A LIOUVILLE PROPERTY FOR GRADIENT GRAPHS AND A BERNSTEIN PROBLEM FOR HAMILTONIAN STATIONARY EQUATIONS.

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Abstract. Using a rotation of Yuan, we observe that the gradient graph of any semi-convex function is a Liouville manifold, that is, does not admit non-constant bounded harmonic functions. As a corollary, we find that any solution of the fourth order Hamiltonian stationary equation satisfying

$$\theta \geq (n-2)\frac{\pi}{2}+\delta$$

for some $\delta > 0$ must be a quadratic.

In this short note, we record the following.

Theorem 1. *Suppose that u is semi-convex. Then the gradient graph*

$$\Gamma = \{(x, Du(x)) : x \in \mathbb{R}^n\}$$

with the induced submanifold metric, is a Liouville manifold.

Recall that a manifold has the Liouville property if all bounded harmonic functions are constant. A rotation of Yuan [Yua02] allows us to write the Laplace operator as a uniformly elliptic divergence operator. The result then follows readily from the De-Giorgi-Nash-Moser theory.

We are interested in studying a fourth order special Lagrangian type equation. Let λ_i be the eigenvalues of the Hessian D^2u. The lagrangian phase is given by

$$\theta = \sum_{i=1}^{n} \arctan \lambda_i.$$

We extend the following generalization of theorems of Yuan [Yua02][Yua06].

Theorem 2. *Let g be the metric induced on $\Gamma = (x, Du(x))$. Suppose that u is an entire solution of the fourth order Hamiltonian stationary equation*

$$\Delta_g \theta = 0 \tag{0.1}$$

with

$$|\theta(u)| > (n-2)\frac{\pi}{2}+\delta \tag{0.2}$$

or

$$D^2u \geq 0.$$

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Then u is a quadratic function, that is, Γ is a plane.

For a fixed bounded domain $\Omega \subset \mathbb{R}^n$ consider the volume functional for the gradient graph $\Gamma = \{(x, Du(x)) : x \in \mathbb{R}^n\}$:

$$F_\Omega(u) = \int_\Omega \sqrt{\det\left(I + \left(D^2u\right)^T D^2u\right)} dx.$$

A function u is critical for $F_\Omega(u)$ under compactly supported variations of the scalar function if and only if u satisfies the equation (0.1) c.f [Oh93] [SW03, Proposition 2.2]. In other words, the gradient graph of u has smallest volume compared with other gradient graphs. Recall that if u satisfies the special Lagrangian equation [HL82]

$$D\theta = 0 \tag{0.3}$$

then the surface is critical under *all* compactly supported variations of the surface Γ.

The Liouville property, together with (0.2) will force θ to satisfy (0.3), so Γ is a minimal surface. It then follows immediatly from a result of Yuan that Γ is a plane. For Bernstein results for (0.1) with a volume growth constraint, and more discussion of the problem, see [Mes01].

1. Preliminary: Lewy-Yuan rotations

In this section we will discuss the Lewy-Yuan rotation from [Yua06, section 2] in detail, as it will be essential to the proof.

Heuristically, the Lewy-Yuan rotation is a change of coordinates on the gradient graph, $(x, Du(x))$ resulting from a unitary rotation of $\mathbb{C}^n$. That is, given an angle $\beta > 0$, consider the unitary map

$$e^{i\beta} I_n : \mathbb{C}^n \to \mathbb{C}^n$$

explicitly given by

$$\begin{aligned} \bar{x} &= \cos(\beta)\, x + \sin(\beta)\, y \\ \bar{y} &= -\sin(\beta)\, x + \cos(\beta)\, y. \end{aligned} \tag{1.1}$$

Now suppose u is a semi-convex function on some domain Ω, satisfying, for some $\delta > 0$

$$D^2u \geq \tan\left(\beta + \delta - \frac{\pi}{2}\right) \tag{1.2}$$

and consider the gradient graph $\Gamma = (x, Du(x))$ as a Riemannian manifold with metric induced from $\mathbb{C}^n$. The map

$$\bar{x}(x) = \cos(\beta)\, x + \sin(\beta)\, Du(x)$$

satisfies

$$\begin{aligned}\frac{d\bar{x}}{dx} &= \cos(\beta)\, I + \sin(\beta)\, D^2u(x) \\ &\geq \cos(\beta)\, I + \sin(\beta)\tan\left(\beta + \delta - \frac{\pi}{2}\right) I \\ &= \cos(\beta)\left[1 + \tan(\beta)\tan\left(\beta + \delta - \frac{\pi}{2}\right)\right] I \\ &= \cos(\beta)\frac{\tan\beta - \tan\left(\beta + \delta - \frac{\pi}{2}\right)}{\tan(\frac{\pi}{2} - \delta)} I \geq C_{\beta,\delta} I > 0\end{aligned}$$

where in the last line we have used the standard formula

$$1 + \tan A \tan B = \frac{\tan A - \tan B}{\tan(A - B)}. \tag{1.3}$$

It follows that

$$\frac{d\bar{x}}{dx} \geq C_{\beta,\delta} I > 0 \tag{1.4}$$

and the map $\bar{x}(x)$ is a diffeomorphism from the Ω onto its image - in particular, this is a change of coordinates on the manifold Γ.

Next, consider the function $\bar{u}$ on the manifold Γ, which we describe as a function of the coordinates x :

$$\bar{u}(x) = u(x) - \sin^2(\beta) Du(x) \cdot x + \cos(\beta)\sin(\beta)\frac{|Du(x)|^2 - |x|^2}{2}.$$

A computation yields that

$$\frac{d\bar{u}}{dx}(x) = [-\sin(\beta)\, x + \cos(\beta) Du(x)] \cdot \frac{d\bar{x}}{dx}$$

It follows from the chain rule that

$$\frac{d\bar{u}}{d\bar{x}} = -\sin(\beta)\, x + \cos(\beta) Du(x).$$

Now observe that the rotation (1.1) maps $(x, Du(x))$ as follows:

$$(x, Du(x)) \mapsto (\bar{x}(x), -\sin(\beta)\, x + \cos(\beta) Du(x)).$$

In particular the isometric image of Γ under the rotation (1.1) is precisely

$$(\bar{x}, \frac{d\bar{u}}{d\bar{x}}(\bar{x}))$$

for the function $\bar{u}$.

Now we can compute the Hessian

$$\begin{aligned}\frac{D^2\bar{u}}{d\bar{x}d\bar{x}} &= \frac{d^2\bar{u}}{d\bar{x}dx} \cdot \frac{dx}{d\bar{x}} \\ &= \left[-\sin(\beta)\, I + \cos(\beta) D^2u(x)\right] \cdot \left[\cos(\beta)\, I + \sin(\beta)\, D^2u(x)\right]^{-1}.\end{aligned}$$

One can check that $D^2\bar{u}$ is diagonalized precisely when D^2u is diagonalized. Doing this, we can derive an easy relation between the eigenvalues

$$\frac{D^2\bar{u}}{d\bar{x}d\bar{x}} = \begin{pmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \bar{\lambda}_n \end{pmatrix} = \begin{pmatrix} \frac{-\sin(\beta)+\cos(\beta)\lambda_1}{\cos(\beta)+\sin(\beta)\lambda_1} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \frac{-\sin(\beta)+\cos(\beta)\lambda_n}{\cos(\beta)+\sin(\beta)\lambda_n} \end{pmatrix}$$

where $\lambda_1, .., \lambda_n$ are the eigenvalues of D^2u.

In particular,

$$\bar{\lambda}_j = \frac{-\sin(\beta) + \cos(\beta)\lambda_j}{\cos(\beta) + \sin(\beta)\lambda_j}.$$

By (1.3) this is equivalent to

$$\bar{\lambda}_j = \tan(\arctan(\lambda_j) - \beta). \tag{1.5}$$

It follows that $\bar{\lambda}_j$ are uniformly bounded above, directly by (1.5) :

$$\bar{\lambda}_j \leq \tan(\frac{\pi}{2} - \beta). \tag{1.6}$$

We may also bound $\bar{\lambda}_j$ below by inserting (1.2) into (1.5) to get

$$\bar{\lambda}_j \geq \tan\left(\delta - \frac{\pi}{2}\right).$$

2. Proof

2.1. Proof of Theorem 1. If u is semi-convex, there exist β,δ satisfying (1.2). Thus we may perform the rotation as above. The manifold

$$\Gamma = (x, Du(x))$$

is isometric to a graph

$$\bar{\Gamma} = (\bar{x}, D\bar{u}(\bar{x})).$$

Further, if the coordinates x are $\mathbb{R}^n$, then by (1.4) the coordinates $\bar{x}$ must also be $\mathbb{R}^n$. Thus $\bar{\Gamma}$ can be represented by the gradient graph of an entire function with

$$-C_0I \leq D^2\bar{u} \leq C_0I.$$

In particular the induced metric

$$g = I + \left(D^2\bar{u}\right)^T \left(D^2\bar{u}\right)$$

satisfies

$$I_n \leq g \leq (1 + C_0^2)I_n.$$

It follows that the Laplacian, given by

$$\Delta_g f = \frac{\partial_j \left(\sqrt{\det g}g^{ij}\partial_i f\right)}{\sqrt{\det g}},$$

is a uniformly elliptic divergence type operator, in this choice of coordinates.

The remaing proof is standard, but we include it for completeness.

We recall the Harnack inequality of De-Giorgi-Nash-Moser (cf [GT01, Theorem 8.20]).

Theorem 3. *Let $u \geq 0$ be a solution of*

$$\partial_j \left(a^{ij}(x) \partial_i f(x) \right) = 0$$

on $B_4(0)$ with

$$0 < \varepsilon I_n \leq a^{ij} \leq \frac{1}{\varepsilon} I_n.$$

There exists a universal constant $C(\varepsilon, n)$ such that

$$\sup_{B_1(0)} f \leq C(\varepsilon, n) \inf_{B_1(0)} f.$$

To finish the proof of the theorem, assume that we have a bounded harmonic function f. We may assume that either f or $(-f)$ is bounded below, and we may add a constant and assume $\inf f = 0$. Notice that

$$f_R(x) := f\left(\frac{x}{R}\right)$$

is a solution of the equation

$$\partial_j \left(a^{ij}\left(\frac{x}{R}\right) \partial_i f_R(x) \right) = 0$$

so satisfies the hypotheses of the Harnack inequality. In particular

$$\sup_{B_R(0)} f \leq C \inf_{B_R(0)} f$$

for every ball $B_R(0)$, with a fixed constant C. Taking $R \to \infty$ gives $\sup f = 0$.

In fact, we can state a slightly more general theorem.

Theorem 4. *Suppose that $F(D^2u)$ is an elliptic functional, and let g be the induced metric on the gradient graph. If u is semi-convex and*

$$\Delta_g F(D^2u) = 0$$

then

$$F(D^2u) = const.$$

Proof. If u is semiconvex, there exists a value M such that

$$D^2u \geq -MI_n.$$

Recall that (weak) ellipticity of a nonlinear operator F is equivalent to the statement that

$$F(M + P) \geq F(M)$$

for all positive matrices P. It follows by ellipticity then that

$$F(D^2u) \geq F(-MI_n) > -\infty.$$

The quantity F is bounded below and the result follows immediately from our main theorem. $\square$

2.2. **Proof of Theorem 2.** The function θ is odd in u, so we need only show that

$$\theta(u) > (n-2)\frac{\pi}{2} + \delta \tag{2.1}$$

implies that u is semiconvex.

We claim that

$$\lambda_i \geq \arctan(\delta - \frac{\pi}{2}).$$

Suppose instead that

$$\lambda_i < \arctan(\delta - \frac{\pi}{2}).$$

Then we must have

$$\sum_{i \neq j} \arctan \lambda_j > (n-2)\frac{\pi}{2} + \delta - (\delta - \frac{\pi}{2}) = (n-1)\frac{\pi}{2}$$

which is clearly a contradicton as

$$\arctan \lambda_j \leq \frac{\pi}{2}.$$

We conclude that

$$D^2 u \geq \arctan(\delta - \frac{\pi}{2})$$

and u is semiconvex. Thus $\theta(D^2u) = const$, by Theorem 1. The result follows by the main results in [Yua02][Yua06], which state that entire solutions of the special Lagrangian equation $\theta = const$ must be quadratic when $\theta > (n-2)\frac{\pi}{2}$ or when u is convex.

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