CONTINUUM NASH BARGAINING SOLUTIONS

MICAH WARREN

Abstract. Nash's classical bargaining solution suggests that N players in a non-cooperative bargaining situation should find a solution that maximizes the product of each player's utility functions. We consider a special case: Suppose that the players are chosen from a continuum distribution μ and suppose they are to divide up a resource ν that is also on a continuum. The utility to each player is determined by an integral of the exponential of a distance type function. The maximization problem becomes an optimal transport type problem, where the target density is the minimizer to the functional

$$F(\beta) = H_\nu(\beta) + W^2(\mu, \beta)$$

where $H_\nu(\beta)$ is the entropy and W^2 is the 2-Wasserstein distance. This minimization problem is also solved in the Jordan-Kinderlehrer-Otto scheme. Thanks to optimal transport theory, when the measures are supported on convex regions of the same Euclidean space, the solution may be described by a potential that solves a fourth order nonlinear elliptic PDE, similar to Abreu's equation. Using the PDE, we see solutions are smooth when the measures have smooth positive densities.

1. Introduction

In the 1950's [8] John Nash characterized a solution to the bargaining problem that has since been central to the theory. Namely, the Nash bargaining solution is the allocation that maximizes the product of the utilities to each player, over the total space of possible allocations of a surplus. The Nash bargaining solution is not only mathematically natural but can be achieved by strategic approaches, see [2]. Bargaining solutions represents a class of allocations that are neither centrally planned nor wholly decentralized. For a general overview of the economic theory, see [7].

In this paper, we consider utility functions that are given as an integral of a utility density function, namely

$$U_i(\nu_i) = \int_Y s(p_i, y) d\nu_i(y). \tag{1}$$

Here Y is a surplus space, X is a player space, ν_i is a measure on the space Y that determines the allocation of resources to player $p_i \in X$, and

$$s : X \times Y \to \mathbb{R}^+$$

is a utility density function. More details are found in section 2.

We begin by extending an observation of Schumacher [10] to continua. For utility of the form

$$s(x, y) = e^{-c(x,y)}$$

The author's work was supported in part by the NSF via DMS-1161498.

for a cost function c, Nash bargaining solutions will be optimal transport plans between a measure μ which is known a priori, and some measure β which is determined by the solution. Thus, for the purpose of maximizing the product of the utility functions U_i, we need only to parameterize a space of measures on the resource space, or equivalently, a set of Kantorivich potentials. This allows us to do analysis in the spirit of Benamou-Carlier-Mérigot-Oudat [1], in order to show that discrete solutions have continuous limits.

This paper has two objectives. First, we would like to frame the Nash Bargaining Problem in terms of theory of measures on product spaces. We see that much of theory applicable to the optimal transport problem fits very nicely in this setting.

Second, we demonstate the problem has nice continuous solutions that arise by passing to the limit, by considering the particular case when the spaces $X, Y \subset \mathbb{R}^n$ are both convex compact domains, players are distributed according to a probability measure μ, and surplus is distributed according to a measure ν, both of which are absolutely continuous with respect to Lebesgue measure, and the utility density is given by

$$s(x,y) = e^{-|x-y|^2/2}.$$

Following the arguments in [1], we find that the limit of solutions can be described by a gradient mapping, whose potential satisfies a fourth order PDE. The problem becomes equivalent to minimizing

$$G(\varphi) = \int_X \ln\left(\frac{\det D^2\varphi(x)}{\nu(\nabla\varphi(x))}\right) d\mu(x) + \frac{1}{2}\int_X \|x - \nabla\varphi\|^2 \, d\mu, \tag{2}$$

which gives a fourth order nonlinear PDE similar to Abreu's equation [4]. In particular, the quantity $\det D^2\varphi(x)$ satisfies a second order elliptic equation. By arguments such as in [9, Proposition 8.7] $\det D^2\varphi(x)$ is bounded and Lipshitz continuous; from here we can apply Caffarelli's Schauder theory for Monge-Ampère equations. Smoothness will follow by the general Schauder theory, see Section 5. Abreu's equation involves minimizing a functional of the form (2), where the gradient term is replaced by a integral involving the potential φ. Details on regularity for Abreu's equation can be found in [6].

Our main result (stated roughly for now) is as follows

Theorem 1.1. *Under appropriate conditions on the measures μ, ν and for $c(x,y) = |x-y|^2/2$, solutions to finite player Nash bargaining solutions (say, by sampling points according to μ) converge to a continuum Nash bargaining solution.*

We also have a regularity result:

Theorem 1.2. *Suppose that $X, Y \subset \mathbb{R}^n$ are smooth, bounded, convex domains. Suppose that μ and ν and smooth measure densities on X and Y respectively, that are bounded and bounded away from zero. Then, the minimizers of the functional (2) are smooth.*

2. Setup

Our goal in this section is to describe the bargaining problem in terms of measures.

Suppose that Y is any topological space, and ν is a Borel probability measure on Y. We call the pair (Y, ν) the surplus. When there are N players vying for portions of the surplus, we can think of the space of possible allocations of the surplus as N-tuples of non-negative Borel measures $(\nu_1, ..., \nu_N)$, such that

$$\sum_{i=1}^{N} \nu_i \leq \nu.$$

The function

$$s : X \times Y \to \mathbb{R}^+$$

is called a utility density function. The value $s(x, y)$ gives the utility to player at $x \in X$ for a unit of $y \in Y$. As we will be integrating the density, it makes most sense to assume the utility is linear in terms of a fixed resource y, in the sense that the marginal utility to a player at x of a unit y is determined only by the utility density function, and not by a function of the same or other player's allocations.

Example 2.1. *Suppose two emporers have set up capital cities at points x_1 and x_2 in a region $\Omega \subset \mathbb{R}^2$, and must negotiate how to split the region. Each emporer decides that the value of any unit y of land to their kingdom is given by $e^{-|x_i - y|^2/2}$. The possible allocations of the region are measures ν_1, ν_2 such that*

$$\nu_1 + \nu_2 = dy_{|\Omega}.$$

The utility function to each is

$$U_i(\nu_i) = \int_Y e^{-|x_i - y|^2/2} d\nu_i(y).$$

Notice that in the N-player case, when the utility density is positive, Pareto-optimal solutions are a set of N measures with $\sum_{i=1}^{N} \nu_i = \nu$. If $X = \{p_1, ..., p_n\}$ and $\pi \in P(X \times Y)$ is a probability measure such that $\pi_Y = \nu$ (here $\pi_Y = (\mathrm{Proj}_Y)_\# \pi$, i.e. the right marginal) we may consider the measures $\nu_i = \pi_{|\{p_i\} \times Y}$. In this case, we can define the utility to each player as follows

$$U_i(\pi) = \int_{\{p_i\} \times Y} s(p_i, y) d\pi. \tag{3}$$

The Nash bargaining solution is determined by maximizing the Nash product, namely

$$\mathcal{N} = \prod_{i=1}^{N} U_i.$$

(Note that we are assuming the disagreement point is the 0-allocation, that is, all players get nothing when they do not agree.) Equivalently, one can maximize the logarithm

$$\ln \mathcal{N} = \sum_{i=1}^{N} \ln U_i.$$

With this in mind, for the case of N players, we define the Nash bargaining problem as follows:

Problem 2.2. *Find a measure π that maximizes the functional*

$$F(\pi) = \ln N + \frac{1}{N} \sum_{i=1}^{N} \ln \int_{\{p_i\} \times Y} s(p_i, y) d\pi \tag{4}$$

over the space of measures $\pi \in P(X \times Y)$, under the constraint

$$\pi_Y = \nu.$$

The term $\ln N$ and factor $1/N$ do not affect the arg max, however, they are present for normalization reasons which will become clear when we attempt to build a continuous solution.

2.1. **Reformulating the functional.** Working with (4):

$$\begin{aligned} F(\pi) &= \ln N + \frac{1}{N}\sum_{i=1}^{N}\left[\ln\left(\pi\left(\{p_i\}\times Y\right)\right) + \ln\left(\frac{1}{\pi\left(\{p_i\}\times Y\right)}\int_{\{p_i\}\times Y} s(p_i,y)d\pi\right)\right] \\ &= \ln N + \frac{1}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{\pi\left(\{p_i\}\times Y\right)}{1/N}\right) + \ln(1/N) + \ln\left(\frac{1}{\pi\left(\{p_i\}\times Y\right)}\int_{\{p_i\}\times Y} s(p_i,y)d\pi\right)\right] \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{\pi\left(\{p_i\}\times Y\right)}{1/N}\right) + \ln\left(\frac{1}{\pi\left(\{p_i\}\times Y\right)}\int_{\{p_i\}\times Y} s(p_i,y)d\pi\right)\right]. \end{aligned}$$

At this point, we define a measure on $X = \{p_1, ..., p_N\}$ as

$$\alpha = \pi_X$$

or equivalently,

$$\alpha(p_i) = \pi\left(\{p_i\}\times Y\right).$$

Defining

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{p_i} \tag{5}$$

we get

$$\begin{aligned} F(\pi) &= \frac{1}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{\alpha(p_i)}{\mu_N(p_i)}\right) + \ln\left(\frac{1}{\alpha(p_i)}\int_{\{p_i\}\times Y} s(p_i,y)d\pi\right)\right] \\ &= \int_X \ln\left(\frac{d\alpha}{d\mu_N}\right)\frac{d\mu_N}{d\alpha}d\alpha + \int_X A\mu_N \end{aligned} \tag{6}$$

where

$$A(p_i) = \ln\left(\frac{1}{\alpha(p_i)}\int_{\{p_i\}\times Y} s(p_i,y)d\pi\right).$$

Note that if $\alpha(p_i) = 0$, then $\int \ln\left(\frac{d\alpha}{d\mu}\right)d\mu = -\infty$, in which case $A(p_i)$ may be undefined, but we agree that $F(\pi) = -\infty$.

2.2. **Solutions are Optimal Transport Plans.** (Cf. [10, section 3].) Suppose that π is a maximizer for F, and s is a positive, continuous function. Define

$$c(x,y) = -\ln s(x,y).$$

Recall that a measure $\pi \in P(X \times Y)$, is a solution to the optimal transportation problem pairing the measures μ and ν, when π minimizes

$$\int_{X\times Y} c(x,y)d\pi$$

under the constraint

$$\pi_X = \mu$$
$$\pi_Y = \nu.$$

By Kantorovich duality, we have that, [11, Theorem 5.10 (iii)]

$$\min_{\pi\in P(X\times Y)} \int c(x,y)d\pi = \max_{\varphi\in L^1(X,\alpha_N)} \int_Y \varphi^c(y)d\nu - \int_X \varphi(x)d\mu.$$

where φ^c is the "cost-transform" of φ. This is a general fact. For many cost functions of interest (for example $c(x,y) = |x-y|^2/2$) the functions φ^c and φ are determined uniquely up to a constant.

Proposition 2.3. *The measure π is an optimal plan, pairing α and ν.*

Proof. We proceed as in [11, proof of Theorem 5.10, step 3 on page 65]. We show that π is c-cyclically monotone.

Suppose that

$$\{(x_1,y_1),...,(x_n,y_n)\} \subset \mathrm{Supp}(\pi).$$

We would like to show that

$$\sum_{i=1}^{N} c(x_i,y_i) \leq \sum_{i=1}^{N} c(x_i,y_{i-1})$$

(using convention that $y_0 = y_N$). Moving a bit of mass from (x_1,y_1) to (x_2,y_1) will preserve the right marginal condition, but cannot increase the functional (4), by maximality. Because $(x_1,y_1)\in \mathrm{Supp}(\pi)$, there is some mass available to move. Choose an arbitrarily small set B_ε near (x_1,y_1) in y and consider the competing family of measures for $t\in[0,1]$:

$$\pi(t) = \{\tilde{\nu}_1(t), \tilde{\nu}_2(t), \nu_3, ..., \nu_N\}$$
$$\tilde{\nu}_1 = \nu_1 - t\nu_1|_{B_\varepsilon}$$
$$\tilde{\nu}_2 = \nu_2 + t\nu_1|_{B_\varepsilon}.$$

Clearly $\pi(t)$ is a path of admissible measures, so we can take a one-sided derivative of (4) with respect to t :

$$0 \geq \frac{dF}{dt}|_{t=0} = \frac{1}{N}\left[\frac{-\int_{\{p_1\}\times B_\varepsilon} s(x_1,y)d\nu_1}{\int_{\{p_1\}\times Y} s(x_1,y)d\nu_1} + \frac{\int_{\{p_1\}\times B_\varepsilon} s(x_2,y)d\nu_1}{\int_{\{p_2\}\times Y} s(x_2,y)d\nu_2}\right].$$

That is

$$0 \geq \left[\frac{-\frac{1}{\nu_1(B_\varepsilon)}\int_{\{p_1\}\times B_\varepsilon} s(x_1,y_1)d\nu_1}{\int_{\{p_1\}\times Y} s(x_1,y)d\nu_1} + \frac{\frac{1}{\nu_1(B_\varepsilon)}\int_{\{p_1\}\times B_\varepsilon} s(x_2,y_1)d\nu_1}{\int_{\{p_2\}\times Y} s(x_2,y)d\nu_2}\right].$$

Choosing B_ε small, since s is continuous, the average values in the numerator must converge to point values, and in the limit we see

$$\frac{s(x_1,y_1)}{\int_{\{p_1\}\times Y} s(x_1,y)d\nu_1} \geq \frac{s(x_2,y_1)}{\int_{\{p_2\}\times Y} s(x_2,y)d\nu_2}. \tag{7}$$

Then we have

$$\ln s(x_1,y_1) - \ln s(x_2,y_1) \geq \ln \int_{\{p_1\}\times Y} s(x_1,y)d\nu_1 - \ln \int_{\{p_2\}\times Y} s(x_2,y)d\nu_2$$

or

$$c(x_1, y_1) - c(x_2, y_1) \leq \ln \kappa_1 - \ln \kappa_2.$$

By relabeling,

$$c(x_2, y_2) - c(x_3, y_2) \leq \ln \kappa_2 - \ln \kappa_3,$$
$$\ldots$$
$$c(x_N, y_N) - c(x_1, y_N) \leq \ln \kappa_N - \ln \kappa_1.$$

Summing, we have

$$\sum_{i=1}^{N} c(x_i, y_i) - \sum_{i=1}^{N} c(x_i, y_{i-1}) \leq 0.$$

□

Corollary 2.4. *Suppose that*

$$s(x, y) = e^{-|x-y|^2/2}$$

and $X, Y \subset \mathbb{R}^n$. *Then the solution maximizing (4) can be described as*

$$\nu_i = \nu_{|E_i}$$

where each E_i *is a Laguerre cell, defined as the subgradient of a convex function* φ *at point* x_i.

In particular solutions to this type of bargaining solution are pure, in the sense that a generic resource will not be split between two players.

We won't prove this directly, but refer the reader to [11, Chapter 5] and [1, Section 2]. We include a condensed recap of [1, Section 2] which will help our discussion moving forward: Suppose $P \subset \mathbb{R}^n$ is a finite set of points, and $Y \subset \mathbb{R}^n$. Given a function φ defined on P, define

$$\varphi_{\mathcal{K}_Y} = \max \left\{ \psi \in \mathcal{K}_Y ; \psi_{|P} \leq \varphi_{|P} \right\}$$

where

$$\mathcal{K}_Y = \left\{ \psi^* ; \psi : Y \to \bar{\mathbb{R}} \right\}$$

is the set of functions which are Legendre-Fenchel transforms of functions on Y, which are necessarily convex. Define

$$\mathcal{K}_Y(P) = \{ \varphi : P \to \mathbb{R} ; \varphi = \varphi_{\mathcal{K}_Y}|_P \}$$

and

$$Lag_P^{\varphi}(p) := \{ y \in \mathbb{R}^n ; \forall q \in P, \varphi(q) \geq \varphi(p) + \langle q - p | y \rangle \}.$$

For normalization, we define

$$\mathcal{K}_Y(P)_0 = \{ \varphi \in \mathcal{K}_Y(P) ; \min \varphi = 0 \}.$$

Lemma 2.5. [1, Lemma 2.2] *Let* P *be a finite point set. A function* φ *on* P *belongs to* $\mathcal{K}_Y(P)$ *if and only if for every* p *in* P, *the intersection* $Lag_P^{\varphi}(p) \cap Y$ *is non-empty. Moreover, if this is the case, then*

$$\partial \varphi_{\mathcal{K}}(p) = Lag_P^{\varphi}(p) \cap Y.$$

2.3. **Reformulating the problem again.** Now that we have established that the solution must be an optimal transport plan, we may formulate the problem over the space of optimal transport plans. First, we need

Lemma 2.6. *A maximizing solution to (4) exists, and is unique.*

Proof. The constraint $\pi_Y = \nu$ is linear on the set of nonnegative probability measures, which form a compact convex set. For each i, the functional

$$f_i = \int_{\{p_i\}\times Y} s(p_i, y)d\pi$$

is clearly linear in the measure π. It follows that the sum of logarithms is strictly concave. A concave function on a convex compact set achieves its maximum value, which is unique by strict concavity. $\square$

It is clear that the maximizing measure must be supported on $P \times Y$: Moving any mass from a point not in P to a point on P will increase the value of F. By Kantorovich duality, the optimal transport plans with target ν can be parameterized by convex functions on the set P, which is just a set of N values. The functional can be expressed as follows.

$$\tilde{F}_N(\varphi) = \frac{1}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{\nu(E_i)}{\mu_N\left(p_i\right)}\right) + \ln\left(\frac{1}{\nu(E_i)}\int_{E_i} s(p_i, y)d\nu\right)\right] \tag{8}$$

where

$$E_i = Lag_P^{\varphi}(p_i).$$

We now offer a second formulation of the problem.

Problem 2.7. [Nash bargaining problem, Version 2] *: Maximize (8) over the space of functions $\mathcal{K}_Y(P)_0$.*

2.4. **Absolutely continuous pushforward measures.** Given a potential function $\varphi \in \mathcal{K}_Y(P)_0$, we have a Laguerre decomposition of Y. The subgradient map is set-valued, so one cannot define the pushforward directly. However, following [1], if we have chosen a background measure ν, we can "average" over the Laguerre cell to define an absolutely continuous pushforward measure.

Given any decomposition of Y into cells E_i, each with positive measure, define the following probability measure on Y : For measurable $Z \subset Y$,

$$\beta_\varphi(Z) = \frac{1}{N}\sum_{i=1}^{N}\frac{\nu(E_i\cap Z)}{\nu(E_i)}. \tag{9}$$

In the measure π, which is optimal between α and ν, each cell $\{p_i\}\times E_i$ has measure $\nu(E_i)$. By multiplying each piece by

$$\frac{1}{N}/\nu(E_i)$$

we get a new measure, π', which has the same support as π, but now has marginals

$$(\pi')_X = \mu_N$$
$$(\pi')_Y = \beta_\varphi.$$

This means that the set-valued mapping induced by the subgradient of φ is an optimal transport mapping not only between α and ν , but also between μ_N and β_φ.

3. Limits of finite source solutions

Our next goal is to take a limit of solutions, when the set of points is drawn from a distribution μ and the number of points becomes infinite. In the following we assume that $X, Y \subset \mathbb{R}^n$ are both bounded convex regions, and that μ, ν are probability measures on X, Y respectively, each of which are absolutely continuous with respect to Lebesgue measure, with densities bounded away from zero. For each N, choose a set of points P_N from X and define the measure

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{p_i}.$$

In the sequel we will assume that $\mu_N \to \mu$ weakly on X. By arguments in the previous section, for each N we have

$$\begin{aligned} \pi_N &\in P(X \times Y) \\ \varphi_N &\in \mathcal{K}_Y(P)_0 \\ \alpha_N &\in P(X) \\ \beta_{\varphi_N} &\in P(Y) \end{aligned} \tag{10}$$

each of which are unique and can be used to identify the solution. We will see that when μ, ν are nice measures, all four of the objects (10) converge to limiting objects, respectively, each of which uniquely defines a solution on the continuous space $X \times Y$.

Inspecting (6), note the following. The first term is the relative entropy of a known measure μ with respect to a measure α, to be determined. The second term is the integral of the natural log of the average of an exponential function, over a small cell. As the size of the cells becomes smaller, one expects the average to recover the value, and the second term will become the negative total cost of the mass transport plan between μ and its image β under the mapping, which is also to be determined. Under changes of measures

$$\begin{aligned} \mu &\to \beta \\ \alpha &\to \nu \end{aligned}$$

the first term becomes the negative relative entropy of β with respect to the measure ν. Thus we may formulate the problem by trying to minimize the following function over the set of probability measures on Y .

$$\hat{F} : P(Y) \to \mathbb{R}^-$$

$$\hat{F}(\cdot) = -H_v(\cdot) - W^2(\mu, \cdot). \tag{11}$$

The concave functional $\hat{F}$ will have a unique maximizer on $P(Y)$. Define

$$\hat{\beta} = \arg\max \hat{F}(\cdot)$$

and choose $\hat{\varphi}$ such that

$$\hat{\beta} = (\nabla \hat{\varphi})_{\#}\, \mu.$$

Our main theorem is the following.

Theorem 3.1. *Let π_N be a sequence of maximizers to (4), and let β_N be the associated absolutely continuous right marginals (9), Then*

$$\hat{F}(\beta_N) \to \hat{F}(\hat{\beta}).$$

In particular, because $\hat{F}$ is strictly concave,

$$\beta_N \to \hat{\beta}.$$

In this case, choose $\hat{\varphi}$ such that

$$\hat{\beta} = (\nabla\hat{\varphi})_{\#}\,\mu.$$

We define $\hat{\varphi}$ as the potential solution, $\nabla\hat{\varphi}$ as the map solution, and the measure $\pi = (I \times \nabla\hat{\varphi})_{\#}\,\mu \in P(X \times Y)$ as the measure solution of the Nash bargaining problem.

3.1. **Outline of Proof.** We outline four steps, and then combine these in step 5 to get the proof. The detailed proof of steps 1-4 will appear in section 4.

Note that this result almost follows from [1, Theorem 4.1]. Instead of a pure Wasserstein term, however, in our problem we have a term that *should* converge to a Wasserstein term, provided the measures concentrate on the graph of a map. So we must justify that the maximimizers of our finite problem are indeed concentrating on the graph of map. We reproduce some, but not all of the proof.

Step 1. The functional $\hat{F}$ has maximizer, $\hat{\beta}$. The density $\frac{d\hat{\beta}}{d\nu}$ is Lipschitz continuous.

This is essentially [9, Propostion 7.32, Proposition 8.7] applied when $\tau = 1$

Step 2. There is a sequence of optimal transportation plans pairing μ_N with $\hat{\beta}$. To these we can associate a function $\hat{\varphi}_N$ and an absolutely continuous measure $\hat{\beta}_{N.}$ Then $\hat{\beta}_N \to \hat{\beta}$ and $\hat{F}(\hat{\beta}_N) \to F(\hat{\beta})$. The associated Laguerre cells have diameters which go uniformly to 0.

Step 3. For the maximizers φ_N of the finite problem, we can also associate an absolutely continuous measure β_N (16). The limits of both exist, and the diameters of the associated Laguerre cells go uniformly to 0.

Step 4. Because the diameters of the Laguerre cells go to zero, the second term in the $\tilde{F}_N$ functional (8) (defined on φ_N or $\hat{\varphi}_N$) converges to the second term in in the $\hat{F}$ functional (defined on the associated β_N). That is

$$\left| \frac{1}{N} \sum_{i=1}^{N} \ln\left(\frac{1}{\nu(E_i)} \int_{E_i} s(p_i, y)\, d\nu(y) \right) - W^2(\mu_N, \beta_N) \right| \to 0. \tag{12}$$

It follows that

$$\left| \hat{F}(\beta_N) - \tilde{F}_N(\varphi_N) \right| \to 0, \tag{13}$$

$$\left| \hat{F}(\hat{\beta}_N) - \tilde{F}_N(\hat{\varphi}_N) \right| \to 0. \tag{14}$$

Step 5. **Proof of Theorem.** Choose $\varepsilon > 0$: Because we have chosen φ_N as a maximizer for (8) we have

$$\tilde{F}_N(\varphi_N) \geq \tilde{F}_N(\hat{\varphi}_N). \tag{15}$$

Thus for N large depending on ε,

$$\begin{aligned}\hat{F}\left(\beta_N\right) &\geq \tilde{F}_N(\varphi_N) - \varepsilon \text{ by (13)} \\ &\geq \tilde{F}_N(\hat{\varphi}_N) - \varepsilon, \text{ by (15)} \\ &\geq \hat{F}\left(\hat{\beta}_N\right) - 2\varepsilon, \text{by (14)} \\ &\geq \hat{F}\left(\hat{\beta}\right) - 3\varepsilon, \text{ by step 2.}\end{aligned}$$

Thus,

$$\lim_{N\to\infty} \hat{F}\left(\beta_N\right) \geq \hat{F}\left(\hat{\beta}\right).$$

But $\hat{F}\left(\hat{\beta}\right)$ is the maximum. By strict concavity, it follows that $\beta_N \to \hat{\beta}$. Thus $\beta = \hat{\beta}$.

4. Proof details

Unless otherwise specified, c will refer to an arbitrary cost function

$$c : X \times Y \to \mathbb{R}$$

and W_c will refer to the Wasserstein distance, given the cost c. The cost c is assumed to have global bounds and global Lipschitz bounds. The results of Step 1 should hold for general cost c, whereas the results in Step 2 and 3 rely on Caffarelli's regularity theory for $c = |x-y|^2/2$.

4.1. **Step 2.**

This is essentially [1, Steps 4 and 5, section 4]. In short: by Brenier's Theorem, we may find ϕ_N such that $\nabla\phi_N\hat{\beta} = \mu_N$. Because $\mu_N \to \mu$, the potentials ϕ_N converge uniformly. It follows that the Legendre transforms $\hat{\varphi}_N$ converge uniformly as well, to $\hat{\varphi}$. If the diameter of the Laguerre cells does not shrink to zero, there will be a line segment on which the convex potential $\hat{\varphi}$ is linear. This contradicts Caffarelli's regularity result [3], which ensures that the potential defining the mapping between the absolutely continuous measures μ and $\hat{\beta}$ must be strictly convex.

Each ϕ_N defines a set of Laguerre cells $\{E_i\}$. We can define a measure:

$$\hat{\beta}_N(Z) = \frac{1}{N}\sum_{i=1}^{N}\frac{\nu(E_i \cap Z)}{\nu(E_i)}$$

which has the property that

$$\nabla\phi_N\hat{\beta}_N = \mu_N.$$

The densities $\frac{d\hat{\beta}_N}{d\nu}$ will converge pointwise uniformly to $\frac{d\hat{\beta}}{d\nu}$, because the density $\frac{d\hat{\beta}_N}{d\nu}(y)$ is simply the average of the continuous density of $\frac{d\hat{\beta}}{d\nu}$ over the cell containing y, and the cell diameters are shrinking to zero. It follows quickly that $\hat{F}(\hat{\beta}_N) \to \hat{F}\left(\hat{\beta}\right)$.

4.2. **Step 3.** For each finite player Nash bargaining problem, we choose φ_N maximizing (8), and obtain a set of Laguerre cells $\{E_i\}$. As before, we construct a probability measure on Y

$$\beta_N(Z) = \frac{1}{N}\sum_{i=1}^{N}\frac{\nu(E_i \cap Z)}{\nu(E_i)}. \tag{16}$$

We would like to conclude these $\beta_N \to \beta$, but we do not yet have a unique limit β. We need to use properties of β_N to conclude properties of possible limits β.

Claim 4.1. *The measures α_N are uniformly mutually absolutely continuous with respect to μ_N.*

Proof. Proceeding as in the proof of Proposition 2.3, we have

$$\frac{\int_{B_\varepsilon} s(p_1,y)d\nu}{\int_{E_1} s(p_1,y)d\nu} \geq \frac{\int_{B_\varepsilon} s(p_2,y)d\nu}{\int_{E_2} s(p_2,y)d\nu}$$

or

$$\frac{\int_{E_2} s(p_2,y)d\nu}{\int_{E_1} s(p_1,y)d\nu} \geq \frac{\int_{B_\varepsilon} s(p_2,y)d\nu}{\int_{B_\varepsilon} s(p_1,y)d\nu_1} \geq \frac{\min s(x,y)}{\max s(x,y)} > 0.$$

Similarly,

$$\frac{\int_{E_1} s(p_1,y)d\nu}{\int_{E_2} s(p_2,y)d\nu} \geq \frac{\min s(x,y)}{\max s(x,y)} > 0$$

so also

$$\frac{\nu(E_1)}{\nu(E_2)} \geq \frac{\frac{1}{\max s(x,y)}\int_Y s(p_1,y)d\nu_2}{\frac{1}{\min s(x,y)}\int_Y s(p_2,y)d\nu_1} > \left[\frac{\min s(x,y)}{\max s(x,y)}\right]^2 = a_0 > 0.$$

This will be true for any pair, so the ratios of the measures is uniformly bounded:

$$\frac{a_0}{a_0+N-1} \leq \alpha(p_i) \leq \frac{1}{1+(N-1)a_0}. \tag{17}$$

□

Claim 4.2. *The measures α_N have a weak limit α which is absolutely continuous with respect to μ. The density $d\alpha/d\mu$ is bounded away from zero.*

Proof. On the compact space X, the Wasserstein metric is compact, and equivalent to weak topology, so there is a weak limit α. A straightforward argument in the spirit of Littlewood's principles using (17) shows that for all measurable E,

$$a_0\mu(E) \leq \alpha\,(E) \leq \frac{1}{a_0}\mu(E).$$

□

Next we consider the convergence of φ_N. In order to extend these to X, we denote

$$\bar{\varphi}_N = \max\left\{\psi \in \mathcal{K}_Y : \psi_{|_P} \leq \varphi_{N|_P}\right\}.$$

Claim 4.3. *The functions $\bar{\varphi}_N$ converge uniformly on compact subsets of X.*

Proof. This follows from the fact that $(\mathcal{K}_Y)_0$ is compact: The space of convex functions with subgradients in a bounded set is compact up to addition of a constant. □

Claim 4.4. *Let* $\varphi = \lim \bar{\varphi}_N$. *Then*

$$(\nabla\varphi)_{\#}\,\alpha = \nu.$$

Proof. We consider the dual problem (cf. [11, Theorem 5.10 (iii)]). For each N, we have

$$\min_{\substack{\pi\in P(X\times Y)\\ \pi_X=\alpha_N\\ \pi_Y=\nu}} \int c(x,y)d\pi = \max_{\varphi\in L^1(X,\alpha_N)} \int_Y \varphi^c(y)d\nu - \int_X \varphi(x)d\alpha_N,$$

which is realized by

$$\int c(x,y)d\pi_N = \int_Y \varphi_N^c(y)d\nu - \int_X \varphi_N(x)d\alpha_N.$$

Now the set of cost-transpose functions $\{\varphi^c; \varphi : X \to \mathbb{R}\}$ is also compact up to addition of a constant. Thus the functions φ_N^c converge uniformly as well. Taking all limits and letting π be the weak limit of π_N, we have

$$\int c(x,y)d\pi = \int_Y \varphi^c(y)d\nu - \int_X \varphi(x)d\alpha.$$

By duality, it follows that φ describes the optimal map between α and ν. In particular, for $c = |x-y|^2/2$ we conclude that $(\nabla\varphi)_{\#}\,\alpha = \nu$. $\square$

Now, we simply define

$$\beta := (\nabla\varphi)_{\#}\,\mu. \tag{18}$$

Despite knowing less about the regularity of β than we did for $\hat{\beta}$ in Step 2, we may repeat the essential portion of the argument found in [1]. Caffarelli's strict convexity result [3] only requires the densities are bounded and bounded away from zero, which is true by Claim 4.1 for α and ν. We conclude that the potential φ is strictly convex and that the Laguerre cells associated to φ_N must have vanishing diameters.

4.3. **Step 4.** For either φ_N or $\hat{\varphi}_N$, note that for each cell E_i the following holds for p_i and any $y_i \in E_i$:

$$\left| s(p_i,y_i) - \frac{1}{\nu(E_i)} \int_{E_i} s(p_i,y)d\nu(y) \right| \leq \mathrm{osc}_{y\in E_i}\ s(p_i,y).$$

The average value of the cost over any set must be larger than

$$b_0 = e^{-\max_{X\times Y} c(x,y)}.$$

It follows immediately by the fundamental theorem of calculus that

$$\left| \ln\left(s(p_i,y_i) - \frac{1}{\nu(E_i)} \int_{E_i} s(p_i,y)d\nu(y) \right) - \ln s(p_i,y_i) \right| \leq \frac{1}{b_0}\mathrm{osc}_{y\in E_i}\ s(p_i,y).$$

Now for any set of choices of $\{y_i \in E_i\}$

(19)

$$\left| \frac{1}{N}\sum_{i=1}^N \ln\left(\frac{1}{\nu(E_i)} \int_{E_i} s(p_i,y)d\nu(y) \right) - \frac{1}{N}\sum_{i=1}^N \ln s(p_i,y_i) \right| \leq \frac{1}{N}\sum_{i=1}^N \frac{1}{b_0}\mathrm{osc}_{y\in E_i}\ s(p_i,y).$$

The utility function is continuous, so as the diameters of E_i shrink, so does the right hand side of (19). On the other hand

$$\frac{1}{N}\sum_{i=1}^{N}\ln s(p_i,y_i) = -\int_X c(x_i,y_i)d\mu_N.$$

The cost function is continuous, and the values y_i are being chosen from the subgradient of $\nabla\varphi_N(p_i)$, and $\mu_N \to \mu$, so we conclude

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\ln\left(\frac{1}{\nu(E_i)}\int_{E_i}s(p_i,y)d\nu\right) = -\int_X c(x,\nabla\varphi(x))d\mu(x). \tag{20}$$

This proves the first claim in Step 4.

Next, note that

$$\begin{aligned}
-H_\nu(\beta_N) &= -\sum_{i=1}^{N}\int_{E_i}\ln\left(\frac{1}{N}/\nu(E_i)\right)\frac{1}{N}/\nu(E_i)d\nu \\
&= -\sum_{i=1}^{N}\ln\left(\frac{1}{N}/\nu(E_i)\right)\frac{1}{N} \\
&= \frac{1}{N}\sum_{i=1}^{N}\ln\left(\frac{\nu(E_i)}{\mu_N(p_i)}\right). \qquad (21)
\end{aligned}$$

Thus, using (20),(18), recalling (11), and (8)

$$\begin{aligned}
\lim_{N\to\infty}\left(\tilde{F}_N(\varphi_N)-\hat{F}(\beta_N)\right) &= \lim_{N\to\infty}\left(\frac{1}{N}\ln\left(\frac{1}{\nu(E_i)}\int_{E_i}s(p_i,y)d\nu\right)+W^2(\mu,\beta_N)\right) \\
&= -\int c(x,\nabla\varphi(x))d\mu + \lim_{N\to\infty}W^2(\mu,\beta_N) \\
&= -W^2(\mu,(\nabla\varphi)_\#\,\mu)+W^2(\mu,\beta) \\
&= 0.
\end{aligned}$$

This proves (13). A nearly identical argument proves (14).

5. A Fourth Order PDE and smoothness of solutions

In this section we observe that the function φ satisfies an elliptic quasilinear fourth order PDE and enjoys derivative estimates of all orders.

Given a smooth probability measure μ on X, one can parameterize the space of probability measures on Y by convex potentials φ on $X:$ For any β, we can solve the optimal transportation problem pairing μ with β, obtaining φ such that

$$\beta = (\nabla\varphi)_\#\,\mu. \tag{22}$$

On the other hand, for any convex φ the subgradient mapping defines a probability measure via (22). Using this, we can insert φ into the functional $\hat{F}:$

$$\begin{aligned}
\hat{F}(\varphi) &= -\int_Y\ln\left(\frac{d\beta}{d\nu}(y)\right)\frac{d\beta}{d\nu}(y)d\nu(y)-\frac{1}{2}\int_X\|x-\nabla\varphi(x)\|^2\,d\mu(x). \\
&= -\int_Y\ln\left(\frac{d\beta}{d\nu}(y)\right)d\beta(y)-\frac{1}{2}\int_X\|x-\nabla\varphi(x)\|^2\,d\mu(x).
\end{aligned}$$

Now $\nabla\varphi$ is a change of measure, so

$$\begin{aligned}\hat{F}(\varphi) &= -\int_X \ln\left(\frac{d\beta}{d\nu}\left(\nabla\varphi(x)\right)\right)d\mu(x) - \frac{1}{2}\int_X \left\|x-\nabla\varphi(x)\right\|^2 d\mu(x)\\ &= -\int_X \ln\left(\frac{\mu(x)}{\det(D^2\varphi(x))\nu(\nabla\varphi(x))}\right)d\mu(x) - \frac{1}{2}\int_X \left\|x-\nabla\varphi(x)\right\|^2 d\mu(x).\\ &= -\int_X \left[\ln\mu(x) - \ln\det(D^2\varphi(x)) - \ln\nu(\nabla\varphi(x))\right]d\mu(x) - \frac{1}{2}\int_X \left\|x-\nabla\varphi(x)\right\|^2 d\mu(x).\end{aligned}$$

Now consider a compactly supported variation:

$$\varphi_t = \varphi(x) + t\eta(x)$$

for some compactly supported smooth test function η. We compute

$$\begin{aligned}\frac{d\hat{F}(\varphi_t)}{dt}|_{t=0} &= -\int_X \left[-\varphi^{ij}\eta_{ij} - \frac{1}{\nu(\nabla\varphi)}\nabla\nu(\nabla\varphi)\cdot\nabla\eta\right]d\mu - \int_X (x-\nabla\varphi)\cdot\nabla\eta d\mu\\ &= \int_X \left\{\partial_i\partial_j\left(\mu\varphi^{ij}\right) + \operatorname{div}\left(-\mu\frac{1}{\nu(\nabla\varphi)}\nabla\nu(\nabla\varphi) + \mu\left(x-\nabla\varphi\right)\right)\right\}\eta dx.\end{aligned}$$

Here φ^{ij} is the inverse of the Hessian matrix φ_{ij}. Now if the measure $\beta = (\nabla\varphi)_{\#}\,\mu$ is a maximizer, any compactly supported variation will not change the functional to first order, so we have an Euler-Lagrange equation:

$$\partial_i\partial_j\left(\mu\varphi^{ij}\right) + \operatorname{div}\left(\mu\frac{1}{\nu(\nabla\varphi(x))}\nabla\nu(\nabla\varphi(x)) + \mu\left(x-\nabla\varphi\right)\right) = 0.$$

This becomes an equation on $\det(D^2\varphi(x))$: Write the first term as

$$\partial_i\partial_j\left(\mu\varphi^{ij}\right) = \partial_i\partial_j\left(\mu\frac{C_\varphi^{ij}}{\det D^2\varphi(x)}\right)$$

where

$$C_\varphi^{ij} = \det D^2\varphi(x)\varphi^{ij}$$

is the (divergence-free) cofactor matrix. We see

$$\partial_i\partial_j\left(\mu\varphi^{ij}\right) = L\left(\frac{\mu(x)}{\det D^2\varphi(x)}\right)$$

where

$$L = C^{ij}\partial_i\partial_j$$

the equation becomes

$$L\left(\frac{\mu(x)}{\det D^2\varphi(x)}\right) = \operatorname{div}\left(\mu(x)\frac{1}{\nu(\nabla\varphi(x))}\nabla\nu(\nabla\varphi(x)) - \mu(x)\left(x-\nabla\varphi(x)\right)\right). \tag{23}$$

Now the density $\frac{d\hat{\beta}}{d\nu}$ must be Lipschitz continuous [9, Prop. 8.7]. It follows that the potential φ satisfying

$$\det(D^2\varphi(x)) = \frac{\mu(x)}{\hat{\beta}(\nabla\varphi(x))}$$

will be $C^{2,\alpha}$, for any $\alpha<1$, with estimates on any interior set [3]. In particular, the cofactor matrix defining L is uniformly elliptic. Thus the equation (23) is uniformly elliptic with Hölder coefficients. Also note that because $\varphi\in C^{2,\alpha}$, the right hand side of (23) is $C^{0,\alpha}$. Thus we can apply the classical Schauder theory

[5, Theorem 6.19] and conclude that $\det D^2\varphi(x)$ is itself $C^{2,\alpha}$. Repeating these two steps gives estimates of arbitrarily high order.

References

[1] Jean-David Benamou, Guillaume Carlier, Quentin Mérigot, and Édouard Oudet. Discretization of functionals involving the Monge-Ampère operator. *Numer. Math.*, 134(3):611–636, 2016.

[2] Ken Binmore, Ariel Rubinstein, and Asher Wolinsky. The Nash bargaining solution in economic modelling. *Rand J. Econom.*, 17(2):176–188, 1986.

[3] Luis A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.*, 5(1):99–104, 1992.

[4] S. K. Donaldson. Interior estimates for solutions of Abreu's equation. *Collect. Math.*, 56(2):103–142, 2005.

[5] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[6] Nam Q. Le. $W^{4,p}$ solution to the second boundary value problem of the prescribed affine mean curvature and Abreu's equations. *J. Differential Equations*, 260(5):4285–4300, 2016.

[7] Abhinay Muthoo. *Bargaining Theory with Applications*. Cambridge University Press, New York, NY, USA, 1999.

[8] John F. Nash, Jr. The bargaining problem. *Econometrica*, 18:155–162, 1950.

[9] Filippo Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[10] Johannes M. Schumacher. A multi-objective interpretation of optimal transport. *J. Optim. Theory Appl.*, 176(1):94–119, January 2018.

[11] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

University of Oregon, Fenton Hall, University of Oregon, Eugene, OR 97403
E-mail address: micahw@uoregon.edu