \dots \hat{\lambda}_i \hat{\lambda}_j \dots \lambda_n$$

which is strictly positive unless $a_{ij} = 0$, for all i, j. We conclude

$$\det(Q) = P_n + P_{n-2} + \dots + P_0 \geq P_n = \det(S) = \det(\frac{Q + Q^T}{2})$$

with equality if and only if $Q = Q^T$. $\square$

Proof of Proposition. If ξ is a space-like tangent plane, the projection onto $\mathbb{R}^n_x$ is full rank, and we can take a basis for ξ of the form

$$\xi_i = \partial_{x_i} + w_i^{\,j}\partial_{y_j}, \quad i = 1, ..., n.$$

Let Q be the matrix given by $Q_{ij} = w_i^{\,j}$. If g is the induced metric, then with respect to this frame, the tensor $g_{ij} = g(\xi_i, \xi_j)$ becomes $\left(Q + Q^T\right)_{ij}/2$, so

$$Vol(\xi_1 \wedge ... \wedge \xi_n) = \sqrt{\det\left(\frac{Q + Q^T}{2}\right)}.$$

Now

$$\Phi_c(\xi_1 \wedge ... \wedge \xi_n) = \frac{1}{2}[c + \frac{1}{c}\det(Q)] \geq \sqrt{\det Q}$$

with equality if and only if $\det(Q) = c^2$, so

$$\Phi_c(\xi_1 \wedge ... \wedge \xi_n) \geq \sqrt{\det(Q)}.$$

From the above lemma

$$\sqrt{\det(Q)} \geq \sqrt{\det\left(\frac{Q + Q^T}{2}\right)} = Vol(\xi_1 \wedge ... \wedge \xi_n)$$

with equality only if Q is symmetric, that is, if ξ is Lagrangian. Hence

$$\Phi_c(\xi_1 \wedge ... \wedge \xi_n) \geq Vol(\xi_1 \wedge ... \wedge \xi_n)$$

with equality if and only if $\det(Q) = c^2$ and $Q = Q^T$. $\square$

We prove a more general result than Theorem 10.

Theorem 11. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded region, and $u \in C^2(\Omega)$ is a convex solution to the Monge-Ampère equation*

$$\det(D^2 u) = c^2.$$

Then in the pseudo-Euclidean space $(\mathbb{R}^n_x \times \mathbb{R}^n_y, dxdy)$, the gradient graph of $u(x)$, $\Gamma = (x, \nabla u(x))$, is volume maximizing in the following sense:

Let Σ be an oriented n-surface which is homologous to Γ in $\mathbb{R}^n \times \mathbb{R}^n$, with $\partial\Sigma = \partial\Gamma = \Gamma \cap \{x \in \partial\Omega\}$. If any of the following hold

- *$\partial\Omega$ is connected and Σ is C^1 and space-like,*

- *Σ is connected, C^1 and space-like, or*

- *Σ is C^1 and space-like except on a set Σ_0, which has n-dimensional Hausdorff measure zero, such that $\Sigma \backslash \Sigma_0$ is connected,*

then $Vol(\Gamma) \geq Vol(\Sigma)$. If Ω is simply connected with C^1 connected boundary, and Σ is C^1, then equality holds only if $\Sigma = \Gamma$.

Proof of Theorem 11, Theorem 10. Suppose $u \in C^2(\Omega)$ with $\det(D^2u) = c^2$ on Ω. Then

$$\Phi_c(\partial_1 \wedge ... \wedge \partial_n) = \frac{1}{2}[c + \frac{1}{c}c^2] = c,$$

$$Vol(\partial_1 \wedge ... \wedge \partial_n) = \sqrt{\det(D^2u)} = c.$$

So

$$Vol|_\Gamma = \Phi_c|_\Gamma,$$

and Γ is a calibrated submanifold. For any homologous n-surface Σ, we know that

$$\int_\Sigma \Phi_c = \int_\Gamma \Phi_c = Vol(\Gamma),$$

so our task is to show that each of the listed conditions imply that the oriented planes for Σ are oriented with respect to Φ_c, giving

$$\Phi_c|_\Sigma \geq Vol|_\Sigma.$$

We begin by showing that the first condition implies the second condition. Regarded as a linear map, $g_0 \in GL(2n, \mathbb{R})$ has two eigenvalues, $+1$ and -1, so we can decompose $\mathbb{R}^{2n}$ into the eigenspaces corresponding to these two eigenvalues. Let P^+ be the the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto the eigenspace corresponding to the eigenvalue $+1$. If Σ is C^1 and space-like, the projection $P^+|_\Sigma \to \mathbb{R}^n_+$ must be full rank, so must be an open map. It follows that each component of Σ must have non-empty boundary. Since $\partial\Sigma = \partial\Omega$, $\partial\Sigma$ must be connected. Each component of Σ intersects $\partial\Sigma$, so Σ consists of a single component.

It is clear that the second condition implies the third. So now assume Σ is C^1 and space-like except on a set Σ_0, which has n-dimensional Hausdorff measure zero, and $\Sigma\backslash\Sigma_0$ is connected. The integral $\int_\Sigma \Phi_c$ is positive, so for the induced orientation on Σ, $\Phi_c(\xi)$ is positive for some oriented tangent plane $\xi_s = T_s\Sigma$, at some point $s \in \Sigma$. Since Vol does not vanish on $\Sigma\backslash\Sigma_0$, it follows from the C^1 assumption that $\Phi_c(\xi_s) \geq Vol(\xi_s) > 0$ for all $s \in \Sigma\backslash\Sigma_0$. Hence

$$Vol(\Sigma) = \int_\Sigma dVol \leq \int_\Sigma \Phi_c = Vol(\Gamma).$$

□

Uniqueness. It is clear by Proposition 2.2.1 that equality will only occur if Σ is a special Lagrangian surface, locally described by gradient graphs of $\det D^2u = c^2$. In order to use the comparison principle, we first show that Σ is globally described by a single graph over Ω.

Let P^x be the the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n_x$, and let $\Omega_0 = P^x(\Sigma)$. We observe that, due to the space-like condition, the projection P^x is open on the interior of Σ, so $\partial\Omega_0 \subset P^x(\partial\Sigma) = \partial\Omega$.

Let $p = (x_0, \nabla u(x_0)) \in \partial\Omega = \partial\Sigma$, for x_0 an extreme point of Ω. From the open condition on the map P^x on Σ, it follows that x_0 is also an extreme point for Ω_0, and that the inward pointing normals for $\partial\Omega$ and $\partial\Omega_0$ must agree at x_0. The regions Ω_0 and Ω then must intersect on a non-trivial open set near x_0. Using the openness of P^x, together with the boundary condition $\partial\Omega_0 \subset \partial\Omega$, it is then easy to check that $\Omega_0 \cap \Omega$ is relatively closed and relatively open as a subset of Ω, hence $\Omega \subset \Omega_0$.

Take a cover of Σ° by open sets Σ_i, (open with respect to Σ) where each is a gradient graph $\Sigma_i = \{(x, \nabla v_i) | x \in U_i\}$, for v_i solutions to $\det D^2 v_i = c^2$ on U_i. Let $\hat{U}_1 = U_1$, then recursively define $\hat{U}_i = U_i\backslash\hat{U}_{i-1}$, and define $\hat{\Sigma}_i = \{(x, \nabla v_i) | x \in \hat{U}_i\}$. The disjoint union $\bigcup_i \hat{U}_i$ is then an open set that contains Ω, and the disjoint union $\bigcup_i \hat{\Sigma}_i$ is a subset of Σ.

Suppose that $\bigcup_i \hat{\Sigma}_i \neq \Sigma^\circ$. Then there exists an open subset $U_{jk} \subset U_j \cap U_k$, with $j < k$, such that Σ_j and Σ_k are both graphs over U_{jk}, but are disjoint. Then Σ must contain the disjoint union

$$\Sigma' = \bigcup_i \{(x, \nabla v_i) | x \in \hat{U}_i\} \bigcup \{(x, \nabla v_k) | x \in U_{jk}\}.$$

However, integrating over Σ gives

$$\int_{\Sigma} dVol \geq \int_{\Sigma'} dVol \geq \int_{\Omega} c\,dx + \int_{U_{jk}} c\,dx > c|\Omega| = Vol(\Gamma)$$

contradicting the inequality in the conclusion of Theorem 11. We conclude that $\Sigma^\circ = \bigcup_i \hat{\Sigma}_i$ and $P(\Sigma^\circ) = \Omega$. The gradients ∇v_j and ∇v_k must agree on all overlaps $U_j \cap U_k$, so we can extend any of the $\mathbb{R}^n$-valued functions ∇v_j to an $\mathbb{R}^n$-valued function F on all of Ω. We have assumed that Ω is simply connected, so $F = \nabla v$, for some v satisfying $\det D^2 v = c^2$, on all of Ω. To compare u with v, we note that $\nabla u = \nabla v$ is already fixed around the boundary of Ω, which is connected, so we may integrate u and v around the boundary and conclude that u and v differ by a constant. Applying the comparison principle for nonlinear elliptic equations (cf. [16], Theorem 17.1) gives uniqueness. □

Counterexamples. We give some examples to show that the inequality in Theorem 11 fails, if we do not assume any of the conditions on Σ. For small ϵ, let $\Omega \subset \mathbb{R}^2$ be the annulus

$$\{x = (x_1, x_2) \mid |x|^2 \in [1, 1+\epsilon]\}$$

and

$$\Gamma = \{(x_1, x_2, x_1, x_2) | x \in \Omega\} \subset \mathbb{R}^4$$

the gradient graph of $|x|^2/2$. Let $\Sigma = \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \{(x_1, x_2, x_1, x_2) \,|\, x \in \overline{B_1}\}$$

and

$$\Sigma_2 = \{(x_1, x_2, x_1 + \eta(x), x_2 + \eta(x)) \,|\, x \in \overline{B_{1+\epsilon}}\}$$

where $\eta(x)$ is a small function which is positive on the interior of $B_{1+\epsilon}$, and vanishes on $\partial B_{1+\epsilon}$. With a suitable orientation, the disconnected set Σ is homologous to Γ. We see that $Vol(\Sigma)$ is very close to 2π, whereas $Vol(\Gamma)$ is very close to 0, as we have chosen ϵ and η small.

To obtain a connected Σ for which Theorem 11 fails, we alter the previous example slightly. With Ω as above, define a small "bridge" region

$$\Omega' = \Omega \cap \{x_1 > 0\} \cap \{x_2 \in [-\epsilon, \epsilon]\}.$$

Let Γ be the gradient graph of $|x|^2/2$ over $\Omega\backslash\Omega'$, let Σ_1 be the gradient graph of $|x|^2/2$ over $B_1 \cup \Omega'$, and let Σ_2 be as above. Then $\Sigma = \Sigma_1 \cup \Sigma_2$ is connected and homologous to Γ, and is space-like except on a singular set which is one-dimensional, but nonetheless disconnects Σ. Again we have $Vol(\Sigma)$ close to 2π and $Vol(\Gamma)$ close to 0, so the inequality in Theorem 11 does not hold.

2.3 From the Viewpoint of Lie Groups

One can also approach the theory of special Lagrangian calibrations for pseudo-Eulidean space on the level of homogeneous spaces, following the approach of Harvey and Lawson. We use some notation and terminology of [12]. First, noting that $U(n) = O(2n) \cap Sp(2n)$, we consider the "para-unitary" group $U^\pi = O_{g_0}(n,n) \cap Sp(2n)$, where

$$O_{g_0}(n,n) = \{A \in M(2n,\mathbb{R}) | (Au, Av)_{g_0} = (u,v)_{g_0}, \forall u, v \in \mathbb{R}^n \times \mathbb{R}^n\}.$$

Some standard Lie algebra computations will show that (Proposition 4 of [12])

$$U^\pi = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} | AB^T = I \in M(n,\mathbb{R})\}.$$

Next, consider the space L^+ of all space-like Lagrangian planes. Given $\xi \in L^+$, choose an oriented basis, $\xi_1, ..., \xi_n$, with each $\xi_i \in \mathbb{R}^n \times \mathbb{R}^n$, which is orthonormal with respect to g_0, and let (A, B) be the $n \times 2n$ matrix with rows ξ_i. Then associate to each $\xi \in L^+$ the element

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in U^\pi,$$

which is unique up to a factor of $SO(n)$, where the $SO(n)$ action on U^π is defined by

$$S \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} SA & 0 \\ 0 & SB \end{pmatrix}.$$

This gives the set of space-like Lagrangian planes L^+ the structure of a homogeneous space $L^+ \approx U^\pi/SO(n)$. The homomorphism $\Theta^+ : U^\pi \to O_{g_0}(1,1)$ given by

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} \det(A) & 0 \\ 0 & \det(B) \end{pmatrix}$$

descends to a map $L^+ \to O_{g_0}(1,1) \approx H^1$ where H^1 is the pseudo-circle $\{(s,t) | \, t > 0, \; st = 1\}$. A special Lagrangian submanifold is one whose tangent planes lie in a single fiber of Θ^+. We can see then that the calibrations are

$$\Phi_c(\xi) = \frac{1}{2}(c \det A + \frac{1}{c} \det B) \geq Vol(\xi)$$

and the special Lagrangian equations are $\det D^2 u = c^2 > 0$, where the value c^2 is the pseudo-phase analogous to the phase c in section 2.1.

2.4 A Family of Nonlinear Equations

By taking linear combinations of the metrics δ_0 and $g_0 = 2dxdy$, we obtain a family of metrics on $\mathbb{R}^n \times \mathbb{R}^n$

$$g_t = \cos t \, g_0 + \sin t \, \delta_0 \tag{2.3}$$

We find that the extremal Lagrangian surfaces in $(\mathbb{R}^n \times \mathbb{R}^n, g_t)$ arise as solutions to a family of special Lagrangian equations

$$For \ t = 0 \quad \sum_i \ln \lambda_i = c \tag{2.4}$$

$$For \ t \in (0, \frac{\pi}{4}) \quad \sum_i \ln(\frac{\lambda_i + a - b}{\lambda_i + a + b}) = c \tag{2.5}$$

$$For \ t = \frac{\pi}{4} \quad \sum_i \frac{1}{1 + \lambda_i} = c \tag{2.6}$$

$$For \ t \in (\frac{\pi}{4}, \frac{\pi}{2}) \quad \sum_i \arctan(\frac{\lambda_i + a - b}{\lambda_i + a + b}) = c \tag{2.7}$$

$$For \ t = \frac{\pi}{2} \quad \sum_i \arctan \lambda_i = c \tag{2.8}$$

where $a = \cot t$ and $b = \sqrt{|\cot^2 t - 1|}$. Further, we have the following extremal volume property of special Lagrangian graphs in $(\mathbb{R}^n \times \mathbb{R}^n, g_t)$

Theorem 12. *i) Suppose $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, is a solution to (2.5). If the metric g_t restricts to a positive definite metric on $\Gamma = (x, \nabla u(x))$, then Γ is volume maximizing among homologous, C^1, space-like n-surfaces in $(\mathbb{R}^n \times \mathbb{R}^n, g_t)$, as in Theorem 10.*

ii) Suppose $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, is a solution to (2.6). Then the volume of $\Gamma = (x, \nabla u(x))$ is equal to the volume of any homologous, C^1, space-like n-surface in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\frac{\pi}{4}})$.

iii) Suppose $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, is a solution to (2.7). Then $\Gamma = (x, \nabla u(x))$ is absolutely volume minimizing among all homologous n-surfaces in $(\mathbb{R}^n \times \mathbb{R}^n, g_t)$.

Using a change of variables, we may restate the following Bernstein-type results of Jörgens-Calabi-Pogorelov, Flanders, and Yuan.

Bernstein Theorem. *i)*(Jörgens [24], Calabi [9], Pogorelov [37]) *Suppose $u \in C^2(\mathbb{R}^n)$, $D^2u \geq -\cot t$, and $\Gamma = (x, \nabla u(x))$ is a maximal space-like surface in $(\mathbb{R}^n \times \mathbb{R}^n, g_t)$. Then $u(x)$ is a quadratic polynomial.*

ii)(Flanders [14]) *Suppose $u \in C^2(\mathbb{R}^n)$ is a convex solution to (2.6). Then $u(x)$ is a quadratic polynomial.*

iii)(Yuan [56] [58]) *There is a value C_t such that if $u \in C^2(\mathbb{R}^n)$ is a solution to (2.7), with either*

a) $D^2u \geq -\cot t$ or

b) $c > C_t + \frac{(n-2)\pi}{2}$,

then $u(x)$ is a quadratic polynomial.

For $t \in [0, \frac{\pi}{2}]$, let $M_t = (\mathbb{R}^n \times \mathbb{R}^n, g_t)$, with g_t as defined in (2.3). For $t < \frac{\pi}{4}$, M_t is a pseudo-Euclidean space of index n. For $t > \frac{\pi}{4}$, M_t is a Euclidean space. For $t = \frac{\pi}{4}$, M_t carries a degenerate metric of rank n. In any case, an extremal (minimal or maximal) submanifold Σ satisfies $H = 0$ along Σ, where H is the mean curvature vector along Σ.

Lemma 2.4.1. *Suppose $u \in C^3(\Omega)$ and $\Gamma = (x, \nabla u(x))$ defines an extremal surface in M_t, $t \neq \pi/4$. If D^2u is diagonalized at a point $p = (x_0, \nabla(x_0))$, the mean curvature satisfies*

$$H = 0 \stackrel{p}{=} \sum_{i,k} \left(\frac{u_{kii}}{\sin t(1 + \lambda_i^2) + 2\cos t \lambda_i} \partial_{y_k}\right)^N.$$

where ^{N} denotes orthogonal projection onto $N(\Gamma)$, the normal bundle of Γ.

Proof. For an embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$f : \Omega \to \Gamma \hookrightarrow M_t,$$

we can compute the mean curvature vector H by

$$H = (g^{ij} f_{ij})^N$$

where g is the induced metric on Γ, and $g^{ij} = (g^{-1})_{ij}$. In this case,

$$f(x) = (x_1, ..., x_n, u_1(x), ...u_n(x)),$$

so

$$H = (g^{ij}(0, 0, ...0, u_{1ij}, ...u_{nij}))^N.$$

When $D^2u = \mathrm{diag}(\lambda_1, ..., \lambda_n)$,

$$H = \sum_{i,k} (\frac{u_{kii}}{\sin t(1+\lambda_i^2) + 2\cos t\lambda_i} \partial_{y_k})^N = 0,$$

and $(\partial_{y_k})^N$ is a linear combination of ∂_{x_k} and ∂_{y_k}. For $k = 1, ..., n$ the vectors $(\partial_{y_k})^N$ are independent and form a basis for the normal space at p. It follows that

$$\sum_i \frac{u_{kii}}{\sin t(1+\lambda_i^2) + 2\cos t\lambda_i} = 0$$

for all k. □

We now define the nonlinear operators $F^t(D^2u)$ by the equations (2.5) - (2.7).

At a point $p = (x_0, \nabla u(x_0))$, we may diagonalize D^2u and differentiate $F^t(D^2u)$. If the eigenvalues λ_i are all distinct, then each is differentiable, and we have

$$\frac{\partial}{\partial x_k}\lambda_i \overset{p}{=} \frac{\partial}{\partial x_k} u_{ii} = u_{iik}$$

and

$$\frac{\partial}{\partial x_k} F^t(D^2u(x)) = \sum_i \frac{1}{\sin t(1+\lambda_i^2) + 2\cos t\lambda_i} \frac{\partial}{\partial x_k}\lambda_i$$

$$\overset{p}{=} \sum_i \frac{u_{kii}}{\sin t(1+\lambda_i^2) + 2\cos t\lambda_i}. \tag{2.9}$$

If the eigenvalues are not distinct, then one can use an eigenspace projection argument to verify that

$$\frac{\partial}{\partial x_k} \sum_{i \in I_j} \lambda_i \overset{p}{=} \sum_{i \in I_j} u_{iik}$$

where $I_j = \{i \in 1, ..., n | \lambda_i \stackrel{p}{=} \lambda_j\}$, so that (2.9) holds in this case as well. It follows that the mean curvature H vanishes if and only if $\nabla F^t(D^2u) = 0$, that is, if and only if $F^t(D^2u)$ is constant.

For this family of nonlinear equations,

$$\frac{\partial F^t}{\partial \lambda_i} = g^{ii}$$

so we see that solutions are elliptic precisely when the Lagrangian submanifold is space-like. If we differentiate a second time with respect to λ_i, we see that the equation is convex when all $\lambda_i < -a$, and concave when all $\lambda_i > -a$. We note that as $t \to \frac{\pi}{4}$, equation (2.5) with $c = n\ln(a-b)$ becomes (2.6) with $c = \frac{n}{2}$. Equation (2.6) has been studied by Flanders [14], who obtained Bernstein Theorem ii).

2.4.1 Calibrations for M_t

Gradient graphs for solutions to (2.5) and (2.7) give rise to calibrated submanifolds of M_t, that are in fact isometric to calibrated submanifolds of M_0 and $M_{\frac{\pi}{2}}$. The calibrations for M_t may be obtained by pulling back the calibrations on M_0 and $M_{\frac{\pi}{2}}$ via an isometry.

Throughout this section we will again be using the constants $a = \cot t$ and $b = \sqrt{|\cot^2 t - 1|}$, for $t \in [0, \frac{\pi}{2}]$, as well as the constants defined by

$$\sigma = \frac{\sqrt{\cos t + \sin t} + \sqrt{|\cos t - \sin t|}}{2}$$

$$\tau = \frac{\sqrt{\cos t + \sin t} - \sqrt{|\cos t - \sin t|}}{2}.$$

We start with the pseudo-Euclidean metrics. For $t < \frac{\pi}{4}$, the map $\varphi_t : M_t \to M_0$, represented by the $2n \times 2n$ matrix

$$\varphi_t = \begin{pmatrix} \sigma I & \tau I \\ \tau I & \sigma I \end{pmatrix}$$

is an isometry (up to a constant factor). The Lagrangian condition is preserved under φ_t, so this isometry maps Lagrangian n-surfaces to Lagrangian n-surfaces. Pulling back the calibrations on M_0 described in section 2.2, M_t becomes a calibrated manifold. The isometry gives an equivalence between the homogeneous space structure for the space-like Lagrangian planes of M_t and the homogeneous space structure for the space-like Lagrangian planes of M_0 presented in section 3.

Proof of Theorem 12 i). Suppose $u(x)$ is a solution to (2.5). Let $\Gamma = (x, \nabla u(x)) \subset M_t$ be the graph of ∇u. Our goal is to show that Γ is isometric to a special Lagrangian graph $\hat{\Gamma} \subset M_0$. Take $\hat{\Gamma} = \varphi_t(\Gamma) \subset M_0$. At a point $p = (x_0, \nabla(x_0)) \in \Gamma$, the tangent space of Γ can be described by the span of the vectors

$$\partial_i = \partial_{x_i} + \sum_j u_{ij} \partial_{y_j}, \quad i = 1, ..., n.$$

Take $D^2 u$ to be diagonalized at p, and push forward. The tangent space $T_{\varphi_t(p)} \hat{\Gamma} = (\varphi_t)_*(T_p \Gamma)$ is the span of

$$\hat{\partial}_i = (\varphi_t)_* \partial_i = (\sigma + \tau \lambda_i) \partial_{x_i} + (\tau + \sigma \lambda_i) \partial_{y_i}.$$

The space-like condition on Γ imposes restrictions on the values of λ_i, particularly, $\lambda_i \neq -\sigma/\tau$, so we may multiply each $\hat{\partial}_i$ by $1/(\sigma + \tau\lambda_i)$ and see that $T_{\varphi_t(p)} \hat{\Gamma}$ is spanned by

$$\hat{\partial}_{x_i} + \frac{\tau + \sigma\lambda_i}{\sigma + \tau\lambda_i} \hat{\partial}_{y_i}.$$

The image $\hat{\Gamma}$ is a Lagrangian submanifold of M_0, so arises locally as gradient graph $\hat{\Gamma} = (\hat{x}, \nabla \hat{u}(\hat{x}))$. From the above expression, the eigenvalues of $D^2 \hat{u}$ are given by

$$\hat{\lambda}_i = \frac{\tau + \sigma\lambda_i}{\sigma + \tau\lambda_i} = \frac{(\lambda_i + \tau/\sigma)}{(\lambda_i + \sigma/\tau)} \frac{(\sigma)}{(\tau)}. \tag{2.10}$$

The λ_i's satisfy (2.5), so noting that $a + b = 1/(a - b) = \sigma/\tau$, we have

$$\prod_{i=1}^{n} \frac{\lambda_i + \tau/\sigma}{\lambda_i + \sigma/\tau} = e^c > 0\,,$$

and may conclude that $\hat{u}(\hat{x})$ satisfies

$$\prod_{i=1}^{n} \hat{\lambda}_i = [\frac{\sigma}{\tau}]^n \, e^c > 0,$$

that is, $\hat{u}(\hat{x})$ satisfies the Monge-Ampère equation (2.1). It follows that $\hat{\Gamma}$ is a calibrated submanifold. The property of being calibrated is local and is preserved under isometries of the ambient manifolds, so Γ is calibrated. Theorem 12 i) then follows by the same reasoning as in the proof of Theorem 11. $\square$

The function $\hat{u}$ obtained above is only local. In order to study solutions to (2.5) further, in particular to obtain uniqueness for Theorem 12 i) and the Bernstein-type result, we transform u into a solution to (2.1) which describes $\hat{\Gamma}$ globally, when possible.

The isometry φ_t acts via

$$\varphi_t(x, y) = (\sigma x + \tau y, \tau x + \sigma y) = (p(x, y), q(x, y))$$

so the image $\hat{\Gamma} = \varphi_t(\Gamma)$ lies in the set $p(\Gamma) \times q(\Gamma) \subset \mathbb{R}^n \times \mathbb{R}^n$. As Γ is parameterized by Ω, the isometry $\Gamma \to \hat{\Gamma}$ locally amounts to a change of coordinates $\Omega \to p(\Gamma)$, wherever Dp is invertible. If $p|_\Gamma$ is a bijection, then $\hat{\Gamma}$ is globally parameterized by $p(\Gamma)$. In the case where $p|_\Gamma$ is a bijection, $\hat{\Gamma}$ is a graph over $p(\Gamma)$ of the function

$$r = q \circ p^{-1} : p(\Gamma) \to \mathbb{R}^n.$$

Using Ω as coordinates for Γ

$$p(x) = \sigma x + \tau \nabla u(x)$$
$$q(x) = \tau x + \sigma \nabla u(x)$$

hence

$$Dp = \sigma I + \tau D^2 u$$
$$Dq = \tau I + \sigma D^2 u$$

so p is locally invertible if and only if no eigenvalues of $D^2 u$ attain the value $-\sigma/\tau = -(a+b)$, which is precluded by the space-like condition. Diagonalizing $D^2 u$, we see that Dr is symmetric, so $r = \nabla \hat{u}$ for some $\hat{u}$, and the eigenvalues of $D^2 \hat{u}$ become

$$\hat{\lambda}_i = \frac{\tau + \sigma \lambda_i}{\sigma + \tau \lambda_i}$$

as in (2.10). Now if $D^2 u \geq -a > -(a+b)$, then $Dp > 0$ and p is injective on Ω. The function $\hat{u}$ is then a solution to the Monge-Ampère equation defined on all of $p(\Gamma)$. Further, the inequality is uniform, $Dp \geq \epsilon > 0$. Thus if $\Omega = \mathbb{R}^n$, p is a bijection on $\mathbb{R}^n$, so $\hat{u}$ will then be a convex solution to (2.1) on all of $\mathbb{R}^n$. Bernstein Theorem i) follows from the famous result

Theorem (Jörgens [24], Calabi [9], Pogorelov [37]). *Any convex solution to (2.1) on all of $\mathbb{R}^n$ is a quadratic polynomial.*

Using a recent, more general result, we see that we may drop the restriction that $D^2 \geq -a$.

Theorem (Jost-Xin, ([25] Theorem 4.2)). *Let M be a space-like extremal m-surface in $\mathbb{R}^{n+m}_n$. If M is closed with respect to the Euclidean topology, then M must be a linear subspace.*

Uniqueness. From the above discussion, if we assume that $D^2u \geq -a$, the surface Γ can be described globally as the gradient graph of a solution to the Monge-Ampère equation. In this case, that Γ is the unique surface with this volume follows from Theorem 11. The proof of Theorem 11 relied heavily on the fact that the space-like condition forced the projection $\Gamma \to \mathbb{R}^n_x$ to be an open map. This is not the case for gradient graphs of solutions to equation (2.5), so the corresponding proof of uniqueness will not work. We do not have a proof of uniqueness for equation (2.5) with no restrictions on D^2u, nor for the analogous result for solutions to (2.7). As in ([19], Theorem 5.8), we do have uniqueness for solutions to (2.5) and (2.7) whenever the boundary data is analytic, as an application of the Cauchy-Kowaleswki Theorem.

Proof of Theorem 12 ii) Let $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the map

$$(x, y) \mapsto \frac{x+y}{2}.$$

Given the degenerate metric $g_{\frac{\pi}{4}}$ on $\mathbb{R}^n \times \mathbb{R}^n$, this map is an "isometry" in the sense that $g_{\frac{\pi}{4}} = P^*\delta_0$. The graph $\Gamma = (x, F(x))$ is isometric to $P(\Gamma)$ for any $F(x)$ such that DF avoids the eigenvalue $\lambda_i = -1$. Let Σ be any space-like surface with boundary $\partial\Sigma = \Gamma \cap \{x \in \Omega\}$. Then if $\omega = x_n dx_1 \wedge ... \wedge dx_{n-1}$, so that $dw = dVol$ on $\mathbb{R}^n$,

$$Vol(\Sigma) = \int_\Sigma dVol = \int_\Sigma P^* dVol = \int_\Sigma P^* dw =$$

$$\int_\Sigma dP^* w = \int_{\partial\Sigma} P^* w$$

which depends only on the boundary $\partial\Gamma = \partial\Sigma$. □

Proof of Theorem 12 iii). As in the pseudo-Euclidean case, for $t > \frac{\pi}{4}$, the map $\varphi_t : M_t \to M_{\frac{\pi}{2}}$, is an isometry. For a given n-plane ξ,

$$Vol^2_{g_t}(\xi) = Vol^2_{\delta_0}(\varphi\xi) \geq Vol(\varphi\xi \wedge J\varphi\xi)$$

$$= \det_{\mathbb{R}} A = ||\det_{\mathbb{C}} A||^2 = \alpha^2(\xi) + \beta^2(\xi) \geq \alpha^2(\xi),$$

where $A \in M(n, \mathbb{C}) \subset M(2n, \mathbb{R})$ is the map sending $\partial_{x_i} \mapsto \varphi\xi_i$, extended by complex linearity as before. By Hadamard's Inequality, we have equality if and only if ξ is Lagrangian and $\beta(\xi) = 0$.

If ξ is Lagrangian, then $\varphi\xi_1, ..., \varphi\xi_n$ is an orthonormal basis for $\mathbb{C}^n$, so $A \in U(n)$. Using the map

$$\Theta : L \approx U(n)/SO(n) \stackrel{\det_{\mathbb{C}}}{\to} S^1.$$

the special Lagrangian equations (2.7) can be deduced from

$$\arg\det(\sigma + \tau D^2 u + \sqrt{-1}(\tau + \sigma D^2 u)) = \sum \arctan(\frac{\tau + \sigma\lambda_i}{\sigma + \tau\lambda_i}) = c.$$

If follows that any gradient graph $\Gamma = (x, \nabla u(x))$ for $u(x)$ satisfying this equation is calibrated, and therefore an absolutely volume minimizing submanifold. □

By differentiating with respect to λ, one can verify the identity

$$\sum \arctan\frac{\lambda_i + a}{b} + C_t = \sum \arctan(\frac{\tau + \sigma\lambda_i}{\sigma + \tau\lambda_i})$$

where

$$C_t = \arctan(\frac{\tau}{\sigma}) - \arctan\frac{a}{b}.$$

It follows that a solution to (2.7) also satisfies

$$\sum \arctan\frac{\lambda_i + a}{b} = c - C_t,$$

and the function

$$v(x) = \frac{u(x)}{b} + \frac{a}{2b}|x|^2$$

is a solution to (1.1). If $v(x)$ is convex, which is the case if $D^2 u \geq -a$, we may conclude that $v(x)$ is a quadratic polynomial by applying the following result.

Theorem (Yuan [56]). *Suppose $u \in C^2(\mathbb{R}^n)$ is a convex solution to (1.1). Then $u(x)$ is a quadratic polynomial.*

Similarly, any solution to (2.7) with $c > (n-2)\pi/2 + C_t$ is a quadratic polynomial, by the theorem

Theorem (Yuan[58]). *Suppose $u \in C^2(\mathbb{R}^n)$ is a solution to (1.1) with $c > \frac{(n-2)\pi}{2}$. Then $u(x)$ is a quadratic polynomial.*

2.4.2 Example

The equations (2.5)-(2.7) can be manipulated to take the form

$$f(\lambda_1, ..., \lambda_n) = 0,$$

where f is a polynomial of degree no higher than n. For the equation (2.5) with $c = 0$, the polynomial f has degree $n-1$. When $n = 2$, (2.5) with $c = 0$ becomes a linear equation, taking the form $\Delta u = -2a$. We exploit this degeneracy in order to write down an explicit solution to (2.5).

Fix some $t \in (0, \frac{\pi}{4})$, and define $u(x_1, x_2)$ on $\Omega = \{x_1 > 0 \subset \mathbb{R}^2\}$

$$u(x_1, x_2) = -\frac{a}{2}(x_1^2 + x_2^2) + ke^{x_1}\cos x_2,$$

for k some large constant, and $a = \cot t$. The eigenvalues of D^2u are

$$\lambda_1 = -a - ke^{x_1},$$

$$\lambda_2 = -a + ke^{x_1},$$

Clearly, u is a solution to (2.5) with $c = 0$, that is, $\Delta u = -2a$, and one can check that the resulting surface $\Gamma = (x, \nabla u(x))$ is space-like in $(\mathbb{R}^2 \times \mathbb{R}^2, g_t)$. We transform this to a maximal submanifold of $(\mathbb{R}^n \times \mathbb{R}^n, g_0)$, using φ_t. This isometry acts via

$$\varphi_t(x, y) = (\sigma x + \tau y, \tau x + \sigma y)$$

so the image $\varphi_t(\Gamma) = \hat{\Gamma} \subset (\mathbb{R}^4, g_0)$ lies in $p(\Gamma)\times\mathbb{R}^2$, where $p(x,y) = \sigma x + \tau y$. Parameterizing Γ by Ω, the map $p|_\Gamma$ takes the form

$$p(x_1, x_2) = \sigma \cdot (x_1, x_2) + \tau \cdot \nabla u(x_1, x_2),$$

$$\text{or } p(z) = (\sigma - \tau a)z + \tau k e^{\bar{z}},$$

where $z = x_1 + i x_2$. For very large k, a simple volume computation over a vertical strip shows that p can not be injective on Γ, so $\hat{\Gamma}$ will not be described globally as a graph over $p(\Gamma)$. This surface $\hat{\Gamma}$ is a maximal Lagrangian surface which is globally described as a solution to (2.5) and locally described as a solution to (2.1), but which is not globally described as a solution to (2.1). This surface is not complete, as we have restricted the domain to a half-plane, but by the result of Jost and Xin ([25] Theorem 4.1) one would not expect to find a complete surface with this property.

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Vita

Micah was born in Fort Collins, Colorado, but moved to Seattle at a young age. He has remained in the Puget Sound area since. He was homeschooled in elementary school, attended Heritage Christian School in Bothell for junior high, and then Lake Washington High School in Kirkland. He accepted a music scholarship from Pacific Lutheran University in Tacoma, where he went on to earn degrees in Math and Physics, before coming to the University of Washington. He never once has been skiing. He enjoys a competitive game of basketball as well as playing music on trumpet or guitar.

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