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A McLean Theorem for the moduli space of Lie solutions to mass transport equations

Micah Warren [1]

Department of Mathematics, Princeton University, Princeton, NJ, USA

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ABSTRACT

On compact manifolds which are not simply connected, we prove the existence of "fake" solutions to the optimal transportation problem. These maps preserve volume and arise as the exponential of a closed 1-form, hence appear geometrically like optimal transport maps. The set of such solutions forms a manifold with dimension given by the first Betti number of the manifold. In the process, we prove a Hodge–Helmholtz decomposition for vector fields. The ideas are motivated by the analogies between special Lagrangian submanifolds and solutions to optimal transport problems.

1. Introduction

In [9], graphs of solutions to the optimal transportation problem were shown to solve a volume maximization problem, using a calibration argument. With the appropriate metric on the product space $M \times \bar{M}$, maximality is equivalent to the vanishing of certain differential forms along the graph of the optimal map. In this note, we discuss the converse and see that, at least in the smooth case, topology allows for maximizers of the volume problem which do not arise as solutions to the optimal transportation problem, despite locally having the same geometric properties. These maximizers are special Lagrangian in the sense of Hitchin [3] and Mealy [13], a pseudo-Riemannian analogue of the special Lagrangian geometry of Harvey and Lawson [6].

In the case when the cost is given by Riemannian distance squared, Delanoë [2] introduced the notion of "Lie solutions of Riemannian transport equations" (see also [5]). The graphs of Delanoë's solutions are maximizers of the volume maximization problem discussed in [9].

We recall the McLean Theorem [12, Theorem 3.6], [10, Theorem 3.21].

Theorem 1.1 *(McClean). Suppose L is a smooth embedded special Lagrangian submanifold of a Calabi–Yau manifold. The moduli space $\mathcal{M}$ of special Lagrangian submanifolds near L is a manifold of dimension $b_1(L)$. The tangent space to $\mathcal{M}$ is identified with the harmonic 1-forms on L, which has a naturally induced L^2 metric.*

E-mail address: mww@math.princeton.edu.

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The seminal paper of Harvey and Lawson [6] shows that special Lagrangian submanifolds are minimal submanifolds in Calabi–Yau manifolds, and that a manifold is special Lagrangian if and only if the Kähler form and a certain n-form vanishes along the submanifold. Hitchin [3] analyzed the metric on the moduli space in McLean's theorem and showed that it arises via a Lagrangian embedding into a pseudo-Riemannian space. Hitchin's notion of special Lagrangian, that the Kähler form and certain combinations of n-forms vanish, describes Lie solutions of the mass transport problems.

The optimal transportation problem is the following. Given probability volume forms ρ and $\bar{\rho}$ on manifolds M and $\bar{M}$, and a cost function $c : M \times \bar{M} \to \mathbb{R}$, find a map $T : M \to \bar{M}$ which minimizes a cost integral,

$$\int_M c\big(x, T(x)\big)\, d\rho$$

among all maps T which preserve the measure, i.e.

$$T_\# \, d\rho = d\bar{\rho}. \tag{1}$$

The work of Brenier [1] and McCann [11] shows that given standard conditions on the cost function, the unique solution will be the map satisfying (1) and arising as the cost exponential (see Definition 2.1) of the gradient of a potential function u:

$$T(x) = c\text{-}\exp_x du(x). \tag{2}$$

When the map T is a diffeomorphism, Eq. (1) can be expressed as

$$T^* \bar{\rho} = \rho \tag{3}$$

which is equivalent to the vanishing of the form

$$\bar{\rho}(\bar{x})\, d\bar{x} - \rho(x)\, dx$$

along the graph $(x, T(x)) \subset M \times \bar{M}$, and (2) implies the vanishing of a Kähler form (17) defined in [8]. Here and in the sequel when we use $\rho(x)$ rather than ρ we distinguish between a density and an n-form. In particular

$$\rho = \rho(x)\, dx$$

in local coordinates, also for $\bar{\rho}$.

The general problem we attack here is to find maps which locally solve the optimal transport problem, that is, satisfy (3) and arise as the exponential of a closed form

$$T(x) = c\text{-}\exp_x\big(\eta(x)\big), \quad d\eta = 0. \tag{4}$$

The result is the following.

Theorem 1.2. *Suppose M, $\bar{M}$ are compact manifolds with nowhere vanishing smooth volume forms ρ and $\bar{\rho}$. Let c be a continuous cost function on $M \times \bar{M}$, which away from a cut locus C, is smooth and satisfies local and global twist assumptions* (A2) *and* (A2) *(see Section 2). If a diffeomorphism T is a smooth Lie solution of the mass transport equation (i.e. satisfying (3) and (4)) avoiding the cut locus, then there is a moduli space of smooth Lie solutions near T which is a smooth manifold of dimension $b_1(M)$.*

Remark. There are cost functions such that even for some smooth densities, the optimal solutions will not be smooth, as was demonstrated by Loeper [4], when the (A3) (see [14,4]) assumption on the cost is not satisfied. Also, in the absence of the (A3) assumption, local c-convexity does not imply global, so even on simply connected domains, Lie solutions are not necessarily solutions to the optimal transport problem.

Remark. Harvey and Lawson [7] have recently given a version of McLean's theorem which holds in a stronger setting, Ricci-flat Kähler $\mathbb{D}$-manifolds, which are the natural analogue of Calabi–Yau manifolds in the pseudo-Riemannian setting. The proof of the result [7, Theorem 13.1] follows easily once the elegant geometry is put in place. However, our setting of optimal transport maps for more general costs is not so elegant.

The core of our result is a Hodge–Helmholtz type decomposition holding at Lie solutions when $n \geqslant 3$, which decomposes deformations of maps into those which preserve property (3), those which preserve property (4), and the harmonic deformations, which preserve both. Unfortunately, this is only a decomposition of the vector fields along solutions, unlike in the Euclidean setting where one can decompose the maps themselves, and also vector fields along arbitrary maps. The metric used is the one induced by the linearized operator to the optimal transport equation, multiplied by an explicit conformal factor. This metric and concomitant Laplacian are defined in Section 2, and their Hodge–Helmholtz properties are shown in Section 3. We prove the theorem in Section 4. Due to some dimensional considerations arising with the conformal factor, we have to prove the 2-dimensional case after the cases $n \geqslant 3$.

2. Preliminaries

For n-dimensional manifolds M and $\bar{M}$, let $c: N \subset M \times \bar{M} \to \mathbf{R}$ be a continuous cost function which is smooth almost everywhere, except on a set $\mathcal{C} = M \times \bar{M} - N$ which we call the "cut locus". (The reason for this terminology is clear if we take the distance squared function on a manifold as the cost function.) Let $D\bar{D}c$ be the $n \times n$ matrix given by (in local coordinates)

$$(D\bar{D}c)_{i\bar{j}}(x,\bar{x}) = c_{i\bar{j}}(x,\bar{x}) = \frac{\partial^2}{\partial x^i \partial \bar{x}^j} c(x,\bar{x}).$$

On N we will require that

$$\det(D\bar{D}c) \neq 0 \tag{A2}$$

which is a local version of the standard *twist* condition: (A1) For each x and $\bar{x}$ respectively,

$$\bar{x} \to Dc(x,\bar{x}) \tag{5}$$

is invertible, with an inverse depending continuously on x. To be clear with our conventions, we recall the following Kantorovich problem: If

$$J(u,v) = \int_M (-u)\,d\rho + \int_{\bar{M}} v\,d\bar{\rho},$$

the problem is to maximize J over all $-u(x) + v(\bar{x}) \leqslant c(x,\bar{x})$. One also considers a dual problem: If

$$I(\pi) = \int_{M \times \bar{M}} c(x,y)\,d\pi,$$

find the minimum of I over all measures π on the product space $M \times \bar{M}$ which have marginals ρ and $\bar{\rho}$. It is well known (cf. [16]) that

$$\sup_{-u(x)+v(\bar{x}) \leqslant c(x,\bar{x})} J(u,v) = \inf_{\pi \in \Pi(\rho,\bar{\rho})} I(\pi).$$

With this setup in mind we can derive the optimal map T from u as follows: Suppose $(x_0, \bar{x}_0)$ is a point where the equality $-u(x_0) + v(\bar{x}_0) = c(x_0, \bar{x}_0)$ occurs. The function

$$z_{\bar{x}_0}(x) = c(x, \bar{x}_0) + u(x)$$

must have a minimum at x_0.

Definition 2.1. Suppose that a cost c satisfies (A1). Then for a covector η which is in the range of the map (5), we define the cost exponential of η as

$$c\text{-}\exp_x(\eta) = \{\bar{x} \mid \eta = -Dc(x,\bar{x})\}.$$

For a covector η we define

$$T(x,\eta) = c\text{-}\exp_x(\eta).$$

If differentiable, from the fact that $z_{\bar{x}_0}(x_0)$ is at a minimum we have

$$u_i(x_0) + c_i(x_0, \bar{x}_0) = 0 \tag{6}$$

where c_i refers to differentiation in the first variable, thus

$$\bar{x}_0 = T(x_0, du(x_0)).$$

(One can check that $T(x,u) = -\exp_x \nabla u$ when $c(x,\bar{x}) = d^2(x,\bar{x})/2$.) This only depends locally on the function u, and requires that Du stay inside the range of $Dc(x,\cdot)$. The elliptic optimal transportation equation can be derived by taking another derivative and then a determinant:

$$u_{ij} + c_{ij} + c_{i\bar{s}} T^{\bar{s}}_j = 0, \tag{7}$$

$$\det(u_{ij} + c_{ij}) = \det(-c_{i\bar{s}} T^{\bar{s}}_j) = \det(-c_{i\bar{s}}) \frac{\rho(x)}{\bar{\rho}(T(x))} \tag{8}$$

using the fact that T locally pushes ρ forward to $\bar{\rho}$, so satisfies

$$\bar{\rho}\big(T(x)\big)\det DT = \rho(x). \tag{9}$$

Here and in the sequel we use the following conventions: Indices i, j, k, etc. will be coordinates on M while $\bar{p}, \bar{s}, \bar{v}$, etc. will be coordinates on $\bar{M}$. The variable t will be reserved for variations. We assume on any local product chart that $c_{i\bar{s}}$ is negative definite, and let $b_{i\bar{s}} = -c_{i\bar{s},}$ with $b^{\bar{s}i}$ its inverse. The tangent space is denoted by $\mathfrak{T}M$. We use $w_{ij}(x) = u_{ij}(x) + c_{ij}(x, T(x))$. To be clear, we will use $\partial_k w_{ij}$ to denote coordinate derivatives of w, otherwise a subscript will denote differentiation. We note that most of the objects we deal with are coordinate invariant. In particular w_{ij} represents an honest tensor. For future use, we make note of the following restatement of the above identity (7)

$$\begin{aligned} &w_{ij} = b_{i\bar{s}} T^{\bar{s}}_j, \\ &T^{\bar{s}}_j w^{kj} = b^{\bar{s}k}. \end{aligned} \tag{10}$$

2.1. Linearizing the elliptic equation

In order to linearize (8), take a variation,

$$u(x) + tv(x). \tag{11}$$

First, we insert (11) into (6) and differentiate at $t = 0$ to obtain

$$v_i - b_{i\bar{s}}\big(x, T(x, du)\big) D_t T^{\bar{s}} = 0$$

hence

$$D_t T^{\bar{s}} = T^{\bar{s}}_t = v_i b^{\bar{s}i}.$$

Now, taking a logarithm of (8)

$$F\big(x, Du, D^2u\big) = \ln\det\big(u_{ij} + c_{ij}\big(x, T(x)\big)\big) - \ln\det\big(b_{i\bar{s}}\big(x, T(x)\big)\big) - \ln\rho(x) + \ln\bar{\rho}\big(T(x)\big)$$

which is linearized

$$\begin{aligned} Lv = \frac{d}{dt}F(u+tv) &= w^{ij}\big(v_{ij} + c_{ij\bar{s}} T^{\bar{s}}_t\big) - b^{\bar{s}i} b_{\bar{s}i\bar{p}} T^{\bar{p}}_t + \big(\ln\bar{\rho}\big(T(x)\big)\big)_{\bar{s}} T^{\bar{s}}_t \\ &= w^{ij} v_{ij} - w^{ij} b_{ij\bar{s}} b^{\bar{s}k} v_k - b^{\bar{s}i} b_{\bar{s}i\bar{p}} b^{pk} v_k + \big(\ln\bar{\rho}\big(T(x)\big)\big)_{\bar{s}} b^{\bar{s}k} v_k. \end{aligned} \tag{12}$$

A version of this linearized operator was introduced by Trudinger and Wang [15, 2.18].

2.2. The KM and modified KM metrics and a related Laplace–Beltrami

Kim and McCann [8] considered the following pseudo-metric on the product space $M \times \bar{M}$:

$$h(x, \bar{x}) = \begin{pmatrix} 0 & b_{i\bar{s}}(x, \bar{x}) \\ b^T_{i\bar{s}}(x, \bar{x}) & 0 \end{pmatrix}$$

and symplectic form

$$\omega = b_{i\bar{s}}\, dx^i \wedge d\bar{x}^{\bar{s}}.$$

In this metric, the graph $(x, T(x))$ of the cost exponential of any locally c-convex potential u is space-like and Lagrangian. The induced metric, in terms of coordinates on M, is given by

$$\big(x, T(x)\big)^* h = u_{ij} + c_{ij} = w_{ij}.$$

For given densities $\rho(x)$ and $\bar{\rho}(\bar{x})$ representing the forms ρ and $\bar{\rho}$, consider the following metric in [9]

$$h = \frac{1}{2}\left(\frac{\rho(x)\bar{\rho}(\bar{x})}{\det b_{is}(x, \bar{x})}\right)^{1/n} \begin{pmatrix} 0 & b_{i\bar{s}}(x, \bar{x}) \\ b^T_{i\bar{s}}(x, \bar{x}) & 0 \end{pmatrix}$$

and the calibrating form

$$\Omega = \frac{1}{2}(\rho + \bar{\rho}).$$

The graph of any solution to the optimal transportation problem will be a calibrated maximal Lagrangian surface with respect to this metric, but the converse is not true in general. Calibrated maximal Lagrangian surfaces are what we are studying in this paper.

We now define another metric. We intend to show that with respect to this metric, the tangent space of deformations of calibrated submanifolds coincides with harmonic 1-forms, which is the same situation that occurs in McLean's theorem. The metric we use is yet another conformal factor of the metric used by Kim and McCann, differing by a power of the conformal factor from the metric associated to the calibration in [9].

For $n \geqslant 3$, a given frame on M, and a cost exponential T define

$$g_{ij}(x) = w_{ij}(x)\left(\frac{\rho(x)\bar{\rho}(T(x))}{\det b_{i\bar{s}}(x, T(x))}\right)^{1/(n-2)}. \tag{13}$$

We also define (coordinate invariant) quantities

$$\theta = \ln\det w_{ij} - \ln\rho(x) + \ln\bar{\rho}\big(T(x)\big) - \ln\det b_{i\bar{s}}, \tag{14}$$

and

$$\lambda = \left(\frac{\rho(x)\bar{\rho}(T(x))}{\det b_{i\bar{s}}(x, T(x))}\right)^{1/(n-2)}.$$

Our first claim is the following.

Lemma 2.2. *Let $n \geqslant 3$. Let $T = T(x, \eta)$ be the cost exponential of a 1-form locally given by $\eta = du$. Taking the metric (13) and any choice of coordinates, the following holds involving Christoffel symbols*

$$w^{ij}\Gamma_{ij}^{k} = \frac{1}{2}w^{kl}\theta_l + w^{ij}b_{ij\bar{s}}b^{\bar{s}k} + b^{\bar{s}i}b_{\bar{s}i\bar{p}}b^{pk} - (\ln\bar{\rho})_{\bar{s}}b^{\bar{s}k}.$$

Proof. To begin, recall

$$\Gamma_{ij}^{k} = \frac{1}{2}w^{kl}(\partial_i w_{lj} + \partial_j w_{il} - \partial_l w_{ij}) + \frac{1}{2}\left\{\delta_i^k(\ln\lambda)_j + \delta_j^k(\ln\lambda)_i - w^{kl}w_{ij}(\ln\lambda)_l\right\}.$$

Tracing,

$$\begin{aligned}
w^{ij}\Gamma_{ij}^{k} &= w^{ij}w^{kl}\partial_i w_{lj} - \frac{1}{2}w^{kl}w^{ij}\partial_l w_{ij} + \frac{(2-n)}{2}w^{kl}(\ln\lambda)_l \\
&= \frac{1}{2}w^{ij}w^{kl}\partial_l w_{ij} + w^{ij}w^{kl}(\partial_i w_{lj} - \partial_l w_{ij}) + \frac{(2-n)}{2}w^{kl}(\ln\lambda)_l \\
&= \frac{1}{2}w^{kl}\partial_l(\ln\det w_{ij}) + w^{ij}w^{kl}\big(c_{lj\bar{s}}T_i^{\bar{s}} - c_{ij\bar{s}}T_l^{\bar{s}}\big) + \frac{(2-n)}{2}w^{kl}(\ln\lambda)_l
\end{aligned} \tag{15}$$

where we have used the local expression

$$w_{ij}(x) = u_{ij}(x) + c_{ij}\big(x, T(x)\big)$$

to compute the middle term of (15).

Now from (14)

$$\begin{aligned}
&\ln\det w_{ij} = \theta + \ln\rho(x) - \ln\bar{\rho}\big(T(x)\big) + \ln\det b_{i\bar{s}}, \\
&(n-2)\ln\lambda = \ln\rho(x) + \ln\bar{\rho}\big(T(x)\big) - \ln\det b_{i\bar{s}},
\end{aligned}$$

we can combine the first and last terms in (15) to get

$$\begin{aligned}
w^{ij}\Gamma_{ij}^{k} &= \frac{1}{2}w^{kl}\partial_l\theta - w^{kl}\partial_l\big\{\ln\bar{\rho}\big(T(x)\big) - \ln\det b_{i\bar{s}}\big\} + w^{kl}c_{lj\bar{s}}b^{\bar{s}j} - w^{ij}c_{ij\bar{s}}b^{\bar{s}k} \\
&= \frac{1}{2}w^{kl}\partial_l\theta - w^{kl}(\ln\bar{\rho})_{\bar{s}}T_l^{\bar{s}} + w^{kl}b^{\bar{s}i}\big(b_{i\bar{s}l} + b_{i\bar{s}\bar{p}}T_l^{\bar{p}}\big) + w^{kl}c_{lj\bar{s}}b^{\bar{s}j} - w^{ij}c_{ij\bar{s}}b^{\bar{s}k} \\
&= \frac{1}{2}w^{kl}\partial_l\theta - b^{\bar{s}k}(\ln\bar{\rho})_{\bar{s}} + b^{\bar{s}i}b^{\bar{p}k}b_{i\bar{s}\bar{p}} + w^{ij}b_{ij\bar{s}}b^{\bar{s}k}
\end{aligned}$$

as promised. We have used $b = -c$ and the relations (10) repeatedly. $\square$

This lemma allows us to easily compute geometric quantities. In particular we have

Proposition 2.3. *Let $n \geqslant 3$. Let $T = T(x,\eta)$ be the cost exponential of a 1-form locally given by $\eta = du$. Then with the above metric (13) on M,*

$$Lv = \lambda\left(\triangle_g v + \frac{1}{2}\langle\nabla\theta, \nabla v\rangle_g\right) \tag{16}$$

where $\triangle_g$ is Laplace–Beltrami operator with respect to g and L is the linearized operator defined by (12).

Proof. By Lemma 2.2, together with our initial expression (12) of L, we have

$$\begin{aligned} Lv &= w^{ij}v_{ij} - \left(w^{ij}\Gamma_{ij}^k - \frac{1}{2}w^{kl}\theta_l\right)v_k \\ &= \lambda\left\{g^{ij}v_{ij} - \left(w^{ij}\Gamma_{ij}^k - \frac{1}{2}w^{kl}\theta_l\right)v_k\right\} \end{aligned}$$

which is precisely (16). □

3. Deformations

Let $T : M \to \bar{M}$ be a Lie solution of the transport equation. This means that T is locally a solution of the optimal transport equation, or equivalently, T is a cost exponential of a closed 1-form, satisfying

$$\det DT = \frac{\rho(x)}{\bar{\rho}(T(x))}.$$

Given a 1-form η on M we can define a (vertical) deformation vector field along the graph of T as

$$V = -\eta_i c^{\bar{s}i}\partial_{\bar{s}}$$

where $\partial_{\bar{s}}$ is the coordinate tangent frame for $\mathfrak{T}\bar{M}$. To be precise, V is a section of the pullback bundle $T^*\mathfrak{T}\bar{M}$. For a vector field in this bundle, define a deformation of the map T via

$$T_V(x) = \exp_{T(x)}^{\bar{M}}\big(V(x)\big).$$

(As a convenience, we fix arbitrarily a metric on $\bar{M}$ in order to define $\exp^{\bar{M}}$.)

Smoothing out c in a small neighborhood of the cut locus $\mathcal{C}$, we may define the 1-form $\sigma = c_i(x,\bar{x})\,dx^i$, which differs from the total differential of c by $c_{\bar{s}}(x,\bar{x})\,d\bar{x}^{\bar{s}}$. Then we get the following exact Kähler form

$$\omega = d\sigma = -c_{i\bar{s}}(x,\bar{x})\,dx^i \wedge d\bar{x}^{\bar{s}} \tag{17}$$

on $M \times \bar{M}$.

At a solution T, for $n \geqslant 3$, we define the map

$$\Phi : \Lambda^1(M) \to \Lambda^2(M) \oplus \Lambda^0(M)$$

via

$$\Phi(\eta) = \big((Id \times T_V)^*\omega, *\big(T_V(x)^*\bar{\rho} - \rho\big)\big).$$

Here, for $n \geqslant 3$, the operator $*$ is defined with respect to the metric (13). Note that the level set $\Phi^{-1}(0,0)$ consists of forms whose corresponding maps T_V are Lie solutions. Equivalently, the image of $Id \times T_V$ is a calibrated Lagrangian submanifold of $M \times \bar{M}$ with respect to the metric in [9].

Lemma 3.1. *The image of Φ lies in exact 2-forms and coexact 0-forms.*

Proof. The first factor is a pullback of an exact form. For the second factor,

$$\int T_V(x)^*\bar{\rho}(\bar{x})\,d\bar{x} - \rho(x)\,dx = 0,$$

which follows from the fact that the diffeomorphism T is simply a change of integration variables. We are assuming the total mass of both densities is 1. □

Lemma 3.2. *When $n \geqslant 3$, at $0 \in C^{k+1,\alpha}(\Lambda^1(M))$, the differential $D\Phi$ satisfies*

$$D\Phi(\eta) = \left(d\eta, d^*\eta\right).$$

In particular, $D\Phi$ is a topological linear isomorphism

$$d^* C^{k+2,\alpha}\left(\Lambda^2(M)\right) \oplus dC^{k+2,\alpha}\left(\Lambda^0(M)\right) \to dC^{k+1,\alpha}\left(\Lambda^1(M)\right) \oplus d^* C^{k+1,\alpha}\left(\Lambda^1(M)\right).$$

Proof. Define a variation by $V(t) = -t\eta_i c^{\bar{s}i}\partial_{\bar{s}}$. To compute the derivative for the first factor of the map Φ, we use the Lie derivative on the manifold $M \times \bar{M}$ (cf. [10, Section 2.2.2]).

$$\begin{aligned}
\frac{d}{dt}(Id \times T_V)^*\omega|_{t=0} &= \mathcal{L}_{(-c^{\bar{s}m}\eta_m\partial_{\bar{s}})}(Id \times T_V)^*\omega \\
&= (Id \times T_V)^*\left[-c^{\bar{s}m}\eta_m\partial_{\bar{s}} \lrcorner d\omega - d\left(c^{\bar{s}m}\eta_m\partial_{\bar{s}} \lrcorner \omega\right)\right] \\
&= (Id \times T_V)^*\left[d\left(-c^{\bar{s}m}\eta_m\partial_{\bar{s}} \lrcorner \omega\right)\right] \\
&= (Id \times T_V)^*\left[d\left(\eta_m dx^m\right)\right] \\
&= d\eta.
\end{aligned}$$

For the second factor, the volume element is given by

$$\begin{aligned}
Vol_g &= \sqrt{\det w_{ij}(x)\left(\frac{\rho(x)\bar{\rho}(T(\bar{x}))}{\det b_{i\bar{s}}(x, T(x))}\right)^{n/n-2}}\,dx \\
&= \sqrt{\frac{\rho(\bar{x})}{\bar{\rho}(T(\bar{x}))}\det b_{i\bar{s}}\left(\frac{\rho(x)\bar{\rho}(T(\bar{x}))}{\det b_{i\bar{s}}(x, T(x))}\right)^{n/n-2}}\,dx \\
&= \lambda\rho(x)\,dx,
\end{aligned}$$

thus

$$*\left(T_v^*\bar{\rho} - \rho\right) = \frac{1}{\rho(x)\lambda}\left(\det DT_V\,\bar{\rho}\left(T(x)\right) - \rho(x)\right).$$

Differentiating

$$\begin{aligned}
&\left(\frac{1}{\rho(x)\lambda}\det DT_V\,\bar{\rho}\left(T(x)\right) - \frac{1}{\lambda}\right)' \\
&\quad = \frac{\det DT_V\,\bar{\rho}(T(x))}{\rho(x)\lambda}\left\{(\ln\det DT)' + \left(\ln\bar{\rho}\left(T(x)\right)\right)' - \left(\ln\rho(x)\right)' - \ln\lambda'\right\} + \frac{1}{\lambda}(\ln\lambda)'.
\end{aligned} \tag{18}$$

Noting that at $t = 0$, because T is a Lie solution,

$$\det DT = \frac{\rho(x)}{\bar{\rho}(T(\bar{x}))}$$

the expression (18) becomes

$$\frac{1}{\lambda}\left\{(\ln\det DT)' + \left(\ln\bar{\rho}\left(T(x)\right)\right)' - \left(\ln\rho(x)\right)'\right\}. \tag{19}$$

Now in particular, differentiating with respect to t we have

$$\begin{aligned}
\frac{d}{dt} * \left(T_V^*\bar{\rho} - \rho\right)\big|_{t=0} &= \frac{1}{\lambda}\left[\left(DT^{-1}\right)^j_{\bar{s}} T^{\bar{s}}_{jt} + \left(\ln\bar{\rho}\left(T(x)\right)\right)_{\bar{s}} T^{\bar{s}}_t\right] \\
&= \frac{1}{\lambda}\left[\left(DT^{-1}\right)^j_{\bar{s}}\partial_j\left(b^{\bar{s}k}\eta_k\right) + \left(\ln\bar{\rho}\left(T(x)\right)\right)_{\bar{s}} b^{\bar{s}k}\eta_k\right] \\
&= \frac{1}{\lambda}\left[b_{i\bar{s}}w^{ij}b^{\bar{s}k}\partial_j\eta_k - b_{i\bar{s}}w^{ij}b^{\bar{p}k}b^{\bar{s}m}\left(b_{m\bar{p}j} + b_{m\bar{p}\bar{r}}b^{\bar{r}l}w_{lj}\right)\eta_k + \left(\ln\bar{\rho}\left(T(x)\right)\right)_{\bar{s}} b^{\bar{s}k}\eta_k\right] \\
&= \frac{1}{\lambda}\left[w^{ij}\partial_j\eta_i - w^{ij}b^{\bar{p}k}b_{i\bar{p}j}\eta_k - b^{\bar{p}k}b_{i\bar{p}\bar{r}}b^{\bar{r}i}\eta_k + \left(\ln\bar{\rho}\left(T(x)\right)\right)_{\bar{s}} b^{\bar{s}k}\eta_k\right] \\
&= \frac{1}{\lambda}\left[w^{ij}\partial_j\eta_i + \left(\frac{1}{2}w^{kl}\theta_l - w^{ij}\Gamma^k_{ij}\right)\eta_k\right]
\end{aligned} \tag{20}$$

using (10) repeatedly and Lemma 2.2. Now choosing normal coordinates, and noting that along a Lie solution θ vanishes everywhere, we have

$$\frac{d}{dt} * \left(T_V^* \bar{\rho} - \rho\right)|_{t=0} = g^{ij} \partial_j \eta_i$$

which is precisely the expression for $d^*\eta$ in normal coordinates. This proves

$$D\Phi(\eta) = \left(d\eta, d^*\eta\right).$$

The latter conclusion of the Lemma follows from standard algebra using the Hodge decomposition. □

4. Proof of Main theorem

4.1. Case $n \geqslant 3$

Proof. Recall the Hodge decomposition

$$C^{k+1,\alpha}\left(\Lambda^1(M)\right) = \mathcal{H}^1 \oplus dC^{k+2,\alpha}\left(\Lambda^0(M)\right) \oplus d^* C^{k+2,\alpha}\left(\Lambda^2(M)\right).$$

Lemma 3.2 shows that at 0, the smooth map Φ has surjective differential

$$D\Phi : C^{k+1,\alpha}\left(\Lambda^1(M)\right) \to d^* C^{k+1,\alpha}\left(\Lambda^1(M)\right) \oplus dC^{k+1,\alpha}\left(\Lambda^1(M)\right)$$

onto the product of exact and coexact forms. The kernel at 0 is the harmonic forms, which splits by the Hodge decomposition. It follows by the Implicit Function Theorem (see [10, Thm. 2.11]) that for each harmonic form η close enough to 0 there is a unique form $\chi(\eta)$ lying in the orthogonal complement of the harmonic forms so that $\Phi(\eta + \chi(\eta)) = 0$. Thus a neighborhood of 0 in the harmonic forms on M parametrizes the moduli space near T. □

4.2. Case $n = 2$

Proof. Define

$$\Phi : \Lambda^1(M) \to \Lambda^2(M) \oplus \Lambda^2(M)$$

via

$$\Phi(\eta) = \left((Id \times T_V)^*\omega, T_V(x)^*\bar{\rho} - \rho\right).$$

Step 1: Index of $D\Phi$ = Index of $(d, d*)$. We compute $D\Phi$. As in Lemma 3.2, $D\Phi$ on the first factor is simply d. On the second factor we compute as before

$$\left\{\left(T_V^* \bar{\rho}\left(T(x)\right) - \rho(x)\right) dx\right\}' = \rho(x)\left\{(\ln \det DT)' + \left(\ln \bar{\rho}\left(T(x)\right)\right)' - \left(\ln \rho(x)\right)'\right\} dx \tag{21}$$

which is reminiscent of (19). Continuing,

$$\begin{aligned}\frac{d}{dt}\left(T_V^* \bar{\rho} - \rho\right)|_{t=0} &= \rho(x)\left[\left(DT^{-1}\right)_{\bar{s}}^{j} T_{jt}^{\bar{s}} + \left(\ln \bar{\rho}\left(T(x)\right)\right)_{\bar{s}} T_t^{\bar{s}}\right] dx \\ &= \left\{w^{ij}\partial_j\eta_i - w^{ij}b^{\bar{p}k}b_{i\bar{p}j}\eta_k - b^{\bar{p}k}b_{i\bar{p}\bar{r}}b^{\bar{r}i}\eta_k + \left(\ln \bar{\rho}\left(T(x)\right)\right)_{\bar{s}} b^{\bar{s}k}\eta_k\right\}\rho(x)\, dx.\end{aligned} \tag{22}$$

Now we define an operator on 1-forms as

$$\delta\eta = \left[w^{ij}\partial_j\eta_i - w^{ij}b^{\bar{p}k}b_{i\bar{p}j}\eta_k - b^{\bar{p}k}b_{i\bar{p}\bar{r}}b^{\bar{r}i}\eta_k + \left(\ln \bar{\rho}\left(T(x)\right)\right)_{\bar{s}} b^{\bar{s}k}\eta_k\right]\rho(x)\, dx$$

and we have that

$$D\Phi(\eta) = (d\eta, \delta\eta).$$

Now let

$$g_{ij} = \tilde{\lambda} w_{ij}$$

where

$$\tilde{\lambda} = \sqrt{\frac{\rho(x)\bar{\rho}(T(x))}{\det b_{i\bar{s}}(x, T(x))}}.$$

Computing the determinant, we see that

$$dV_g = \sqrt{\frac{\rho(x)\bar{\rho}(T(x))}{\det b_{i\bar{s}}(x, T(x))} \det w_{ij}\, dx}$$

which, at a measure preserving map, is simply $\rho(x)\,dx$. Thus if we take normal coordinates for this metric at a point x, we have $\rho = dx$. With this in mind, one can check that symbol of the elliptic operator $(d, d*)$ is equal to the symbol of (d, δ). It follows that these maps have the same index as differential operators $\Lambda^1(M) \to \Lambda^2(M) \oplus \Lambda^2(M)$.

Step 2: Dim(ker $D\Phi$) = $b_1(M)$. For our given $M^2, \bar{M}^2$ and probability forms ρ, $\bar{\rho}$ consider the transportation problem of finding

$$F(x,\sigma) = \big(T(x,\sigma), \Sigma(x,\sigma)\big) : M^2 \times S^1 \to \bar{M}^2 \times S^1 \tag{23}$$

minimizing the cost

$$\tilde{c}\big[(x,\sigma), (\bar{x}, \bar{\sigma})\big] = c(x, \bar{x}) + \mathrm{dist}^2_{S^1}(\sigma, \bar{\sigma})$$

among all maps (F, Σ) pushing $\rho \wedge d\sigma$ forward to $\bar{\rho} \wedge d\bar{\sigma}$. For any Lie solution $T : M^2 \to \bar{M}^2$ it is clear that

$$F(x,\sigma) = \big(T(x), \sigma + \sigma_0\big) \tag{24}$$

is a Lie solution to the problem (23). It is also clear that translating the solution by changing σ_0 will give new Lie solution. By Theorem 1.2 for $n = 3$ there is a space of deformations of solutions, and it has dimension equal to $1 + b_1(M)$. It follows that

$$\dim\ker D\Phi \leqslant b_1(M).$$

Let T be a Lie solution on M^2 and let (24) be the corresponding Lie solution on $M^2 \times S^1$. Let η be a 1-form which defines a tangent vector to the space of deformations on $M^2 \times S^1$, at the solution F, and write

$$\eta = \eta_1\, dx^1 + \eta_2\, dx^2 + \eta_\sigma\, d\sigma$$

for some local coordinate cotangent frame dx^1, dx^2 for M^2. Differentiating the equation

$$d^*\eta = 0$$

in the σ direction, we have that

$$w^{ij}\partial_\sigma\partial_j\eta_i + \big\{-b_{bj\bar{s}}b^{\bar{s}i}w^{bj} + (\ln\bar{\rho})_{\bar{s}}b^{\bar{s}i} - b^{\bar{s}k}b_{\bar{s}k\bar{p}}b^{\bar{p}i}\big\}\partial_\sigma\eta_i = 0,$$

using the fact that the warped product metric does not depend on σ. Also, because η is a closed form, locally we have $\partial_i\partial_j\eta_\sigma = \partial_\sigma\partial_j\eta_i$ and $\partial_\sigma\eta_i = \partial_i\eta_\sigma$. Thus (the honest function) $z = \eta_\sigma$ locally satisfies an elliptic equation of the form

$$w^{ij}z_{ij} + A^i z_i = 0$$

so enjoys a maximum principle. We conclude that on the compact manifold $M^2 \times S^1$, the S^1 component η_σ of the deformation must be constant. Inspecting $d^*\eta = 0$ using (20) now yields that the other two components satisfy

$$\delta\big(\eta_1\, dx^1 + \eta_2\, dx^2\big) = 0.$$

Thus,

$$\dim\ker D\Phi \geqslant b_1(M)$$

as well, so in particular

$$\dim\ker D\Phi = b_1(M).$$

Step 3: $D\Phi$ is surjective. Consider the space $\mathcal{E}(M) \subset \Lambda^2(M) \oplus \Lambda^2(M)$ of exact 2-forms by exact 2-forms. Recalling Lemma 3.1, the image of both (d, δ) and $(d, d*)$ are in $\mathcal{E}$, and the image of $(d, d*)$ is precisely $\mathcal{E}$, so the cokernel of $(d, d*)$ is a finite-dimensional complement $\mathcal{E}'$. Since the cokernel of (d, δ) must contain $\mathcal{E}'$, by dimensionality it must also be equal to $\mathcal{E}'$.

Now applying the implicit function theorem (see [10, Thm. 2.11]) the result follows. □

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References

[1] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 26 (1991) 375–417.
[2] Ph. Delanoë, Lie solutions of Riemannian transport equations on compact manifolds, Differential Geom. Appl. 26 (3) (2008) 327–338.
[3] N. Hitchin, The moduli space of special Lagrangian submanifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 (XXV) (1997) 503–515.
[4] G. Loeper, Continuity of maps solutions of optimal transportation problems, Acta Math. 202 (2) (2009) 241–283.
[5] V.V. Lychagin, Contact geometry and nonlinear second order differential equations, Russian Math. Surveys 34 (1) (1979) 149–180.
[6] R. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982) 47–157.
[7] R. Harvey, H.B. Lawson, Split special Lagrangian geometry, arXiv:1007.0450v1 [math.DG].
[8] Y.-H. Kim, R.J. McCann, Continuity, curvature, and the general covariance of optimal transportation, J. Eur. Math. Soc. (JEMS) 12 (4) (2010) 1009–1040.
[9] Y.-H. Kim, R.J. McCann, M. Warren, Pseudo-Riemannian geometry calibrates optimal transportation, Math. Res. Lett. 17 (6) (2010) 1183–1197.
[10] S.P. Marshall, Deformations of calibrated submanifolds, Oxford PhD thesis, 2002 (available on D. Joyce's website).
[11] R.J. McCann, Polar factorization of maps on Riemannian manifolds, Geom. Funct. Anal. 11 (2001) 589–608.
[12] R.C. McClean, Deformations of calibrated submanifolds, Comm. Anal. Geom. 6 (4) (1998) 705–747.
[13] J. Mealy, Volume maximization in semi-Riemannian manifolds, Indiana Univ. Math. J. 40 (3) (1991) 793–814.
[14] X.N. Ma, N. Trudinger, X.J. Wang, Regularity of potential functions of the optimal transport problem, Arch. Ration. Mech. Anal. 177 (2) (2005) 151–183.
[15] N.S. Trudinger, X.-J. Wang, On the second boundary value problem for Monge–Ampere type equations and optimal transportation, Ann. Sc. Norm. Super. Pisa 8 (2009) 1–32.
[16] C. Villani, Optimal Transport: Old and New, Grundlehren der mathematischen Wissenscharften, vol. 338, Springer, 2009.