GRIM RAINDROP: A TRANSLATING SOLUTION TO CURVE DIFFUSION FLOW

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Abstract. We show the existence of a properly immersed translating solution to curve diffusion flow in the plane. Curve diffusion flow is a higher order version of curve shortening flow, namely

$$\left(\frac{dX}{dt}\right)^{\perp} = -\kappa_{ss}N.$$

1. Introduction

In this note we present a properly immersed translating solution to the curve diffusion flow.

In [EGBM+15] it was shown that:

- (Corollary 4) The only open, properly immersed stationary solutions to the flow are lines;
- There are many translators that are not properly immersed (Euler spirals); and
- (Proposition 11) If an open translator has curvature vanishing at $+\infty$ and $-\infty$ and the angle the translating vector makes with the tangent is the same at both $+\infty$ and $-\infty$, then the translator is a stationary line.
- (Proposition 12) If $X : \mathbb{R} \to \mathbb{R}^2$ is an open properly immersed translator satisfying

$$\lim_{\rho\to\infty} \frac{1}{\rho^2} \int_{X^{-1}(B_\rho)} \kappa^2 \, ds + \frac{1}{\rho} \int_{X^{-1}(B_{2\rho})} |\langle \vec{E}, T \rangle| \, ds = 0,$$

  where $\vec{E}$ is the translating velocity and T is the oriented unit tangent vector, then $X(\mathbb{R})$ is a straight line.

The authors asked whether the growth condition or properness are necessary in Proposition 12. Our main result partially answers this question by showing that properness by itself does not imply the translator is a line.

In [Pol96] it was shown that singularities occur for the flow. In particular, an immersed figure 8 curve must develop singularities under the flow. It is a natural question to consider self-similar solutions, and these may help model the formation of singularities.

Here we use a shooting method together with symmetry to show that a bounded yet nontrivial solution to a certain 3rd order nonlinear ODE (2.2) exists. The solution of the ODE will describe the angle of the tangent vector of the profile curve for the translating solution of the flow.

2. Curve diffusion flow

For a one dimensional manifold M a map

$$X(\omega, t) : M \times I \to \mathbb{R}^2$$

which satisfies

$$\left(\frac{dX}{dt}\right)^{\perp} = -\kappa_{ss} N \tag{2.1}$$

is a solution to the curve diffusion flow. Here κ is the curvature, which can be defined as the derivative of the angle θ that the unit tangent vector makes with the x-axis. The unit normal vector N is the complex rotation of the oriented unit tangent vector of the curve.

This flow has a long history of interest in a wide variety of fields in both pure and applied mathematics; see [EGBM$^+$15] and references within. As a flow of hypersurfaces, this is generalized to the surface diffusion flow in higher dimensions [EMS98], while this flow also generalizes to the gradient flow for volume of Lagrangian submanifolds within a given Hamiltonian isotopy class [CW24].

For compact M it is easy to check using the first variational formula that this is the gradient flow for arc length on the space of curves with the same enclosed area, where the tangent space consists of variations of the form

$$\frac{dX}{dt} = J\nabla f$$

for f a function on the curve γ, where J is the standard complex rotation and the tangent space is equipped with the L^2 metric

$$\|f\|^2 = \int_\gamma f^2 ds.$$

2.1. Translating solutions.

Suppose that the unprojected flow satisfies

$$\frac{dX}{dt} = \vec{E}$$

for some fixed unit vector $\vec{E}$. From (2.1) we have

$$\vec{E} \cdot N = -\kappa_{ss}.$$

Now, without loss of generality, let

$$\vec{E} = (0, 1) = J(1, 0)$$

so that

$$\begin{aligned}\vec{E}\cdot N &= J(1,0)\cdot JT\\ &= (1,0)\cdot T\\ &= \cos\theta\end{aligned}$$

and the equation becomes

$$\cos\theta = -\kappa_{ss}$$

or

$$\cos\theta = -\theta_{sss}$$

which we write as a third order nonlinear ODE

$$\theta_{sss} = -\cos\theta. \tag{2.2}$$

An arc-length parameterized path

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$

whose angle θ satisfies (2.2) will be a the profile of a translating solution to the curve diffusion flow. If, in addition

$$\begin{aligned}\theta &\to \frac{\pi}{2} \text{ as } s \to \infty\\ \theta &\to -\frac{\pi}{2} \text{ as } s \to -\infty.\end{aligned}$$

then the solution will be properly immersed and have nonconstant curvature.

2.2. **Strategy of construction.** Our strategy is

(1) Show the existence for $s \geq 0$ of a solution $\theta(s)$ with odd initial conditions to a modified ODE (3.1), and show that this stays bounded as $s \to \infty$
(2) Show that any solution to (3.1) which stays bounded must converge to $\pi/2$ as $s \to \infty$.
(3) Argue that the solution in Step 1 is a solution to the original ODE (2.2) and can be reflected across $s = 0$ to get an entire solution which also converges to $-\pi/2$ as $s \to -\infty$.
(4) Now with θ determined as a function of arclength we may construct the path forwards and backwards. The tangent directions will approach $\pm\pi/2$ so the solution must be properly immersed.

3. Proof

3.1. **Step 1.** We start by modifying the ODE (2.2) slightly so that the right-hand side is monotone: Consider the ODE

$$u''' = f(u) \tag{3.1}$$

with

$$f(u) = \begin{cases} -\cos u & \text{if } 0 \leq u \leq \pi \\ -1 & \text{if } u \leq 0 \\ +1 & \text{if } u \geq \pi. \end{cases}$$

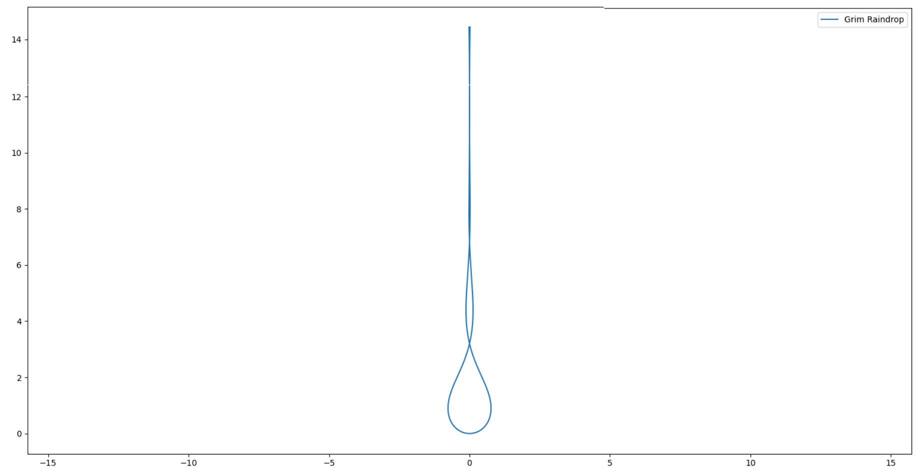

FIGURE 1. It's grim because it travels upwards

It is more straightforward to apply shooting methods to monotone ODE. If we have a solution whose values lie within the range $(0, \pi)$ this will also be a solution of the original ODE (2.2). Our goal is then to shoot to find a solution to the modified ODE, and then argue that this solution satisfies the original ODE.

The following claim sets up a shooting method.

Lemma 3.1. *Let u_a be the solution to the ODE* (3.1)

$$u''' = f(u)$$

with initial conditions

$$\begin{aligned} u(0) &= 0 \\ u'(0) &= a \\ u''(0) &= 0. \end{aligned} \tag{3.2}$$

These have the following order preservation property: If

$$a < b$$

then

$$u_a(s) < u_b(s) \textit{ for all } s > 0.$$

Proof. Let

$$v(s) = u_b(s) - u_a(s)$$

Then

$$\begin{aligned} v(0) &= 0 \\ v'(0) &= b - a \\ v''(0) &= 0 \\ v'''(0) &= 0. \end{aligned}$$

So $v > 0$, and $v' > 0$ for small values of s. Let s_0 be the first value such that $v'(s_0) = 0$. Then

$$0 > \int_0^{s_0} v''(\tau) : d\tau = \int_0^{s_0} \int_0^{\tau} (f(u_b) - f(u_a))\, d\sigma d\tau$$

but this is a contradiction, because on the whole interval we had $f(u_b) - f(u_a) \geq 0$. There is no such s_0; we may conclude $v' > 0$ and hence $u_b(s) > u_a(s)$ for all postive s. □

Lemma 3.2. *Any solution of the IVP* (3.1) *with these initial conditions (3.2) that reaches* 0 *or* π *for some* $s > 0$ *must escape to positive or negative infinite, never returning to* $[0, \pi]$.

This follows from Proposition 3.7 which will be proved directly in Section 3.3.

Lemma 3.3. *There exists a value a such that the solution to*

$$u''' = f(u)$$

with

$$u(0) = 0,$$
$$u'(0) = a,$$
$$u''(0) = 0$$

is bounded for all $s > 0$.

Proof. Fix any $m \in \mathbb{N}$. It's easy to check that there are values of a such that $u_a(m) > 10$ and $u_a(m) < -10$. By the order preservation property Lemma 3.1 and continuous dependence on initial condition, there is a set $I_m = [a_m, b_m]$ such that

$$u_{a_m}(m) = 0,$$
$$u_{b_m}(m) = \pi.$$

By the order preservation property,

$$I_m = \{a \in \mathbb{R}^+ \mid 0 \leq u_a(m) \leq \pi\}.$$

We claim that

$$I_{m+1} \subset I_m.$$

This follows immediately from the Lemma 3.2: If $u_a(m+1) \in [0, \pi]$ then it cannot have left the region prior to that, so $u_a(m) \in (0, \pi)$.

Now we conclude that

$$I_M = \bigcap_{m \leq M} I_m$$

and we can take

$$A = \bigcap_{m \in \mathbb{N}} I_m$$

which is the intersection of compact sets. Now the intersection of compact sets is either nonempty, or must be empty at a finite stage. By the shooting, each $[a_m, b_m]$ is nonempty. It follows that there must be at least one point $a_\infty \in A$. The solution shooting from a_∞ stays bounded for all time. □

3.2. **Step 2.** For cleaner presentation, let $v = u - \pi/2$, and consider instead the following ODE:

$$v''' = f(v) \tag{3.3}$$

with

$$f(u) = \begin{cases} \sin v & \text{if } -\pi/2 \leq u \leq \pi/2 \\ -1 & \text{if } u \leq -\pi/2 \\ +1 & \text{if } u \geq \pi/2. \end{cases}$$

Lemma 3.4. *Suppose that v is a solution to (3.3). Suppose that at $s_{\max}$ the solution v has a positive local maximum and at $s_{\min}$, the next critical point, the solution v has a negative local minimum. Then $v(s_{\max}) > |v(s_{\min})|$.*

Proof. We know $v'(s_{\max}) = 0$ and $v''(s_{\max}) < 0$; if $v''(s_{\max}) \geq 0$ the equation would send v'' and hence v' back upwards. There must be a point $s_0 \in (s_{\max}, s_{\min})$ where $v''(s_0) = 0$. Note $v(s_0) \geq 0$, as v'' can not be decreasing here. Clearly at $s_{\min}$ we have $v''(s_{\min}) \geq 0$ as this is a local minimum.

Thus

$$\begin{aligned} 0 \leq v''(s_{\min}) - v''(s_{\max}) &= \int_{s_{\max}}^{s_{\min}} v'''(s)ds \\ &= \int_{s_{\max}}^{s_{\min}} \sin v(s)ds = \int_{v(s_{\max})}^{v(s_{\min})} \sin v \frac{ds}{dv} dv \\ &= \int_{v(s_{\min})}^{v(s_{\max})} \sin v \left(-\frac{ds}{dv}\right) dv \end{aligned}$$

using a change of variables $s \mapsto v(s)$.

Now let

$$\rho(v) = \left(-\frac{ds}{dv}\right) = -\frac{1}{v'(s)}.$$

Compute

$$\begin{aligned} \frac{d}{dv}\rho(v) &= \frac{d}{ds}\left(\frac{-1}{v'(s)}\right)\frac{ds}{dv} \\ &= \frac{1}{v'(s)^2} v'' \frac{1}{v'(s)} \leq 0 \end{aligned}$$

and strictly negative on an open set. Thus as function of v, ρ is decreasing and we have for any $0 < w < \min\{v_{max}, |v_{min}|\}$

$$\rho(w) < \rho(-w). \tag{3.4}$$

Now for purposes of contradiction assume that

$$v(s_{\max}) \leq |v(s_{\min})|.$$

Then

$$= \int_{v(s_{\min})}^{v(s_{\max})} \sin v\rho(v)dv =$$

$$= \int_{v(s_{\min})}^{-v(s_{\max})} \sin v\rho(v)dv + \int_{-v(s_{\max})}^{v(s_{\max})} \sin v\rho(v)dv$$

with the first term non-positive, so we have so

$$\begin{aligned}
0 &\leq \int_{-v(s_{\max})}^{v(s_{\max})} \sin v\rho(v)dv \\
&= \int_{0}^{v(s_{\max})} \sin v\rho(v) + \int_{v(s_{\max})}^{0} \sin v\rho\,(v)\,dv \\
&= \int_{0}^{v(s_{\max})} \sin v\rho(v) + \int_{0}^{v(s_{\max})} \sin\,(-v)\,\rho\,(-v)\,d(-v) \\
&= \int_{0}^{v(s_{\max})} \sin v\left[\rho(v) - \rho\,(-v)\right] dv < 0
\end{aligned}$$

with the last inequality following from (3.4). This contradicts $v(s_{\max}) \leq |v(s_{\min})|$ and proves the Lemma. □

Noting that an order-reversed proof holds for maximums subsequent to minimums. It follows that if a solution bounces between positive local maximums and negative local minimums, the distance between these must strictly be decreasing. We would like to show it decreases all the way to zero.

Lemma 3.5. *Suppose v is a solution of (3.3) and $K > 0$. Then there is $\varepsilon(K) > 3\delta(K) > 0$ such that if v achieves a local max value in $[K-\delta, K+\delta]$ and a subsequent local minimum if attained, the minimum is at least $-K + \varepsilon(K)$.*

Proof. Solving the initial value problem for

$$\begin{aligned}
v\,(s_0) &= k > 0 \\
v'(s_0) &= 0 \\
v''(s_0) &= b < 0
\end{aligned}$$

we can ask whether the solution will have a negative minimum. Let

$$R = \left\{(k,b) \mid k \in \left[\frac{K}{2}, \frac{3K}{2}\right], b < 0 \text{ and a negative local min occurs for some } s > s_0\right\}.$$

We claim this is compact. Certainly it is bounded: if we choose b large the solution will escape to infinite one way or the other. So we need to show it is closed. Let (k,b) be a value such that v does not attain a future local mininum, in which case v'' escapes to $-\infty$. For some s value, then $|v(s_{100})| = 100$. As this $v\,(s_{100})$ depends continuously on initial conditions, there will be an open set of nearby (k,b) values with $|v(s_{100})| > 99$ for all of these (k,b). Thus R^c is open, and hence R is compact. Because for each $(k,b) \in R$ there is a positive value $\varepsilon(k,b)$ such that

$$v(s_{\min}) = -v\,(s_0) + \varepsilon(k,b)$$

we can find a positive minimum ε over R. Now choose $\delta < \min\left\{\frac{K}{2}, \frac{\varepsilon}{3}\right\}$. □

Proposition 3.6. *Suppose a solution v to (3.3) is bounded for $s > 0$. Then $v(s) \to 0$.*

Proof. Let v be a bounded solution. Either there are infinitely many critical points, or finitely many. If there are finitely many (or none), the solution must either increase or decrease over an infinite interval, so converges to a value. This value can't be anything other than 0; otherwise the third derivatives will be uniformly positive or uniformly negative.

So assume that v has infinitely many critical points. Local maxima must be non-negative and local mins must be non-positive; if not, the equation will force the solution to escape to $\pm\infty$. By Lemma 3.4 these must have strictly decreasing value. We claim the only value they can converge to is 0. If an infinite sequence of local maxima has $k_0 > 0$ as a limit point, apply Lemma 3.5, and obtain a $\delta(k_0)$. Once a maximum falls below $k_0 + \delta(k_0)$ the subsequent minimum is larger than $-k_0 + 2\delta(k_0)$ so the subseqeunt maxima are all less than $k_0 - 2\delta$; thus k_0 cannot be a limit of maxima, a contradiction, unless $k_0 = 0$. □

3.3. **Step 3.** By Lemma 3.3, there exists a value a such that the solution to the IVP exists and is bounded for positive time. By Lemma 3.6 (applied to the translated ODE) the solution must converge to $\pi/2$. Now the initial conditions for the ODE given in (3.2) are odd. The third order equation is even, so we may reflect the solution $u(s) = -u(-s)$ for $s < 0$ to obtain an solution for all s.

Now we need to argue that a solution to (3.1) which is bounded must actually be a solution to (2.2). This by a slight modification of the proof of Lemma 3.4.

Proposition 3.7. *A solution to (3.1) with initial condition (3.2) that remains bounded is also a solutin to (2.2).*

Proof. Observe that if the solution to the (3.1) never leaves the region $[0, \pi]$ for positive s, then it will satisfy both equations. So suffice to show that a bounded solution cannot leave the region $[0, \pi]$. Let u be a bounded solution with initial condition (3.2). We claim first that u cannot reach π without achieving a critical point: Suppose that it does. The solution cannot find an asymptote larger than π because the third derivative will be 1 forever. So in order to stay bounded, u must find a local max, where $u''(s_{\max}) < 0$. Now repeat the same style of argument in proof of Lemma 3.4 (recall $u''(0) = 0$ is an initial condition).

$$\begin{aligned} 0 \geq u''(s_{\max}) - u''(0) &= \int_0^{s_{\max}} f(u)ds \\ &\geq -\int_0^{\pi} \cos(u)\frac{ds}{du}du. \end{aligned} \tag{3.5}$$

As before (with orders reversed) the second derivate must be non-positive on this whole interval, so the function $\frac{ds}{du}$ must be increasing as a function of u and the integral (3.5) must be strictly positive, a contradiction.

This u cannot reach π without finding a local max, so the first local max is strictly less than π. Now as the solution is bounded, this will be followed by a local min, which by the translated reading of Lemma 3.4 must be strictly positive. All subsequent local extrema remain within $[0, \pi]$. $\square$

Corollary 3.8. *There exists a solution to the ODE* (2.2) *for some a which is bounded for all time and converges to $\pi/2$ as $s \to \infty$ and $-\pi/2$ as $s \to -\infty$.*

Proof. A bounded solution exists to (3.1) by Lemma 3.3. By the above Proposition this is a solution to (2.2). Reflecting the solution gives the same properties in the negative direction. $\square$

3.4. **Step 4.** Given a tangent direction $\theta(s)$ we are left to solve the standard ODE

$$\begin{aligned} \frac{d}{ds}\gamma(s) &= F(s) = (\cos\theta(s), \sin\theta(s)) \\ \gamma(0) &= (0,0) \end{aligned}$$

which follows from basic ODE theory. Given that θ solves (2.2), the result will be a curve whose flow will result in translation.

The solution γ is clearly an immersion, the tangent vector is non-vanishing by construction. To see that it must be a proper immersion, note that because $\theta(s) \to \pi/2$ as $s \to \infty$ we can find a value σ_0 such that $\theta(s) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ for $s \geq \sigma_0$. Letting $\gamma(s) = (x(s), y(s))$ we have

$$y(s) - y(\sigma_0) = \int_{\sigma_0}^{s} \sin\theta(s) ds \geq \frac{\sqrt{2}}{2}(s_1 - \sigma_0) \text{ for } s > \sigma_0.$$

Repeating this argument for negative values of s, we see the immersion must be proper. This proves the following.

Theorem 3.9. *There exists a properly immersed translating solution to the curve diffusion flow.*

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