NON-POLYNOMIAL ENTIRE SOLUTIONS TO σ_k EQUATIONS

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ABSTRACT. For $2k = n+1$, we exhibit non-polynomial solutions to the Hessian equation

$$\sigma_k(D^2u) = 1$$

on all of $\mathbb{R}^n$.

1. INTRODUCTION

In this note, we demonstrate the following.

Theorem 1. *For*

$$n \geq 2k - 1,$$

there exist non-polynomial elliptic entire solutions to the equation

$$\sigma_k\left(D^2u\right) = 1 \tag{1}$$

on $\mathbb{R}^n$.

Corollary 2. *For all* $n \geq 3$, *there exist on* $\mathbb{R}^n$ *non-polynomial entire solutions to*

$$\sigma_2\left(D^2u\right) = 1. \tag{2}$$

For $k = 1$, the entire harmonic functions in the plane arising as real parts of analytic functions are classically known. For $k = n$, the famous Bernstein result of Jörgens [5], Calabi [1], and Pogorelov [6] states that all entire solutions to the Monge-Ampère equation are quadratic. Chang and Yuan [2] have shown that any entire convex solution to (2) in any dimension must be quadratic. To the best of our knowledge, for $1 < k < n$, the examples presented here are the first known non-trivial entire solutions to σ_k equations.

The special Lagrangian equation is the following

$$\sum_{i=1}^{n} \arctan \lambda_i = \theta \tag{3}$$

(here λ_i are eigenvalues of D^2u) for

$$\theta \in \left(-\frac{n}{2}\pi, \frac{n}{2}\pi\right)$$

a constant. Fu [3] showed that when $n = 2$ and $\theta \neq 0$ all solutions are quadratic. When $n = 2$ and $\theta = 0$ the equation (3) becomes simply the Laplace equation, which admits well-known non-polynomial solutions. Yuan [8] showed that all convex solutions to special Lagrangian equations are quadratic.

The critical phase for special Lagrangian equations is

$$\theta = \frac{n-2}{2}\pi.$$

The author's work is supported in part by the NSF via DMS-1161498.

Yuan [9] has shown that for values above the critical phase, all entire solutions are quadratic. On the other hand, by adding a quadratic to a harmonic function, one can construct nontrivial entire solutions for phases

$$|\theta| < \frac{n-2}{2}\pi.$$

By [4] when $n = 3$, the critical equation

$$\sum_{i=1}^{3} \arctan \lambda_i = \frac{\pi}{2}$$

is equivalent to the equation (2). Thus Corollary 2 answers the critical phase Bernstein question when $n = 3$. In the process, we also show the following.

Theorem 3. *There exists a special Lagrangian graph in $\mathbb{C}^3$ over $\mathbb{R}^3$ that does not graphically split.*

Harvey and Lawson [4], show that a graph

$$(x, \nabla u(x)) \subset \mathbb{C}^n$$

is special Lagrangian and a minimizing surface if and only if u satisfies (3). We say a graph splits graphically when the function u can be written the sum of two functions in independent variables.

There are still many holes in the Bernstein picture for σ_k equations. To begin with, when $n = 4$ the existence of interesting solutions to $\sigma_3 = 1$. For special Lagrangian equations the existence of critical phase solutions when $n \geq 4$ is open.

2. Proof

We will assume that n is odd and

$$2k = n + 1.$$

We construct a solution u on $\mathbb{R}^n$. The general result will follow by noting that if we define

$$\tilde{u} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

via

$$\tilde{u}(z, w) = u(z)$$

then

$$\sigma_k \left(D^2 \tilde{u}\right) = \sigma_k \left(D^2 u\right) = 1.$$

Consider functions on $\mathbb{R}^{n-1} \times \mathbb{R}$ of the form

$$u(x, t) = r^2 e^t + h(t)$$

where

$$r = \left(x_1^2 + x_2^2 + ... + x_{n-1}^2\right)^{1/2}.$$

Compute the Hessian, rotating $\mathbb{R}^{n-1}$ so that $x_1 = r$:

$$D^2u = \begin{pmatrix} 2e^t & 0 & ... & 0 & 2re^t \\ 0 & 2e^t & 0 & ... & 0 \\ ... & 0 & ... & 0 & ... \\ 0 & ... & 0 & 2e^t & 0 \\ 2re^t & 0 & ... & 0 & r^2e^t + h''(t) \end{pmatrix}$$

$$= e^t \begin{pmatrix} 2 & 0 & ... & 0 & 2r \\ 0 & 2 & 0 & ... & 0 \\ ... & 0 & ... & 0 & ... \\ 0 & ... & 0 & 2 & 0 \\ 2r & 0 & ... & 0 & r^2 + e^{-t}h''(t) \end{pmatrix}. \tag{4}$$

We then compute. The k-th symmetric polyomial is given by the sum of k-minors. Let

$$S = \{\alpha \subset \{1, ..., n\} : |\alpha| = k\}, \tag{5}$$

and let

$$A = \{\alpha \in S : 1 \in \alpha\}$$
$$B = \{\alpha \in S : n \in \alpha\}.$$

We express S as a disjoint union

$$S = (A \cap B) \cup (B \backslash A) \cup (S \backslash B).$$

Define

$$\sigma_k^{(\alpha)} = \det \begin{pmatrix} k \times k \text{ matrix with} \\ \text{row and columns} \\ \text{chosen from } \alpha \end{pmatrix}.$$

For $\alpha \in (A \cap B)$ we have

$$\sigma_k^{(\alpha)} = \det \left(e^t \begin{pmatrix} 2 & 0 & ... & 0 & 2r \\ 0 & 2 & 0 & ... & 0 \\ ... & 0 & ... & 0 & ... \\ 0 & ... & 0 & 2 & 0 \\ 2r & 0 & ... & 0 & r^2 + e^{-t}h'' \end{pmatrix} \right),$$

that is

$$\sigma_k^{(\alpha)} = e^{kt}2^{k-2}\left(2r^2 + 2e^{-t}h'' - 4r^2\right).$$

Next, for $\alpha \in B \backslash A$,

$$\sigma_k^{(\alpha)} = \det \left(e^t \begin{pmatrix} 2 & 0 & ... & 0 \\ 0 & ... & 0 & 0 \\ ... & 0 & 2 & ... \\ 0 & 0 & ... & r^2 + e^{-t}h'' \end{pmatrix} \right),$$

that is

$$\sigma_k^{(\alpha)} = e^{kt}2^{k-1}\left(r^2 + e^{-t}h''\right).$$

Finally, for $\alpha \in (S \backslash B)$ we have

$$\sigma_k^{(\alpha)} = \det \left(e^t \begin{pmatrix} 2 & 0 & ... \\ 0 & ... & 0 \\ ... & 0 & 2 \end{pmatrix} \right),$$

that is
$$\sigma_k^{(\alpha)} = e^{kt}2^k.$$
We sum these up:
$$\sigma_k\left(D^2u\right) = \sum_{\alpha\in(A\cap B)} \sigma_k^\alpha + \sum_{\alpha\in(B\setminus A} \sigma_k^\alpha + \sum_{\alpha\in(S\setminus B\ )} \sigma_k^\alpha.$$
Counting, we get
$$\begin{aligned}\sigma_k\left(D^2u\right) &= \binom{n-2}{k-2}e^{kt}2^{k-1}\left(e^{-t}h'' - r^2\right) \\ &+ \binom{n-2}{k-1}e^{kt}2^{k-1}\left(r^2 + e^{-t}h''\right) \\ &+ \binom{n-1}{k}e^{kt}2^k.\end{aligned} \tag{6}$$
Grouping the terms, we see
$$\begin{aligned}\sigma_k\left(D^2u\right) &= e^{kt}2^{k-1}\left[-\binom{n-2}{k-2} + \binom{n-2}{k-1}\right]r^2 \\ &+ e^{kt}2^{k-1}\left[\binom{n-2}{k-2} + \binom{n-2}{k-1}\right]e^{-t}h'' \\ &+ e^{kt}2^{k-1}2\binom{n-1}{k}.\end{aligned}$$
Now
$$-\binom{n-2}{k-2} + \binom{n-2}{k-1} = -\frac{(n-2)!}{(n-k)!(k-2)!} + \frac{(n-2)!}{(n-k-1)!(k-1)!}.$$
This vanishes if and only if
$$1 = \frac{(n-k)!(k-2)!}{(n-k-1)!(k-1)!} = \frac{(n-k)}{(k-1)},$$
or precisely when
$$n-k = k-1$$
or
$$2k = n+1.$$
Thus for this choice of k, (6) becomes
$$\sigma_k\left(D^2u\right) = A_{n,k}e^{(k-1)t}h'' + B_{n,k}e^{kt}$$
for some constants $A_{n,k}, B_{n,k}$. Setting to this expression to 1, we solve for $h''(t)$
$$h''(t) = \frac{1 - B_{n,k}e^{kt}}{A_{n,k}e^{(k-1)t}}, \tag{7}$$
noting the right-hand side is a smooth function in t. Integrating twice in t yields solutions to (7) and hence to (1).

To see that the equation is elliptic, we first note that inspecting (4) the $n-2$ eigenvalues in the middle must be positive. Of the remaining two, at least one must be positive as the diagonal (of the 2×2 matrix) contains at least one positive entry. We then note the following.

Lemma 4. *Suppose that*

$$\sigma_k(D^2u) > 0$$

and D^2u has at most 1 negative eigenvalue. Then $D^2u \in \Gamma_k^+$.

Proof. Diagonalize D^2u so that $D^2u = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ with $0 \leq \lambda_2 \leq \lambda_3 ... \leq \lambda_n$. Clearly

$$\frac{d}{ds}\sigma_k(diag\{\lambda_1 + s, \lambda_2, ..., \lambda_n\}) \geq 0$$

so we may deform D^2u to a positive definite matrix D^2u+M, with $\sigma_k(D^2u+sM) > 0$ for $s \geq 0$. Thus D^2u is in the component of $\sigma_k > 0$ containing the positive cone, that is, $D^2u \in \Gamma_k^+$. □

Example 5. *When $n = 3$ the function*

$$u(x, y, t) = (x^2 + y^2)e^t + \frac{1}{4}e^{-t} - e^t$$

solves

$$\sigma_2(D^2u) = 1.$$

Remark 6. *This method allows one to construct solutions to complex Monge-Ampère equations as well. See* [7].

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