

Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary

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Abstract We show that on a compact Riemannian manifold with boundary there exists $u \in C^\infty(M)$ such that, $u|_{\partial M} \equiv 0$ and u solves the σ_k -Ricci problem. In the case $k = n$ the metric has negative Ricci curvature. Furthermore, we show the existence of a complete conformally related metric on the interior solving the σ_k -Ricci problem. By adopting results of (Mazzeo and Pacard, Pacific J. Math. 212(1), 169–185 (2003)), we show an interesting relationship between the complete metrics we construct and the existence of Poincaré–Einstein metrics. Finally we give a brief discussion of the corresponding questions in the case of positive curvature.

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1 Introduction

Let $(M^n, \partial M, g)$ be a smooth Riemannian manifold with boundary. Consider the following basic question:

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Question 1.1 Let $\mathcal{S} \subset \mathcal{R}$ be a subset of the space of curvature tensors on M . Does there exist a conformally related metric $\hat{g} = e^{-2u}g$ such that $u|_{\partial M} \equiv 0$ and the curvature tensor of $\hat{g}$ is in $\mathcal{S}$?

It can also be interesting to ask the question with no boundary restriction on u , or to ask for g to be complete on the interior of M .

More specifically, we can ask: Does there exist a conformally related metric with positive/negative scalar curvature, Ricci tensor, Schouten tensor, sectional curvature, curvature operator? On closed manifolds, the answer to all of these questions is “no” in full generality, due to the maximum principle. In the other direction, we note that due to Gromov’s h-principle for open diffeomorphism invariant differential relations on open manifolds, the answer to all of these questions is “yes” if $\mathcal{S}$ is open and we allow ourselves to consider *all* metrics (see [4, 5]). The restriction to a conformal class is interesting in its own right, and can have implications the soft methods do not yield, such as the existence of a metric compatible with a given almost-complex structure with certain curvature properties. Furthermore, our methods below produce *complete* metrics of negative Ricci curvature on manifolds with boundary, a conclusion certainly not forthcoming from the soft methods.

Our first theorem is a solution to the Dirichlet problem for the σ_k -Ricci curvature problem on manifolds with boundary. Before stating the theorem we need a definition.

Definition 1.2 If A is a symmetric matrix $\sigma_k(A)$ is the k -th elementary symmetric polynomial in the eigenvalues of A . Furthermore, let Γ_k^+ be the connected component of the set $\{\sigma_k > 0\}$ which contains the positive definite cone.

Theorem 1.3 Given $(M^n, \partial M, g)$ a compact manifold with boundary and $1 \leq k \leq n$, there exists a unique function $w_k \in C^\infty(M)$ such that $w_k|_{\partial M} \equiv 0$, $-\text{Ric}(e^{2w_k}g) \in \Gamma_k^+$, and $\sigma_k[-g^{-1}\text{Ric}(e^{2w_k}g)] = e^{2kw_k}$.

In particular observe that $\text{Ric}(e^{2w_n}g) < 0$. It is important to note here that many topological obstructions exist for curvature conditions on manifolds with boundary. Specifically in [1] a sphere-type theorem for manifolds with positive Ricci curvature and positive second fundamental form is shown. Results in a similar spirit appear in [10] where the classical Bonnet–Meyers and Cartan–adamard theorems are extended to manifolds with boundary. Further topological obstructions appear in [11, 14]. An interesting geometric conclusion based on curvature and mean curvature conditions appears in [9]. A common feature of all of these results is a (usually quite strong) hypothesis on the second fundamental form. It is to be emphasized that our conformal factors result in metrics with completely unknown second fundamental form, and therefore none of the previous obstructions can apply.

By solving the Dirichlet problem with larger and larger boundary data, we can solve the “infinite boundary data” Dirichlet problem to produce complete metrics with constant σ_k curvature on manifolds with boundary. The case $k = 1$ of this result appeared in [3]. Also, the existence of a complete metric of negative Ricci curvature with constant σ_k -Ricci curvature was shown in [7] with the assumption that the given background metric already has negative Ricci curvature. Negative Ricci curvature of the resulting metric is a *consequence* of our theorem in the case $k = n$.

Theorem 1.4 Given $(M^n, \partial M, g)$ a manifold with boundary and $1 \leq k \leq n$, there exists a unique function $w_k \in C^\infty(M \setminus \partial M)$ such that $w_k \rightarrow \infty$ at ∂M , $-\text{Ric}(e^{2w_k}g) \in \Gamma_k^+$, and $\sigma_k[-g^{-1}\text{Ric}(e^{2w_k}g)] = e^{2kw_k}$. The metric $e^{2w_k}g$ is complete, and if r denotes distance to ∂M , one has

$$\lim_{x \rightarrow \partial M} w_k + \ln r - \frac{1}{2} \ln(n-1) = 0.$$

This theorem has an interesting application to understanding the existence and moduli of Poincaré–Einstein metrics. In particular, in Sect. 6 we adopt results of [13] to our setting and exhibit the space of conformally compact Poincaré–Einstein metrics on a given manifold with boundary as an intersection of finitely many locally closed Banach manifolds in the space of conformally compact metrics. For the statement of this theorem we adopt the notation most commonly used in the study of Poincaré–Einstein metrics. The relevant terminology and the constants $\tilde{\beta}_{k,n}$ are defined in Sect. 6.

Theorem 1.5 *Let (X^{n+1}, g_+) be a conformally compact manifold. Let Θ_k denote the set of conformally compact metrics on X^{n+1} with $\sigma_k[-g_+^{-1} \text{Ric}] = \tilde{\beta}_{k,n}$.*

- (i) *Given a conformally compact metric $g_+ = \rho^{-2} \bar{g}$, and $1 \leq k \leq n+1$, there is a unique conformally compact metric $h_k = e^{2w_k} \bar{g} \in \Theta_k$.*
- (ii) *Let $\mathcal{E}$ denote the space of Poincaré–Einstein metrics. Then*

$$\mathcal{E} = \bigcap_{k=1}^{n+1} \Theta_k,$$

Hence $\mathcal{E}$ is a finite intersection of locally closed Banach submanifolds, and in particular is always closed in the space of conformally compact metrics on X^{n+1} .

In fact, the characterization of $\mathcal{E}$ is much weaker, only requiring that $\Theta_k \cap \Theta_{n+1} \neq \emptyset$ for some $k < n+1$. This fact is captured by a family of nonlocal conformally invariant functions we define in Sect. 6. In principle these invariant functions open a path towards proving existence of new Poincaré–Einstein metrics. Specifically, on Kähler manifolds one has many natural families of conformal classes, and it may be possible to show vanishing of this invariant for carefully chosen conformal classes.

Here is an outline of the rest of the paper. In Sect. 2 we recall some basic formulas and set up the continuity method we use to prove Theorem 1.3. In Sects. 3 and 4 we derive the C^1 and C^2 estimates for the continuity method respectively, and give the proof of Theorem 1.3. In Sect. 5 we prove Theorem 1.4 and in Sect. 6 discuss the relationship of these metrics to Poincaré–Einstein metrics and prove Theorem 1.5. Finally in Sect. 7 we conclude with a brief discussion of the case of positive curvature.

2 Setup for Theorem 1.3

We will explicitly solve the case $k = n$, and discuss the extension to the case $k < n$ at the end of the proof. Fix $(M^n, \partial M, g)$ a compact manifold with boundary and let

$$\rho = -\text{Ric}.$$

We recall that if $\hat{g} = e^{-2u} g$ one has

$$\hat{\text{Ric}} = \text{Ric} + (n-2)\nabla^2 u + \Delta u g + (n-2)(du \otimes du - |du|^2 g).$$

It follows that

$$\hat{\rho} = \rho - (n-2)\nabla^2 u - \Delta u g - (n-2)(du \otimes du - |du|^2 g). \quad (2.1)$$

Thus if we set $w = -u$, one has

$$\hat{\rho} = \rho + (n-2)\nabla^2 w + \Delta w g + (n-2)(|dw|^2 g - dw \otimes dw).$$

Given a conformal factor w , consider the tensor

$$W_t(w) := (1-t)g + t\rho + (n-2)\nabla^2 w + \Delta w g + (n-2)(|dw|^2 g - dw \otimes dw).$$

For the remainder of this section and the next two sections we relabel our dummy variable w as u . Therefore, consider the Dirichlet boundary-value Monge-Ampère equation

$$\begin{aligned} F_t(u) &:= \det W_t(u) - e^{2nu} = 0 & (\star_t) \\ u|_{\partial M} &\equiv 0. \end{aligned}$$

Let $\Omega = \{t \in [0, 1] | \exists u \in C^{4,\alpha}(M) \text{ solving } (\star_t), W_t(u) \in \Gamma_n^+\}$. A few observations are immediate. First of all, equation $(\star_0)$ has the unique solution $w \equiv 0$, thus Ω is nonempty. Also, by construction, it is clear that $W_0 \in \Gamma_n^+$. By the intermediate value theorem it follows that $W_t \in \Gamma_n^+$ for all t , and in particular W_1 will be in Γ_n^+ , as soon as the continuity method is completed. Therefore indeed a solution to $(\star_1)$ is the function required for the theorem. We can show that the set of times t such that $(\star_t)$ is solvable is open (Lemma 2.2), therefore the crux of the matter, as always, is showing a-priori estimates, which we will take up in the next section. Before proving openness of Ω we show a general maximum principle which will be of use to us.

Proposition 2.1 (Maximum principle) *Suppose that u and v are smooth sub and super solutions (respectively) to equation $(\star_t)$. If $u \leq v$ on ∂M , then $u \leq v$ on M .*

Proof Suppose that $u > v$ somewhere. Let C be the maximum of $u - v$ on M , which is attained at some point x_0 in the interior of M . Then $w = u - C$ is a strict subsolution to $(\star_t)$, hence at the point x_0 we conclude

$$\begin{aligned} w(x_0) &= v(x_0) \\ dw(x_0) &= dv(x_0) \\ F_t(w, dw, \nabla^2 w)(x_0) &> F_t(v, dv, \nabla^2 v)(x_0). \end{aligned}$$

It follows immediately that at the point x_0 we have

$$\begin{aligned} &\det[(1-t)g + t\rho + (n-2)\nabla^2 w + \Delta w g + (n-2)(dw \otimes dw - |dw|^2 g)] \\ &> \det[(1-t)g + t\rho + (n-2)\nabla^2 v + \Delta v g + (n-2)(dv \otimes dv - |dv|^2 g)]. \end{aligned}$$

However, note that $v \geq w$ near x_0 , which means that

$$\begin{aligned} \Delta v(x_0) &\geq \Delta w(x_0), \\ \nabla^2 v(x_0) &\geq \nabla^2 w(x_0). \end{aligned}$$

Using these inequalities and the fact that $dw(x_0) = dv(x_0)$ we conclude that at the point x_0 ,

$$\begin{aligned} \mathcal{W} &:= (n-2)\nabla^2 w + \Delta w g + (n-2)(dw \otimes dw - |dw|^2 g) \\ &\leq (n-2)\nabla^2 v + \Delta v g + (n-2)(dv \otimes dv - |dv|^2 g) \\ &=: \mathcal{V} \end{aligned}$$

where the matrix inequality $\mathcal{W} \leq \mathcal{V}$ has the usual interpretation that $\mathcal{V} - \mathcal{W}$ is positive semidefinite. We therefore conclude that

$$\begin{aligned}\det[(1-t)g + t\rho + \mathcal{V}] &= \det[(1-t)g + t\rho + \mathcal{W} + (\mathcal{V} - \mathcal{W})] \\ &\geq \det[(1-t)g + t\rho + \mathcal{W}]\end{aligned}$$

which is a contradiction, and the result follows. $\square$

Note that this maximum principle immediately implies uniqueness of solutions to $(\star_t)$ for all $0 \leq t \leq 1$. Next we observe openness of Ω .

Lemma 2.2 Ω is open in $[0, 1]$.

Proof We compute the linearized operator

$$\begin{aligned}F'_t(u_t)(h) &= T_{n-1}(W_t)^{ij} \left((n-2)\nabla^2 h + \Delta h g_{ij} \right. \\ &\quad \left. + (n-2)(2\langle du_t, dh \rangle g - dh \otimes du_t - du_t \otimes dh) \right. \\ &\quad \left. - 2nhe^{2nu_t} \right).\end{aligned}$$

where $T_{n-1}(W_t)^{ij}$ is the $(n-1)$ Newton transformation, which is positive definite since W_t is by construction. Thus $F'_t(u_t)$ is a strictly elliptic operator with $C^{2,\alpha}$ coefficients and negative constant term, and is hence invertible. The result thus follows by the implicit function theorem. $\square$

3 Construction of subsolutions

In this section we derive a subsolution to the equations $(\star_t)$ which is at the heart of our estimates. We begin with an auxiliary geometric construction. Given $(M, \partial M)$, let $N = \partial M$ and consider the manifold $\overline{M} = M \cup (N \times [0, 1]) / \sim$ where for $x \in N = \partial M$ we have $x \times 1 \sim x$. One should picture an “exterior” collar neighborhood of ∂M . Using a standard partition of unity argument one may extend the metric g to a metric $\overline{g}$ defined on $\overline{M}$ such that $\overline{g}|_M = g$. Consider a point $x_0 \in \partial M$. Fix a point $\overline{x} \in \overline{M} \setminus M$ in the connected component of N which contains x_0 chosen so that x_0 is the closest point to $\overline{x}$ which lies on the boundary. Let r denote geodesic distance from $\overline{x}$. We may arrange things so that $d(\overline{x}, \partial M) > \delta$ where δ only depends on the background metric. See Fig. 1 below.

Fix constants A and p whose exact size will be determined later, and let

$$\underline{u} := A \left(\frac{1}{r^p} - \frac{1}{r(x_0)^p} \right)$$

Our goal is to show that $\underline{u}$ is a subsolution of $(\star_t)$ for all t . It is important to remark here that, as defined above, $\underline{u}$ of course depends on a chosen boundary point. We will not make this dependence explicit in the notation, and it will be clear that the estimates involving $\underline{u}$ will be independent of the choice of x_0 . First we recall the Hessian comparison theorem.

Lemma 3.1 (Hessian comparison theorem) *Let (M^n, g) be a complete Riemannian manifold with $\text{sect} \geq K$. For any point $p \in M$ the distance function $r(x) = d(x, p)$ satisfies*

$$\nabla^2 r \leq \frac{1}{n-1} H_K(r) g$$

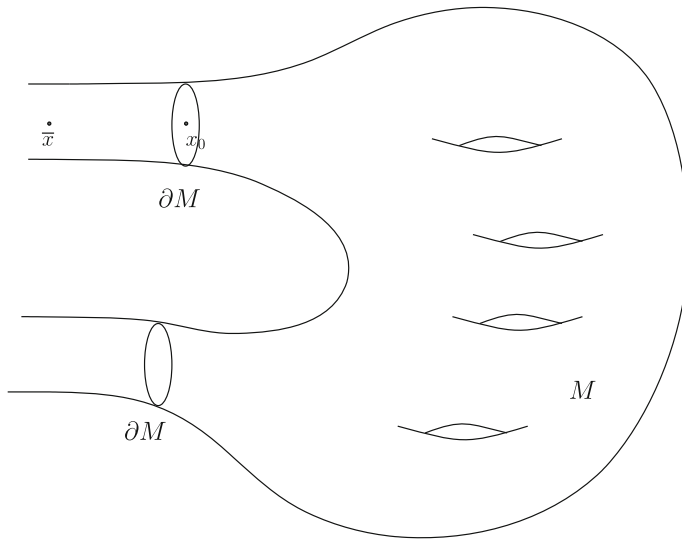


Fig. 1 Exterior collar neighborhood of ∂M

where

$$H_K(r) = \begin{cases} (n-1)\sqrt{K} \cot(\sqrt{K}r) & K > 0 \\ \frac{n-1}{r} & K = 0 \\ (n-1)\sqrt{|K|} \coth(\sqrt{|K|}r) & K < 0 \end{cases}$$

Lemma 3.2 Let $(\bar{M}, \bar{g})$ be the metric we constructed above, and let r denote distance from a point $\bar{x} \in \bar{M} \setminus M$ chosen so that $d(\bar{x}, \partial M) > \delta > 0$ for some fixed small constant δ . Then there exists a constant C such that

$$\nabla^2 r(x) \leq \frac{C}{r(x)} g$$

holds at any point where r is smooth.

Proof If $K \geq 0$ the result follows immediately from Lemma 3.1. Assume $K \leq 0$. The distance of $\bar{x}$ to any point in M is bounded, therefore standard estimates on the coth function give this result away from a controlled ball around $\bar{x}$, which we can assume is contained in $\bar{M} \setminus M$. $\square$

Lemma 3.3 For A and p chosen large enough with respect to constants depending only on g , at any point where r is smooth we have

$$F_t(\underline{u}) > 0.$$

Proof We first compute $(n-2)\nabla^2 v + \Delta v g$. First we compute the action of this operator on $\underline{u}$. Since the action is linear we suppress the constant A and reinsert it at the end.

$$\begin{aligned} \nabla^2 \underline{u} &= p(p+1)r^{-p-2}\nabla r \otimes \nabla r - pr^{-p-1}\nabla^2 r \\ \Delta \underline{u} &= p(p+1)r^{-p-2}|\nabla r|^2 - pr^{-p-1}\Delta r \end{aligned}$$

Since $p > 0$, applying Lemma 3.2 yields

$$-pr^{-p-1}\nabla^2 r \geq -Cpr^{-p-2}g$$

Since the first term in the expression for $\nabla^2 \underline{u}$ above is positive we conclude

$$\nabla^2 \underline{u} \geq -Cpr^{-p-2}g$$

for some constant $C = C(g)$. Similarly applying Lemma 3.2 one has

$$-pr^{p-1}\Delta r \geq -Cpr^{-p-2}$$

for some constant C . Note that $|\nabla r| = 1$ at x_1 since r is smooth here. It follows that

$$\begin{aligned}\Delta \underline{u} &\geq r^{-p-2}(p(p+1) - Cp) \\ &\geq \frac{p^2}{2}r^{-p-2}\end{aligned}$$

for p chosen large with respect to C . In sum we can conclude, reinserting the factor A ,

$$(n-2)\nabla^2 \underline{u} + \Delta \underline{u}g \geq A\frac{p^2}{2}r^{-p-2}g.$$

It is clear then that for p chosen large with respect to universal constants and then A chosen large with respect to the diameter of g we have

$$(n-2)\nabla^2 \underline{u} + \Delta \underline{u}g \geq \frac{p^2}{4}g.$$

Now choose p still larger depending on the ambient Ricci curvature, i.e., so that $p \geq 2\sqrt{-\min_{v \in UTM} \rho(v, v)}$. Observing that the gradient terms in the definition of W_t are always positive, and noting that $\underline{u} \leq 0$, we conclude the result. $\square$

4 Proof of Theorem 1.3

Lemma 4.1 *Given $\underline{u}$ as in Lemma 3.3, for all $t \in [0, 1]$, one has $\underline{u} \leq u_t$.*

Proof Fix a $t \in [0, 1]$ and suppose that $\underline{u} > u_t$ somewhere. We can fix a positive constant C and a point $x_1 \in M$ achieving the maximum of $\underline{u} - u_t$, such that $\underline{u} - C \leq u_t$ and $(\underline{u} - C)(x_1) = u_t(x_1)$. It is clear by construction that this point must be inside of M . We also claim that $\underline{u}$, and equivalently, r , must be smooth at this point x_1 . Indeed, if this were not the case, at x_1 there would be two geodesics γ_1, γ_2 which are each minimizing from $\bar{x}$ to x_1 . Suppose $d(\bar{x}, x_1) = R$. Let γ_1 be given a unit speed parametrization in c . One concludes

$$\lim_{c \rightarrow R^-} \nabla r(\gamma_1(c)) \cdot \gamma_1' = 1. \quad (4.1)$$

We next claim that

$$\lim_{c \rightarrow R^+} \nabla r(\gamma_1(c)) \cdot \gamma_1' < 1. \quad (4.2)$$

The argument of the following paragraph is summarized in Fig. 2. Fix a constant $\epsilon > 0$ so small that $B_\epsilon(x_1)$ is geodesically convex. Consider the point $\tilde{x}_\epsilon = \gamma_1(R + \epsilon)$. Construct a new curve $\tilde{\gamma}$ from $\bar{x}$ to $\tilde{x}_\epsilon$ as follows: follow the geodesic γ_2 from $\bar{x}$ to $\gamma_2(R - \epsilon)$, then connect $\gamma_2(R - \epsilon)$ to $\tilde{x}_\epsilon$ by the unique geodesic in $B_\epsilon(x_1)$ between these two points. Recall that γ_1 and γ_2 are distinct geodesics. In particular, by uniqueness of solutions to ODE, it follows

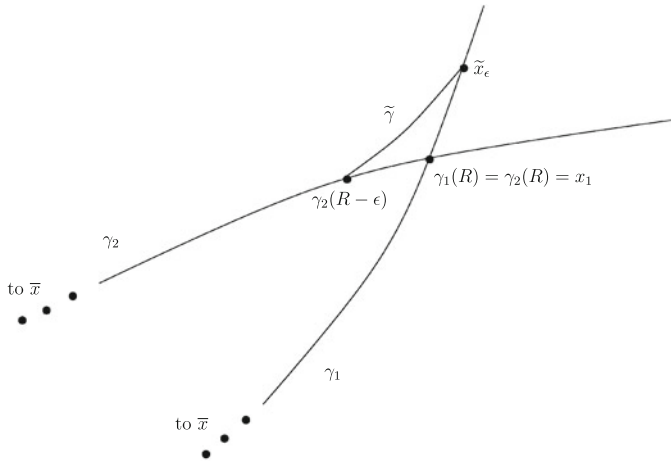


Fig. 2 Geodesics at the cut locus

that $\gamma_1'(R) \neq \gamma_2'(R)$ since $\gamma_1(R) = \gamma_2(R)$. In particular, the triangle formed by the three points $\gamma_2(R - \epsilon)$, $\gamma_1(R) = \gamma_2(R) = x_1$, and $\gamma_1(R + \epsilon) = \tilde{x}_\epsilon$ is nondegenerate. It follows from the Toponogov comparison theorem that $d(\gamma_2(R - \epsilon), \tilde{x}_\epsilon)$ is strictly less than the sum of the lengths of the other two sides of the triangle, with the difference given in terms of a lower bound for the curvature of g . Specifically, there exists a $\delta > 0$ depending on this lower bound and the angles of the triangle so that

$$d(\gamma_2(R - \epsilon), \tilde{x}_\epsilon) \leq (2 - \delta)\epsilon$$

(In fact, since our triangle is very small, the curvature does not need to enter into the bound. One can forgo the Toponogov theorem and get a bound strictly in terms of the angles of the triangle). Using $\tilde{\gamma}$ as a test curve for the distance function, it follows that

$$d(\bar{x}, \tilde{x}_\epsilon) \leq R - \epsilon + (2 - \delta)\epsilon = R + \epsilon - \delta\epsilon.$$

Taking the limit as $\epsilon \rightarrow 0$, we immediately conclude that

$$\begin{aligned} \lim_{c \rightarrow R^+} \nabla r(\gamma_1(c)) \cdot \gamma_1' &= \lim_{\epsilon \rightarrow 0} \frac{r(\gamma_1(R + \epsilon)) - r(\gamma_1(R))}{\epsilon} \\ &\leq \lim_{\epsilon \rightarrow 0} \frac{R + \epsilon - \delta\epsilon - R}{\epsilon} \\ &< 1. \end{aligned}$$

We now finish the argument that $\underline{u}$ is smooth at x_1 . Indeed, it follows from (4.1) and (4.2) by direct calculation that the derivative of the function $f(c) := \underline{u}(\gamma_1(c))$ jumps a certain positive amount at $c = R$. Considering next the smooth function $\psi(c) := u_t(\gamma_1(c))$, by assumption we have that $(\psi - f)(c)$ has a local minimum at $c = R$. Thus

$$\lim_{c \rightarrow R^-} f' - \psi' \geq \lim_{c \rightarrow R^+} f' - \psi'.$$

Since ψ is smooth, we therefore conclude

$$\lim_{c \rightarrow R^-} f' \geq \lim_{c \rightarrow R^+} f'.$$

This contradicts (4.1) and (4.2) since $\frac{du}{dr} < 0$.

Given that $\underline{u}$ is smooth at x_1 , using Lemma 3.3 the argument of Proposition 2.1 applies at this point to yield the required contradiction to the assumption that $\underline{u} > u_t$ somewhere. The lemma follows. $\square$

Lemma 4.2 *The inequality $u_t \leq 0$ holds for all $0 \leq t \leq 1$.*

Proof To get this estimate we exhibit u_t as a subsolution of $(\star_0)$. We may assume without loss of generality that by scaling g in space we have $g \geq \rho$. It follows that

$$\begin{aligned} e^{2nu_t} &= F_t(u_t) + e^{2nu_t} \\ &= \det((1-t)g + t\rho + (n-2)\nabla^2 u_t + \cdots) \\ &\leq \det(g + (n-2)\nabla^2 u_t + \cdots) \\ &= F_0(u_t) + e^{2nu_t}. \end{aligned}$$

Therefore u_t is a subsolution of the equation $F_0(u) = 0$, and the result follows by Proposition 2.1. $\square$

Proposition 4.3 *There exists a constant C such that for all $x_0 \in \partial M$ and for all $0 \leq t \leq 1$ we have $|\frac{\partial}{\partial \nu} u_t| \leq C$ where ν is the unit normal to ∂M at x_0 .*

Proof This follows immediately from Lemmas 4.1 and 4.2 since for instance

$$\frac{u(x) - u(x_0)}{d(x, x_0)} \leq \frac{u(x) - u(x_0)}{d(x, x_0)}.$$

Our construction of $\underline{u}$ is specific to each $x_0 \in \partial M$, but it is clear that the choice of p etc. are all universally controlled, and so the proposition follows. $\square$

Proposition 4.4 *There exists a constant C such that for all $0 \leq t \leq 1$ we have*

$$|u_t|_{C^1} \leq C$$

Proof We have already shown the global C^0 estimate and the boundary C^1 estimate. Suppose that the maximum of $|\nabla u_t|$ occurs at a point in the interior. One may follow the calculation of [8] Proposition 4.1, which is justified at any interior point of M , to yield the a-priori C^1 estimate. The result follows. $\square$

We now proceed with the C^2 estimates. Fix $x_0 \in \partial M$ and let u be a solution to $(\star)$ for some $0 \leq t \leq 1$. Suppose further that $|\nabla_n u| < C$. We will use the indices e_i, e_j to refer to tangent directions to ∂M , and e_n to refer to the unit inward normal at x_0 . We require separate proofs for the different types of boundary second derivatives $\nabla_i \nabla_j u$, $\nabla_i \nabla_n u$, and $\nabla_n \nabla_n u$.

Lemma 4.5 *There exists a constant C depending on $\sup_{0 \leq t \leq 1} |u_t|_{C^1}$ such that for all $x_0 \in \partial M$, for all $0 \leq t \leq 1$ we have*

$$|\nabla_i \nabla_j u(x_0)| < C.$$

Proof We note first, using that $u|_{\partial M} \equiv 0$,

$$\nabla_i \nabla_j u(x_0) = -\nabla_n u(x_0) A(e_i, e_j)$$

where A is the second fundamental form of ∂M . Since $|\nabla_n u(x_0)| < C$ we immediately conclude the result. $\square$

Next we need to bound the derivatives of the form $\nabla_{e_i} \nabla_n u$ at the boundary. For our given $t \in [0, 1]$, let $\mathcal{L}$ denote the linearization of F_t at u . As in Lemma 2.2 we have

$$\begin{aligned} \mathcal{L}(\psi) = & T_{n-1}(W_t)^{ij} \left((n-2) \nabla_i \nabla_j \psi + \Delta \psi g_{ij} \right. \\ & \left. + (n-2) (2 \langle \nabla \psi, \nabla u \rangle g_{ij} - \nabla_i \psi \otimes \nabla_j u - \nabla_i u \otimes \nabla_j \psi) \right) \\ & - 2n \psi e^{2nu}. \end{aligned} \quad (4.3)$$

Fix a point $x_0 \in \partial M$ and let B_δ be the ball of some small radius $\delta > 0$ around x_0 . Pick coordinates in B_δ so that ∂M is the plane $x_n = 0$, and let $\{e_i, e_n\}$ be the corresponding coordinate vector fields. Fix some α and consider the function $\phi = e_\alpha u_t$ defined in B_δ . Note that $\phi|_{\partial M} = 0$. We aim to apply a maximum principle argument similar to the C^1 boundary estimate to yield a bound on the normal derivative of ϕ , which will yield the required bound. The first step is to bound the action of $\mathcal{L}$ on ϕ .

Lemma 4.6 *Using the notation above, there exists a constant C such that*

$$|\mathcal{L}(\phi)| \leq C \left(1 + \sup_{0 \leq t \leq 1} |u_t|_{C^1} \right) \sum F^{ii}.$$

Proof Differentiating equation $(\star_t)$ with respect to e_α yields

$$\begin{aligned} 0 = & F^{ij} (t \nabla_\alpha \rho_{ij} + (n-2) \nabla_\alpha \nabla_i \nabla_j u + \nabla_\alpha \Delta u g_{ij} \\ & + (n-2) (2 \langle \nabla_\alpha \nabla u, \nabla u \rangle g - \nabla_\alpha \nabla_i u \otimes \nabla_j u - \nabla_i u \otimes \nabla_\alpha \nabla_j u)) \\ & - 2n \nabla_\alpha u e^{2nu}. \end{aligned}$$

Commuting derivatives we conclude

$$\begin{aligned} \nabla_\alpha \nabla_i \nabla_j u &= \nabla_i \nabla_\alpha \nabla_j u + \text{Rm} * \nabla u \\ &= \nabla_i \nabla_j \nabla_\alpha u + \text{Rm} * \nabla u \\ &= \nabla_i \nabla_j \phi + \text{Rm} * \nabla u. \end{aligned}$$

Similarly we have

$$\nabla_\alpha \Delta u = \Delta \phi + \text{Rm} * \nabla u.$$

Combining these calculations yields

$$\mathcal{L}(\phi) = F^{ij} (t \nabla_\alpha \rho_{ij} + (\text{Rm} * \nabla u)_{ij}).$$

Applying the C^1 bound we immediately conclude

$$|\mathcal{L}(\phi)| \leq C \left| F^{ij} \right| (1 + |\nabla u|) \leq C \left| F^{ij} \right|$$

where $|F^{ij}|$ refers to the matrix norm. Note that F^{ij} is positive definite, and therefore its norm is dominated by a dimensional constant times its trace. The result follows. $\square$

Lemma 4.7 *There exists a constant C depending on $\sup_{0 \leq t \leq 1} |u_t|_{C^1}$ such that for all $x_0 \in \partial M$, for all $0 \leq t \leq 1$ we have*

$$|\nabla_i \nabla_n u_t(x_0)| \leq C.$$

Proof Fix a point $x_0 \in \partial M$. Choose a small constant $\delta < \frac{1}{2}$ so that $B_\delta(x_0)$ is a geodesically convex ball. Furthermore choose $\bar{x} \in B_{\frac{\delta}{4}}(x_0)$. Following the construction in Sect. 3, let

$$\underline{u} = \frac{1}{r^p} - \frac{1}{r(x_0)^p}.$$

Note that by construction we have that $\underline{u}$ is smooth in $U := M \cap B_\delta(x_0)$. We want to compute $\mathcal{L}(\underline{u})$. Following the calculations of Sect. 3 we conclude

$$(n-2)\nabla^2 \underline{u} + \Delta \underline{u} g \geq \frac{p^2}{4} r^{-p-2} g.$$

Consider one of the linear terms in $\mathcal{L}$ acting on $\underline{u}$. In particular we note that

$$\begin{aligned} \langle \nabla \underline{u}, \nabla u \rangle g &\leq p r^{-p-1} |\nabla u| g \\ &\leq C p r^{-p-2} g \end{aligned}$$

using that $r \leq 1$ and $|\nabla u| \leq C$. All of the linear terms are bounded thusly. Furthermore, $\underline{u} \leq 0$, so we can throw away the constant term and conclude

$$\mathcal{L}(\underline{u}) \geq \frac{p^2}{8} r^{-p-2} \sum F^{ii}$$

when p is chosen large enough with respect to fixed constants. It follows that if we choose p larger still and note that $r \leq 1$, Lemma 4.6 yields

$$\mathcal{L}(\phi - \underline{u}) < \left(C - \frac{p^2}{8}\right) \sum F^{ii} < 0.$$

Thus by the maximum principle we conclude that the minimum of $\phi - \underline{u}$ occurs on the boundary of U . It remains to check these boundary values. There are two components of U to check. First we have $U \cap \partial M$. Here $\phi \equiv 0$ and $\underline{u} \leq 0$ with $\underline{u} = 0$ at x_0 . Next we consider the component $U \cap \partial B_\delta(x_0)$. Here using the C^1 estimate we have $\phi \geq -C$ for some controlled constant C . Since $\bar{x} \in B_{\frac{\delta}{4}}(x_0)$, it follows that for $x \in U \cap \partial B_\delta(x_0)$ one has

$$\begin{aligned} \underline{u}(x) &\leq \frac{1}{\left(\frac{\delta}{2}\right)^p} - \frac{1}{\left(\frac{\delta}{4}\right)^p} \\ &= \left(\frac{1}{\delta}\right)^p (2^p - 4^p). \end{aligned}$$

Thus for p chosen large enough with respect to controlled constants one concludes $\underline{u} < \phi$ on $U \cap \partial B_\delta(x_0)$.

It follows that the minimum of $\phi - \underline{u}$ on ∂U is zero and occurs at x_0 . It follows that the normal derivative of $\phi - \underline{u}$ is nonnegative, and therefore we conclude

$$\nabla_i \nabla_n u \geq -C$$

However, using Lemma 4.6 it is clear that the same argument applies to $-\phi$, and therefore the result follows. $\square$

Lemma 4.8 *There exists a constant C depending on $\sup_{0 \leq t \leq 1} |u_t|_{C^1}$, $\sup_{0 \leq t \leq 1} \sup_{x \in \partial M} |\nabla_i \nabla_j u_t(x)|$, and $\sup_{0 \leq t \leq 1} \sup_{x \in \partial M} |\nabla_i \nabla_n u_t(x)|$ such that for all $x_0 \in \partial M$, for all $0 \leq t \leq 1$ we have*

$$|\nabla_n \nabla_n u(x_0)| \leq C.$$

Proof Orthogonally decompose the matrix W_t at x_0 in terms of n and e_i . Using the assumed bounds this yields

$$W = \begin{pmatrix} (n-1)u_{nn} & 0 \\ 0 & u_{nn}g|_{\partial M} \end{pmatrix} + \mathcal{O}(1) \quad (4.4)$$

It is clear that there exists a constant $C > 0$ such that if $|u_{nn}| > C$ then

$$|\det W| > \frac{1}{10} C^n > e^{2nu_t}$$

which contradicts the equation $(\star_t)$. Thus $|u_{nn}(x_0)| < C$ and the result follows. $\square$

Proposition 4.9 *There exists a constant C such that for all $0 \leq t \leq 1$ we have*

$$|u_t|_{C^2} \leq C$$

Proof By Proposition 4.4 and Lemmas 4.5, 4.7, and 4.8 we conclude uniform global C^1 bounds and boundary C^2 estimates. Suppose that the maximum of $|\nabla^2 u|$ occurs at a point in the interior. One may follow the calculation of [8] Proposition 5.1, which is justified at any interior point of M , to yield the a-priori C^2 estimate. The result follows. $\square$

We can now give the proof of Theorem 1.3.

Proof As discussed in Sect. 2 it suffices to solve our equation $(\star_1)$. Lemma 2.2 yields that the set Ω of t where $(\star_t)$ is solvable is open in $[0, 1]$. Proposition 4.9 yields a uniform C^2 estimate for u_t . Thus $(\star_t)$ becomes a uniformly elliptic equation, and the Evans-Krylov estimates yield uniform $C^{2,\alpha}$ bounds on u_t . Now the Schauder estimates apply to yield uniform C^4 bounds. Thus Ω is closed in $[0, 1]$, and hence $(\star_1)$ is solvable, which completes the proof of existence. By Proposition 2.1, the solution to $(\star_1)$ is unique.

To solve the analogous σ_k problem, $k < n$, it suffices to observe that the subsolution we construct for the determinant equation is also a subsolution to the σ_k problem by Maclaurin's inequality. The proof is otherwise followed line for line to yield this result. $\square$

Remark 4.10 Note that the proof works equally well if we require the boundary condition $u|_{\partial M} \equiv j$ for any constant j . Indeed, with minor modification the proof applies to the general boundary value problem.

5 Proof of Theorem 1.4

We begin by proving a lemma which serves as a subsolution for solutions to the Dirichlet problem with large boundary condition.

Lemma 5.1 *Let $(M, \partial M, g)$ be a manifold with boundary. There exists a function w which is smooth in a controlled neighborhood of the boundary such that the following holds: Fix $0 < \epsilon < 1$, and let u be a solution to*

$$\begin{aligned} F_1(u) &= 0 \\ u|_{\partial M} &\equiv -\ln \epsilon + \frac{1}{2} \ln(n-1). \end{aligned}$$

Let r denote distance from the boundary. Then

$$u \geq -\ln(r + \epsilon) + \frac{1}{2} \ln(n-1) + w.$$

where $w \leq 0$ and $w|_{\partial M} = 0$.

Proof Let r denote distance from ∂M . Fix a small constant $\delta > 0$, constants $A, p > 0$ and let

$$w = A \left(\frac{1}{(r + \delta)^p} - \frac{1}{\delta^p} \right).$$

This choice of w is obviously modeled on our subsolution from Sect. 3, except that here one thinks of $r + \delta$ as distance from an exterior copy of ∂M instead of distance from a point outside of ∂M . The constant δ remains fixed for arbitrarily small values of ϵ .

Fix a small constant $\epsilon > 0$, and let $\tilde{r} = r + \epsilon$. Again, $\tilde{r}$ should be thought of as distance from an exterior copy of ∂M , but this time one getting arbitrarily close to the actual boundary ∂M . Let

$$\underline{u} = -\ln \tilde{r} + \frac{1}{2} \ln(n-1) + w.$$

Our goal is to show that $\underline{u}$ is a subsolution for $(\star_t)$ with boundary condition $u|_{\partial M} \equiv -\ln \epsilon + \frac{1}{2} \ln(n-1)$. The estimate will proceed in two steps. First we estimate $\underline{u}$ in a small collar neighborhood of the boundary, where the $-\ln \tilde{r}$ term dominates the behaviour of $\underline{u}$. Next we exploit the w term using estimates similar to those in Sect. 3 to control $\underline{u}$ on the rest of the manifold.

Fix a point $x_0 \in M \setminus \partial M$ at which r is smooth, and choose coordinates at x_0 as follows: Let $e_1 = \frac{\partial}{\partial r}$, and let $\{e_2, \dots, e_n\}$ be chosen so that $\{e_i\}$ is an orthonormal basis at x_0 . First observe the preliminary calculation

$$(n-2) \left(|d\underline{u}|^2 g - d\underline{u} \otimes d\underline{u} \right) = (n-2) \frac{(\tilde{r}w' - 1)^2}{\tilde{r}^2} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Therefore at x_0 we conclude

$$\begin{aligned} \hat{\rho} &= \rho + (n-2)\nabla^2 \underline{u} + \Delta \underline{u} g + (n-2) \left(|d\underline{u}|^2 g - d\underline{u} \otimes d\underline{u} \right) \\ &= \rho + (n-2)\nabla^2 w + \Delta w g - \frac{1}{\tilde{r}} \left((n-2)\nabla^2 r + \Delta r g \right) \\ &\quad + \frac{1}{\tilde{r}^2} \begin{pmatrix} (n-1) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} + (n-2) \frac{(\tilde{r}w' - 1)^2}{\tilde{r}^2} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \\ &= \rho + (n-2)\nabla^2 w + \Delta w g + (n-2) (w')^2 \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \\ &\quad + \frac{1}{\tilde{r}^2} \left[(n-1)g - 2(n-2)\tilde{r}w' \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} - \tilde{r} \left((n-2)\nabla^2 r + \Delta r g \right) \right]. \end{aligned}$$

We now show that the determinant of the bracketed term above, call it Φ , is positive in a collar neighborhood of ∂M of a fixed width $\eta > 0$. Observe that when $\tilde{r}$ is small this term is

dominating the behaviour of $\hat{\rho}$. We initially choose η small so that the hypersurfaces $\{r = c\}$ are smooth for $c \leq \eta$. First note that $\nabla^2 r$ is simply the second fundamental form of a smooth hypersurface $\{r = c\}$. Therefore it is a tensor with a uniform bound as $r \rightarrow 0$ depending only on g . In particular for η chosen small with respect to constants depending on g we can conclude

$$(n-2)\nabla^2 r + \Delta r g \geq -\lambda g.$$

Also we can directly compute that on the collar neighborhood of radius η

$$w' = -\frac{Ap}{(r+\delta)^{p+1}} \leq -\frac{Ap}{(\eta+\delta)^{p+1}} \leq -Ap$$

provided $\eta + \delta < 1$, which is easily arranged. We therefore conclude that if we choose A_0 and p large with respect to λ , then for any $A \geq A_0$ and $p \geq p_0$ we have

$$\Phi \geq (n-1)g + \tilde{r} \begin{pmatrix} -\lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix}$$

It follows that if η is chosen small with respect to λ , one has $\det \Phi \geq (n-1)^n$ for $\tilde{r} \leq 2\eta$. Since we have chosen $\epsilon \ll 1$ it follows that for $r < \eta$ we have

$$\det(\hat{\rho}) \geq \frac{(n-1)^n}{\tilde{r}^{2n}}.$$

We would like to show this inequality on the rest of M , i.e., for any $r > \eta$. Recall from Sect. 3 that given any constant $C > 0$ we may choose our constants A and p such that

$$(n-2)\nabla^2 w + \Delta w g \geq Cg.$$

Since $\nabla^2 r$ is a bounded tensor as described above, we may therefore choose A and p large so that

$$\begin{aligned} (n-2)\nabla^2 w + \Delta w g &\geq -\rho + \frac{1}{\eta + \epsilon} ((n-2)\nabla^2 r + \Delta r g) \\ &\geq -\rho + \frac{1}{2\eta} ((n-2)\nabla^2 r + \Delta r g). \end{aligned}$$

Note here that the choices of A and p depend on η . We were careful above to ensure that the choice of η only depended on lower bounds for A and p , therefore we are free to choose them still larger, even with respect to η . It follows that at any point where r is smooth we have

$$\det(\hat{\rho}) \geq \left(\frac{1}{\tilde{r}^2} (n-1) \right)^n = \frac{(n-1)^n}{\tilde{r}^{2n}} = e^{2n(-\ln \tilde{r} + \frac{1}{2} \ln(n-1))} \geq e^{2n\underline{u}}$$

where the last inequality follows since $w \leq 0$. It follows that $\underline{u}$ is a subsolution to $(\star_1)$. It is clear by our construction of $\underline{u}$ that the comparison argument of Lemma 4.1 applies to show that any point where $\underline{u} - u$ achieves its maximum must be smooth, and hence the argument of Proposition 2.1 applies to show that $u \geq \underline{u}$. The lemma follows. $\square$

Lemma 5.2 *Let $(M, \partial M, g)$ be a manifold with boundary. Let u be a solution to*

$$F_1(u) = 0$$

with any boundary condition. Then

$$\lim_{x \rightarrow \partial M} \left[u(x) + \ln r(x) - \frac{1}{2} \ln(n-1) \right] \leq 0.$$

Furthermore, there given any small constant $R > 0$ there exists a constant $C(R) > 0$ so that given $x_0 \in M$, $B_R(x_0) \subset M$ one has

$$u(x_0) \leq C(R).$$

Proof The proof is an adaptation of an argument in [12] Theorem 4 to the case where the background geometry is not conformally flat. We observe that by the Maclaurin inequality it suffices to find a supersolution to the equation

$$-S := \sigma_1[-\text{Ric}(e^{2u})] = ne^{2u}$$

to bound the solution to any of the σ_k equations from above. This equation is given by

$$-\frac{S_g}{n-1} + 2\Delta u + (n-2)|du|^2 = \frac{n}{n-1}e^{2u}. \quad (5.1)$$

We proceed to find a local supersolution to this equation.

Take a point x_0 in M with distance d from the boundary. Consider a geodesic running from the point on the boundary which is closest to x_0 , passing through x_0 , and out a small distance R into the manifold to a point z_0 . We will fix a small R and a function $f(t)$ based on the following. Ensure that we choose both d and R small enough so that given any such point z_0 as above a distance $R+d$ from the boundary, then the geodesics inside $B_R(z_0)$ will intersect the boundary only once, and on this ball $\Delta d^2(z_0, \cdot) \geq 1$. Further we would like to choose R small enough so that there is a solution to the differential relation on $[0, R^2]$

$$\begin{aligned} (n-2)(f')^2 + 2f'' &\leq 0, \\ f' &> \max_M |S| + C(g) \\ f(0) &= 0 \end{aligned}$$

where S as above is the scalar curvature on M . In particular, one may choose

$$f(t) = \sqrt{t + \epsilon^2} - \epsilon$$

and then for t and ϵ chosen small with respect to fixed constants the required properties are satisfied.

Let r denote the distance function from the point z_0 , and define a function $\bar{u}$ on $B_R(z_0)$ by

$$\bar{u} = -\ln(R^2 - r^2) + f(R^2 - r^2) + \ln 2 + \ln R + \frac{1}{2} \ln(n-1)$$

One directly computes

$$\begin{aligned} d\bar{u} &= \left(\frac{1}{R^2 - r^2} - f' \right) dr^2 \\ \nabla^2 \bar{u} &= \left(\frac{1}{R^2 - r^2} - f' \right) \nabla^2 r^2 + \left(\left(\frac{1}{R^2 - r^2} \right)^2 + f'' \right) dr^2 \otimes dr^2 \\ \Delta \bar{u} &= \left(\frac{1}{R^2 - r^2} - f' \right) \Delta r^2 + \left(\left(\frac{1}{R^2 - r^2} \right)^2 + f'' \right) |dr^2|^2 \end{aligned}$$

It follows that the left hand side of equation (5.1) becomes

$$\begin{aligned} & -\frac{S_0}{n-1} + 2 \left(\frac{1}{R^2 - r^2} - f' \right) \Delta r^2 + 2 \left(\left(\frac{1}{R^2 - r^2} \right)^2 + f'' \right) |dr^2|^2 \\ & + (n-2) \left(\frac{1}{R^2 - r^2} - f' \right)^2 |dr^2|^2 = \frac{1}{(R^2 - r^2)^2} \\ & \times \left\{ \left[2(R^2 - r^2) - 2f'(R^2 - r^2)^2 \right] (2n + \text{tr}_g K) + 2 \left[1 + (R^2 - r^2)^2 f'' \right] 4r^2 \right\} \\ & + (n-2) \left\{ \left[1 - 2(R^2 - r^2) f' + (R^2 - r^2)^2 (f')^2 \right] 4r^2 - \frac{S_0}{n-1} (R^2 - r^2)^2 \right\} \end{aligned}$$

where $K := \nabla^2 r^2 - 2I$. Now using $(n-2)(f')^2 + 2f'' \leq 0$, we may continue the calculation to bound the above expression. In particular

$$\begin{aligned} & \leq \frac{1}{(R^2 - r^2)^2} \left\{ \left[2(R^2 - r^2) - 2f'(R^2 - r^2)^2 \right] (2n + \text{tr}_g K) + 8r^2 \right\} \\ & + (n-2) \left\{ \left[1 - 2(R^2 - r^2) f' \right] 4r^2 - \frac{S_0}{n-1} (R^2 - r^2)^2 \right\} \\ & = \frac{1}{(R^2 - r^2)^2} \left\{ 4nR^2 - 4nr^2 + 2\text{tr}_g K (R^2 - r^2) - 2f'(R^2 - r^2)^2 \Delta r^2 + 8r^2 \right\} \\ & + 4(n-2)r^2 - 8(n-2)(R^2 - r^2)f'r^2 - \frac{S_0}{n-1}(R^2 - r^2)^2 \\ & = \frac{1}{(R^2 - r^2)^2} \left\{ 4nR^2 + 2\text{tr}_g K (R^2 - r^2) - 2f'(R^2 - r^2)^2 \Delta r^2 \right\} \\ & - 8(n-2)(R^2 - r^2)f'r^2 - \frac{S_0}{n-1}(R^2 - r^2)^2 \\ & \leq \frac{1}{(R^2 - r^2)^2} \left\{ 4nR^2 - (-2\text{tr}_g K + \frac{S_0}{n-1})(R^2 - r^2) + 2f'(R^2 - r^2)(R^2 - r^2) \right\} \end{aligned}$$

where in the last line we used that $f' > 0$ and $\Delta r^2 \geq 1$. Applying the second defining property of f we conclude that

$$\begin{aligned} -\frac{S_g}{n-1} + 2\Delta u + (n-2)|du|^2 & \leq 4nR^2 \frac{1}{(R^2 - r^2)^2} \\ & \leq 4nR^2 \frac{1}{(R^2 - r^2)^2} e^{2f} \\ & = \frac{n}{n-1} e^{2u}. \end{aligned}$$

So, given x_0 as above and noting that $\bar{u}$ is infinite on $\partial B_R(z_0)$, we apply the maximum principle on this ball to conclude that

$$\begin{aligned} u(x_0) & \leq \bar{u}(x_0) \\ & = -\ln(R^2 - (R-d)^2) + \ln 2 + \ln R + \frac{1}{2} \ln(n-1) + f(R^2 - r^2) \\ & = -\ln(2Rd - d^2) + \ln 2 + \ln R + \frac{1}{2} \ln(n-1) + f(2Rd - d^2) \\ & = -\ln d - \ln(2R - d) + \ln 2R + \frac{1}{2} \ln(n-1) + f(2Rd - d^2) \\ & = -\ln d + \frac{1}{2} \ln(n-1) - \ln \frac{(2R-d)}{2R} + f(d(2R-d)). \end{aligned}$$

Taking the limit as d goes to zero yields the first result of the lemma. To see the second statement, note that the supersolution $\bar{u}$ can be constructed as above on any sufficiently small

ball in M . Indeed, given $z_0 \in M$ with $B_R(z_0) \subset M$ with R sufficiently small, the estimates above yield that

$$u(z_0) \leq \bar{u}(z_0) = \ln \frac{(n-1)^{\frac{1}{2}}}{R} + f(R^2) \leq C(R).$$

□

We are now ready to give the proof of Theorem 1.4.

Proof Our proof is similar in nature to [12] Theorem 4, and indeed we will exploit an estimate derived there for our purposes. We reuse the notation of the previous sections. The first step is to construct a solution to the problem

$$\begin{aligned} F_1(u) &= 0 \\ \lim_{x \rightarrow \partial M} u(x) &= \infty. \end{aligned} \quad (5.2)$$

Remark 4.10 guarantees the existence of functions u_j solving

$$\begin{aligned} F_1(u_j) &= 0 \\ u_j|_{\partial M} &\equiv j. \end{aligned} \quad (5.3)$$

We claim that we can extract a subsequence which converges uniformly on compact sets to a solution to (5.2). First observe that $u_j \geq u_0$ by Proposition 2.1. Furthermore, by the second statement of Lemma 5.2 we have that for given $K \subset M \setminus \partial M$, there exists a constant $C = C(K)$ such that $u_j \leq C(K)$ for all $j \geq 0$, the constant depending on $d(K, \partial M)$. Therefore we may apply the interior regularity estimates for solutions to $F_1(u) = 0$ to conclude uniform C^l bounds for any l on any given compact subset $K \subset M \setminus \partial M$. Interior regularity for such equations is well established, and one may see for instance [7] Theorems 2.1, 3.1. By the Arzela-Ascoli theorem, we conclude that a subsequence u_{j_n} converges uniformly on compact sets to a function u_∞ .

To show that u is indeed a solution to (5.2), first note that by Proposition 2.1 the sequence $\{u_j\}$ is monotonically increasing. Applying Lemma 5.1 we conclude

$$\begin{aligned} u_\infty &\geq \lim_{j \rightarrow \infty} u_j \\ &\geq \lim_{j \rightarrow \infty} \left[-\ln(r + e^{-j}) + \frac{1}{2} \ln(n-1) + w \right] \\ &= -\ln r + \frac{1}{2} \ln(n-1) + w. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{x \rightarrow \partial M} u_\infty &\geq \lim_{x \rightarrow \partial M} -\ln r + \frac{1}{2} \ln(n-1) + w \\ &= \lim_{x \rightarrow \partial M} -\ln r + \frac{1}{2} \ln(n-1). \end{aligned}$$

This in fact yields the precise expected asymptotic lower limit for u_∞ . Lemma 5.2 yields the asymptotic upper limit, implying the precise asymptotic behaviour of u_∞ near the boundary.

Finally, we show uniqueness of the solution among solutions approaching ∞ at the boundary. Suppose w is a solution to (5.2). By Lemma 5.2 we conclude that

$$w(x) \leq -\ln r + \frac{1}{2} \ln(n-1) + o(1)$$

where the notation $o(1)$ means $\lim_{x \rightarrow \partial M} o(1) = 0$. Furthermore let u_j be as in (5.3). It follows from Proposition 2.1 that $w \geq u_j$, and hence arguing as above we conclude

$$w \geq u_\infty \geq -\ln r + \frac{1}{2} \ln(n-1) + o(1).$$

Hence w has the same asymptotic behavior as the constructed solution u_∞ . Let $M_\delta = \{r \geq \delta\}$ with boundary $\partial M_\delta = \{r = \delta\}$ for small $\delta > 0$. Both w and u_∞ are solutions to $F_1(\cdot) = 0$ on M_δ with nearly equal boundary conditions. In particular, given $\epsilon > 0$ we may choose δ small such that $w \leq u_\infty + \epsilon$ on ∂M_δ . Furthermore, it is clear that $u_\infty + \epsilon$ is a supersolution to $F_1(\cdot) = 0$, therefore by Proposition 2.1 we conclude that $w \leq u_\infty + \epsilon$ on M_δ . Taking the limit as ϵ goes to 0 we see that $\delta \rightarrow 0$ as well, and so we conclude that $w \leq u_\infty$. However, the argument is symmetric for any two solutions to (5.2), hence $u_\infty \leq w$ and so $w \equiv u_\infty$.

To solve the other σ_k problems, we merely observe as in the proof of Theorem 1.3, we have produced a subsolution to the σ_n problem and a supersolution to the σ_1 problem, hence we have subsolutions and supersolutions to every σ_k problem by Maclaurin's inequality, and the proof may be followed exactly as above. $\square$

6 Conformally compact manifolds

In this section we give an application of Theorem 1.4 to the study of Poincaré–Einstein metrics. The material is inspired by the work of Mazzeo–Pacard, and we will refer to [13] for many of the details. We also change notation for this section to that more commonly used in the study of Poincaré–Einstein metrics. The following definition of conformally compact metrics contains the basic setup.

Definition 6.1 Let $\bar{X}^{n+1}$ be a compact manifold with boundary $M^n = \partial X^{n+1}$. A Riemannian metric g_+ defined in the interior X^{n+1} is said to be *conformally compact* if there is a non-negative defining function $\rho \in C^\infty(X^{n+1})$ with

$$\begin{aligned} \rho &> 0 \text{ in } X^{n+1}, \\ \rho &= 0 \text{ on } M^n, \\ |\nabla_{g_+} \rho| &\neq 0 \text{ on } M^n, \end{aligned}$$

such that $\bar{g} = \rho^2 g_+$ defines a Riemannian metric on $\bar{X}^{n+1}$, and $\bar{g}$ extends at least continuously to M . The manifold $(M^n, \bar{g})$ is called the *conformal infinity* of (X^{n+1}, g_+) .

Note that one can obtain other defining functions through multiplication by a positive function; thus the object naturally associated to a conformally compact manifold is not the metric $\bar{g}$ per se (which depends on ρ) but the conformal class $[\bar{g}]$ of its conformal infinity. The curvature transformation formulas for conformal metrics automatically imply that any conformally compact manifold is *asymptotically hyperbolic*: i.e., all the sectional curvatures of g_+ converge to -1 at infinity.

A conformally compact manifold (X^{n+1}, g_+) satisfying the Einstein condition

$$\text{Ric}(g_+) = -ng_+ \tag{6.1}$$

is called a *Poincaré–Einstein* (P–E) metric. The canonical example of such a metric is $X^{n+1} = B^{n+1} \subset \mathbb{R}^{n+1}$, the unit ball, with g_+ the hyperbolic metric, $\bar{g} = \frac{1}{4}(1 - |x|^2)g_+ = ds^2$ the Euclidean metric, and the conformal infinity is the round sphere $(\mathbb{S}^n, g = \bar{g}|_{\mathbb{S}^n})$. Due to its

connection to the AdS/CFT correspondence (see [15]) there is an extensive literature on the subject of P–E metrics and their physical/geometric properties.

The question of the existence of a P–E metric with given conformal infinity can be interpreted as an asymptotic Dirichlet problem. In [13], Mazzeo–Pacard explored the connection between the existence of P–E metrics and the σ_k -Yamabe problem. More precisely, let g_+ be a P–E metric; by (6.1) the Schouten tensor is given by

$$A(g_+) = -\frac{1}{2}g_+,$$

so that

$$\sigma_k[-A(g_+)] = \frac{1}{2^k} \binom{n+1}{k} \equiv \beta_{k,n}. \quad (6.2)$$

Therefore, a P–E metric is a solution (indeed, the unique solution) of the k -Yamabe problem in its conformal class, for all $1 \leq k \leq n+1$. The converse is also true: a conformally compact metric g_+ satisfying (6.2) for all $1 \leq k \leq n+1$ is obviously P–E. The main result of Mazzeo–Pacard is a more precise statement of this equivalence:

Theorem 6.2 (See [13], Theorems 1, 3) *Let Σ_k denote the set of conformally compact metrics on X^{n+1} with Schouten σ_k -curvature equal to $\beta_{k,n}$.*

- (i) *If $g \in \Sigma_k$, then there is a neighborhood $\mathcal{U}$ of g in the space of conformally compact metrics on X^{n+1} such that $\mathcal{U} \cap \Sigma_k$ is an analytic Banach submanifold of $\mathcal{U}$ (with respect to an appropriate Banach topology).*
- (ii) *In addition,*

$$\mathcal{E} = \bigcap_{k=1}^{n+1} \Sigma_k,$$

where $\mathcal{E}$ is the set of Poincaré–Einstein metrics. Hence, $\mathcal{E}$ is a finite intersection of locally closed Banach submanifolds, and in particular is always closed in the space of conformally compact metrics on X^{n+1} .

This equivalence is not just an algebraic curiosity: as Mazzeo–Pacard point out, the linearization of the P–E condition may have a nontrivial finite dimensional cokernel, while (as we saw in Sect. 2), the linearization of the Schouten equations do not, at least in the negative cone. On the other hand, it is important to point out that when $k \geq 2$, aside from P–E metrics (and perturbations arising from the above Theorem) there are no general existence results for metrics in Σ_k . Indeed, for $k \geq 2$, given a conformal infinity $(M^n, [\bar{g}])$ there may be no conformally compact metrics $g_+ \in \Sigma_k$ with $\rho^2 g_+|_{M^n} = \bar{g}$. Indeed, given a conformal class, the problem of finding a complete metric in the class solving (6.2) is obstructed:

Proposition 6.3 *Let $X^{n+1} = S^n \times (0, 1)$, and $g = g_{S^n} + dt^2$ where t is the coordinate on $(0, 1)$. There is no complete metric g_+ conformally related to g solving (6.2) for $k = n$.*

Proof Suppose g_+ is a solution to (6.2). Since g_+ is asymptotically hyperbolic, the Schouten tensor is strictly negative definite near the boundary of X at infinity. It follows by the maximum principle applied to equation (6.2) that the Schouten tensor is negative definite in X . Since g is conformally flat, it follows that the curvature operator

$$R = S \odot g$$

is negative definite, where $\odot$ denotes the Kulkarni-Nomizu product. By the Cartan Hadamard theorem it follows that the universal cover of X is $\mathbb{R}^{n+1}$, a contradiction. $\square$

By contrast, it follows from Theorem 5.2 that every conformally compact manifold $(X^{n+1}, g_+ = \rho^{-2}\bar{g})$ admits a unique conformal metric $h_k = e^{2w_k}\bar{g}$ with

$$\sigma_k[-h_k^{-1}\text{Ric}(h_k)] = \tilde{\beta}_{k,n} > 0$$

where we define the constants

$$\tilde{\beta}_{k,n} = \sigma_k(ng) = n^k \binom{n+1}{k}, \quad (6.3)$$

as the values of $\sigma_k[-\text{Ric}]$ for a Poincaré Einstein metric normalized as in (6.1). Therefore, it is natural to ask whether the results of Mazzeo–Pacard have a counterpart for symmetric functions of the Ricci tensor. The answer turns out to be yes.

Theorem 6.4 *Let (X^{n+1}, g_+) be a conformally compact manifold. Let Θ_k denote the set of conformally compact metrics on X^{n+1} with $\sigma_k[-g_+^{-1}\text{Ric}]$ equal to $\tilde{\beta}_{k,n}$.*

- (i) *Given a conformally compact metric $g_+ = \rho^{-2}\bar{g}$, and $1 \leq k \leq n+1$, there is a unique conformally compact metric $h_k = e^{2w_k}\bar{g} \in \Theta_k$.*
- (ii) *Let $\mathcal{E}$ denote the space of Poincaré–Einstein metrics. Then*

$$\mathcal{E} = \bigcap_{k=1}^{n+1} \Theta_k,$$

Hence $\mathcal{E}$ is a finite intersection of locally closed Banach submanifolds, and in particular is always closed in the space of conformally compact metrics on X^{n+1} .

Proof We sketch the details as the proof of a straightforward adaptation of the proof of Theorem 6.2. That proof relies on the structure of the linearized operator for the Schouten tensor equations, and except for some differences of constants, the linearized operator of the corresponding equations for the Ricci tensor is the same. To be more precise, let

$$\begin{aligned} \mathcal{H}_k(g_+, w) = & \sigma_k[-\text{Ric}(g_+) + (n-2)\nabla^2 w + \Delta w g_+ \\ & - (n-2)(dw \otimes dw - |dw|^2 g_+)] - \tilde{\beta}_{k,n} e^{2kw}. \end{aligned}$$

Thus, if $\mathcal{H}_k(g_+, w) = 0$ (and $e^{2w}g_+$ is conformally compact) then $e^{2w}g_+ \in \Theta_k$, and conversely. If g_+ is P–E, then the linearization of $\mathcal{H}_k$ with respect to w is given by

$$(\mathcal{L}_{\text{Ric}})_k \phi = \tilde{c}_{k,n} \Delta \phi - 2k \tilde{\beta}_{k,n} \phi,$$

where

$$\tilde{c}_{k,n} = 2(n-1)n^{k-1} \binom{n}{k}.$$

If we linearize the Schouten tensor equations at a P–E metric, the operator is given by

$$(\mathcal{L}_A)_k \phi = c_{k,n} \Delta \phi - 2k \beta_{k,n} \phi,$$

where

$$c_{k,n} = 2^{1-k} \binom{n}{k}.$$

The essential feature is that, in both cases, there is one positive and one negative indicial root of the associated normal operator. This makes it possible to choose an appropriate weighted space on which $(\mathcal{L}_A)_k$ and $(\mathcal{L}_{Ric})_k$ are Fredholm (see Sect. 2 of [13]). After setting up the right function spaces and mappings, both statements of Theorem 6.2 follow from a version of the implicit function theorem in, for example, [6]. $\square$

Finally, we note that the above characterization of $\mathcal{E}$ can be weakened considerably, and this involves the introduction of a family of n potentially interesting conformal invariants.

Definition 6.5 Let $(X^{n+1}, g_+ = \rho^{-2}\bar{g})$, be a conformally compact manifold. Fix $1 \leq k \leq n+1$, and let $h_k = e^{2w_k}\bar{g}$ be the unique conformally compact metric satisfying

$$\sigma_k[-h_k^{-1} \text{Ric}(h_k)] = \tilde{\beta}_{k,n}.$$

Given $1 \leq k \leq n$, let

$$H_k = w_k - w_{n+1}. \quad (6.4)$$

Proposition 6.6 Let $(X^{n+1}, g_+ = \rho^{-2}\bar{g})$ be a conformally compact manifold, and fix $1 \leq k \leq n$.

- (1) H_k is a conformal invariant, that is, the definition above does depend on the choice of conformal background metric $\bar{g}$.
- (2) $H_k \in C^\infty(X^{n+1}) \cap C^0(\bar{X}^{n+1})$
- (3) $H_k = 0$ on $\partial X^{n+1} = M^n$
- (4) $H_k \geq 0$ in X^{n+1} . Moreover, $H(x_0) = 0$ at some point $x_0 \in X^{n+1}$ if and only if $H_k \equiv 0$ and $g_+^1 = g_+^{n+1}$ is a Poincaré–Einstein metric.

Proof (1) If we change $\bar{g}$ to $e^{2\phi}\bar{g}$ for some $\phi \in C^\infty(\bar{X}^{n+1})$, then the corresponding solutions to the σ_k problem are respectively $w_k + 2\phi$ and $w_n + 2\phi$. Therefore, $H = H(g_+)$ is uniquely determined by a given conformally compact metric, independent of the choice of defining function.

(2,3) The solutions w_k are always smooth on the interior. Moreover, by Lemmas 5.1 and 5.2 they have the same asymptotic limit at ∂M , hence $H|_{\partial M} \equiv 0$.

(4) By Maclaurin's inequality w_k is a supersolution of the σ_{n+1} equation, therefore by Proposition 2.1 we conclude that $H \geq 0$, and $H(x_0) = 0$ if and only if H vanishes identically. If this is the case, the characterization of equality in the Maclaurin inequality yields that $g_+^1 = g_+^{n+1}$ is a Poincaré–Einstein metric. $\square$

This proposition says that $\mathcal{E}$ is the set of all conformally compact metrics g_+ for which for some $1 \leq k \leq n$, the function $H_k(g_+)$ vanishes somewhere, and hence everywhere. From this perspective, the conformal invariants H_k carry the same information for different choices of k .

7 Remarks on positive curvature

Remark 7.1 In the notation of the introduction, let $\mathcal{S} = \{S_g > 0\}$ where here S_g is the Schouten tensor of g . Recall the equation for the conformal Schouten tensor.

$$S_{e^{-2u}g} = S_g + \nabla^2 u + du \otimes du - \frac{1}{2} |du|^2 g.$$

If we let $u = \ln w$ this is rewritten as

$$S_{w^{-2}g} = S_g + \frac{1}{w} \nabla^2 w - \frac{1}{2} \frac{|dw|^2}{w^2} g$$

Now consider an open set U in $\mathbb{R}^n$ with nonconvex boundary. Suppose one had a function $w > 0$ such that $S_{w^{-2}g} > 0$ and $w_{\partial U} \equiv 0$. Since the metric background metric on U is flat, it follows that $\nabla^2 w > 0$. Since $w_{\partial U} \equiv 0$, it follows that ∂U is a level set of a convex function, and as such should be convex, which it is not. Thus we can never solve for conformal deformation to positive Schouten tensor with restricted boundary condition. However, it is not yet clear if we can solve without the boundary condition, or whether positive Ricci curvature can be solved for.

However, it is easy to show that on a surface with boundary one can always deform to a metric of positive scalar curvature.

Proposition 7.2 *Let $(M^2, \partial M, g)$ be a compact Riemannian surface with boundary. There exists $u \in C^\infty(M)$ such that $R(e^{-2u}g) > 0$ and $u|_{\partial M} \equiv 0$.*

Proof On a surface one has

$$e^{2u} R(e^{-2u}g) = R(g) - 2\Delta u.$$

Therefore the problem reduces to solving the Dirichlet problem

$$\begin{aligned} -2\Delta u &= 1 - R(g) \\ u &\equiv 0 \quad \text{on } \partial M. \end{aligned}$$

Solvability of this equation is a known result ([2] Theorem 4.8). $\square$

Therefore deformation to positive curvatures remains elusive, and the nature of the obstructions, if any, are not clear. To emphasize the issues here, we formally ask the following question.

Question 7.3 Given $(M^n, \partial M, g)$, $n \geq 3$, can we conformally deform g to a metric with positive Ricci curvature? Scalar curvature? Can we do either while preserving the induced metric on the boundary?

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