RADIAL SOLUTIONS OF A FOURTH ORDER HAMILTONIAN STATIONARY EQUATION

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Abstract. We consider smooth radial solutions to the Hamiltonian stationary equation which are defined away from the origin. We show that in dimension two all radial solutions on unbounded domains must be special Lagrangian. In contrast, for all higher dimensions there exist non-special Lagrangian radial solutions over unbounded domains; moreover, near the origin, the gradient graph of such a solution is continuous if and only if the graph is special Lagrangian.

1. Introduction

An important class of Lagrangian submanifolds in a symplectic manifold is the so-called Hamiltonian stationary submanifolds, which are Lagrangian submanifolds and are critical points of the volume functional under Hamiltonian variations. A well-known subset of this class consists of the special Lagrangian submanifolds. These are Lagrangian and critical for the volume functional for all variations; so they are minimal Lagrangian submanifolds. The special Lagrangians form one of the most distinguished classes in calibrated geometry, and they play a unique role in string theory. In this paper, we shall study the radially symmetric Hamiltonian stationary graphs in the complex Euclidean space $\mathbb{C}^n$, and explore conditions under which the Hamiltonian stationary ones reduce to the special Lagrangians.

For a fixed bounded domain $\Omega \subset \mathbb{R}^n$, let $u : \Omega \to \mathbb{R}$ be a smooth function. The gradient graph $\Gamma_u = \{(x, Du(x)) : x \in \Omega\}$ is a Lagrangian n-dimensional submanifold in $\mathbb{C}^n$, with respect to the complex structure J defined by $z_j = x_j + iy_j$ for $j = 1, \cdots, n$. The volume of Γ_u is given by

$$F_\Omega(u) = \int_\Omega \sqrt{\det\left(I + (D^2u)^T D^2u\right)}dx.$$

A smooth function u is critical for the volume functional $F_\Omega(u)$ under compactly supported variations of the scalar function if and only if u satisfies the equation

$$\Delta_g \theta = 0 \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator on Γ_u for the induced metric g from the Euclidean metric on $\mathbb{R}^{2n}$, (c.f. [Oh93], [SW03, Proposition 2.2]). Here, the Lagrangian phase function is defined by

$$\theta = \operatorname{Im}\log\det\left(I_n + \sqrt{-1}D^2u\right)$$

or equivalently,

$$\theta = \sum_{i=1}^{n} \arctan \lambda_i \tag{1.2}$$

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for λ_i the eigenvalues of D^2u. The mean curvature vector along Γ_u can be written

$$\vec{H} = J\nabla\theta$$

where ∇ is the gradient operator of Γ_u for the metric g (cf. [HL82, 2.19]).

The gradient graph of u which solves (1.1) is called Hamiltonian stationary. A Hamiltonian stationary gradient graph Γ_u is a critical point of the volume functional $F_\Omega(\cdot)$ under Hamiltonian deformations, that is, those generated by $J\nabla\eta$ for some smooth compactly supported function η. On the other hand, recall that if u satisfies the special Lagrangian equation [HL82]

$$\nabla\theta = 0 \tag{1.3}$$

i.e. $\vec{H} \equiv 0$, then the surface is critical for the volume functional under *all* compactly supported variations of the surface Γ_u. In terms of the potential function u, the Hamiltonian stationary equation is of fourth order and the special Lagrangian equation is of second order, both are elliptic. There are Hamiltonian stationary but not special Lagrangian surfaces even when $n = 2$, and this causes serious problems for constructing special Lagrangian surfaces (see [SW03]).

In this paper, we consider radial solutions of the Hamiltonian stationary equation (1.1) on a domain which may not contain the origin. Our first observation is that the fourth order Hamiltonian stationary equation reduces to the second order special Lagrangian equation for radial solutions defined on any unbounded domain in $\mathbb{R}^2$. However, this is not the case for $n > 2$.

We show the following:

Theorem 1.1. *Suppose that $u = u(r)$ is a radial solution of the fourth order Hamiltonian stationary equation (1.1) on $\left\{x \in \mathbb{R}^2 : |x| > a\right\}$ for some $a \geq 0$. Then u is a solution of the special Lagrangian equation (1.3). For any $n > 2$, (1.1) admits a radial solution over any unbounded domain in $\mathbb{R}^n \backslash \{0\}$, which is not special Lagrangian.*

Theorem 1.1 will be proved in two propositions: Proposition 3.2 and Proposition 4.1.

Radial solutions can be characterized according to their behavior near 0. For $n = 3$, we exhibit a strong solution (a classical solution away from the origin of $\mathbb{R}^3$) the closure of whose gradient graph over the unit 3-ball $\mathbb{B}^3$ is a smoothly embedded submanifold with boundary that is diffeomorphic to a half cylinder, yet it is not Hamiltonian isotopic to the gradient graph of any smooth function over $\mathbb{B}^3$. In fact, we are able to show the following:

Theorem 1.2. *Suppose that $u = u(r)$ is a radial solution of the fourth order Hamiltonian stationary equation (1.1) on $\{x \in \mathbb{R}^n : 0 < |x| < b\}$. If*

$$\lim_{r\to 0} u'(r) = 0$$

then u is a solution of the special Lagrangian equation and the completed graph is a flat disk.

If

$$\lim_{r\to 0} u'(r) = c \neq 0$$

then the gradient graph (including the sphere over the origin as its boundary) is a properly embedded half cylinder.

Note that, due to the strong maximum principle applied to (1.1) on any $B_R(0)$, any radial and harmonic θ that extends to a weakly harmonic function across the origin must be constant. It follows from a removable singularity result of Serrin [Ser64] that if a radial solution u is $C^{1,1}$ near the origin, then θ must extend to a weakly harmonic function and must be constant (cf. [CW16]). The corresponding potential function u must be quadratic, as we will see in section 5 where we investigate properties of the ODE and combine them with a calibration argument.

Bernstein type results for special Lagrangian equations in dimension 2 were obtained by Fu [Fu98]: the global solutions are either quadratic polynomials or harmonic functions. For convex solutions and for large phases in higher dimensions Bernstein results were given by Yuan [Yua02, Yua06]. For Bernstein and Liouville results for (1.1) with constraints, and more discussion of the problem, see [Mes01], [War16].

The rest of the paper is organized as follows. In section 2, we will derive an ODE that characterizes radial solutions, and prove that the Dirichlet integral of the phase function is always finite in the radially symmetric case. In section 3, we show that for $n = 2$ all radial solutions on unbounded domains must be special Lagrangian, and write down all radial solutions to special Lagrangian equations. In section 4, we show that, for $n > 2$, non-special Lagrangian solutions to (1.1) exist on $\mathbb{R}^n \setminus \{0\}$. In section 5, we explore the behavior near the origin, and show that continuity of the gradient graph implies a radial solution is special Lagrangian.

2. The radial Hamiltonian stationary equation

We are interested in the gradient graph $\Gamma = \{(x, Du(x)) : x \in \mathbb{R}^n \setminus \{0\}\}$ of a radially symmetric function u. Suppose $u = u(r)$, then

$$Du = u'(r)Dr.$$

In terms of the standard parametrization

$$\begin{aligned}(0,\infty) \times \mathbb{S}^{n-1} &\to \mathbb{R}^n \setminus \{0\}\\ (r,\xi) &\to r\xi\end{aligned}$$

we can parametrize the gradient graph as

$$(r,\xi) \to (r\xi, u'(r)\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2.1}$$

Define the following notations for tangent vectors at a point $(r\xi, u'(r)\xi)$ of Γ

$$\partial_r = (\xi, u''(r)\xi) \in T_{(r\xi,u'(r)\xi)}\Gamma$$

and

$$\partial_V = (rV, u'(r)V) \in T_{(r\xi,u'(r)\xi)}\Gamma$$

for each $V \in T_\xi \mathbb{S}^{n-1}$. Thus, the induced metric g on Γ may be computed as

$$\begin{aligned}
g_{rr} &= \langle \partial_r, \partial_r \rangle = 1 + (u'')^2\\
g_{VV} &= \langle \partial_V, \partial_V \rangle = r^2 + (u')^2\\
g_{V_iV_j} &= \langle \partial_{V_i}, \partial_{V_j} \rangle = 0, \quad \forall i \neq j, V_i \perp V_j, V_i, V_j \in T_\xi \mathbb{S}^{n-1}\\
g_{rV} &= \langle \partial_r, \partial_V \rangle = 0.
\end{aligned}$$

In the (r,ξ)-coordinates, the Hamiltonian stationary equation

$$\sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ij} \theta_i \right)_j = 0$$

reduces to (because only r derivatives persist)

$$(\sqrt{g} g^{rr} \theta_r)_r = 0.$$

This directly gives that

$$\sqrt{g} g^{rr} \theta_r = C,$$

in particular

$$\theta_r = \frac{Cg_{rr}}{\sqrt{g}} = C\sqrt{\frac{1+(u'')^2}{\left(r^2+(u')^2\right)^{n-1}}}. \tag{2.2}$$

To compute θ, we may diagonalize D^2u with respect to the Euclidean coordinates. Taking $x_1 = r, x_2 = 0, \ldots, x_n = 0$, then

$$D^2u = \begin{pmatrix} u'' & 0 & ... & 0 \\ 0 & \frac{u'}{r} & 0 & ... \\ ... & 0 & ... & 0 \\ 0 & ... & 0 & \frac{u'}{r} \end{pmatrix}.$$

Thus by (1.2)

$$\theta = \arctan(u'') + (n-1)\arctan\left(\frac{u'}{r}\right)$$

and

$$\theta_r = \frac{1}{1+(u'')^2}u''' + (n-1)\frac{1}{1+\left(\frac{u'}{r}\right)^2}\left(\frac{u'}{r}\right)'.$$

Finally, in terms of the radial function u, the Hamiltonian stationary equation (2.2) becomes

$$\frac{1}{1+(u'')^2}u''' + (n-1)\frac{1}{1+\left(\frac{u'}{r}\right)^2}\left(\frac{u'}{r}\right)' = C\sqrt{\frac{1+(u'')^2}{\left(r^2+(u')^2\right)^{n-1}}}. \tag{2.3}$$

Note that $C = 0$ corresponds to the special Lagrangian equation (1.3).

Letting

$$v = u',$$

the equation (2.3) becomes a second order ODE in v:

$$\frac{1}{1+(v')^2}v'' + (n-1)\frac{v'r - v}{r^2+v^2} - C\frac{\left(1+(v')^2\right)^{\frac{1}{2}}}{(r^2+v^2)^{\frac{n-1}{2}}} = 0. \tag{2.4}$$

Next, we note a consequence of (2.2): For any radial solution to the Hamiltonian stationary equation, the energy of the Lagrangian phase function θ admits a uniform upper bound, independent of the size of the domain.

Theorem 2.1. *Suppose that $u(r)$ is a radial solution of the fourth order Hamiltonian stationary equation (1.1) on $\{x \in \mathbb{R}^n : a < |x| < b.\}$. Letting $\Gamma = \{(x, Du(x)) : a < |x| < b\}$, then*

$$\int_\Gamma \left|\nabla_g \theta\right|^2 dV_g = C\,\mathrm{Vol}(\mathbb{S}^{n-1})\left[\theta(b) - \theta(a)\right],$$

where C is the constant in (2.2). In particular, the energy of θ is always finite.

Proof. Using the fact that θ is radial,

$$\left|\nabla_g\theta\right|^2 = g^{rr}\theta_r^2.$$

Now using (2.2),

$$\left|\nabla_g\theta\right|^2 = g^{rr}\theta_r\theta_r = g^{rr}\frac{C^2g_{rr}^2}{g} = \frac{C^2g_{rr}}{g}.$$

Thus

$$\int_{\mathbb{B}^n_b(0)\backslash\mathbb{B}^n_a(0)} \left|\nabla_g\theta\right|^2 dV_g = \int_{\mathbb{S}^{n-1}}\int_a^b \frac{C^2 g_{rr}(r)}{g(r)}\sqrt{g(r)}drdV_{S^{n-1}}$$
$$= \mathrm{Vol}(\mathbb{S}^{n-1})\int_a^b \frac{C^2 g_{rr}(r)}{\sqrt{g(r)}}dr.$$

On the other hand, by (2.2) again

$$\theta(b)-\theta(a) = \int_a^b \theta_r dr = \int_a^b \frac{C g_{rr}(r)}{\sqrt{g(r)}}dr.$$

Putting these together finishes the proof. □

3. Reduction of order for $n = 2$

The equation (2.4) always admits a short time solution near 0 for any constant C, provided that $v(0) \neq 0$. In particular, the solutions for $C \neq 0$ are not solutions to the special Lagrangian equation for any $n \geq 2$. However, when $n = 2$, we will show that the solutions to (2.4) which are defined on any *unbounded* domain in $\mathbb{R}^2$ are necessarily special Lagrangian. In the next section, we will exhibit existence of solutions on $\mathbb{R}^n\backslash\{0\}$ for $n > 2$ which are not special Lagrangian. So the reduction of order for radial solutions, namely from the fourth order equation (2.4) to the second order equation (1.3) only happens in dimension 2 and is non-local.

We first investigate the case that the limit of θ at infinity is not zero.

Proposition 3.1. *When $n = 2$, if $u = u(r)$ is a radial solution on (a,∞) to (2.3) with*

$$\lim_{r\to\infty}\theta(r) \neq 0$$

then $C = 0$. In particular, u is a solution of the special Lagrangian equation.

Proof. Recall that θ is bounded and monotone by (2.2), so θ has a finite limit when $r \to \infty$. Without loss of generality we may take $\lim_{r\to\infty}\theta(r) = \delta > 0$. In particular, we may choose R_0 large enough so that

$$\theta(r) > \frac{\delta}{2}, \quad \text{for } r > R_0.$$

Letting

$$\Omega = \mathbb{R}^2\backslash\mathbb{B}^2_{R_0}(0) \subset \mathbb{R}^2\backslash\mathbb{B}^2_a(0)$$

we consider the portion of the gradient graph of u restricted to Ω, namely

$$\Gamma = \left\{(x, Du(x)) : x \in \mathbb{R}^2\backslash\mathbb{B}^2_{R_0}(0)\right\}.$$

Now for λ_1, λ_2, the eigenvalues of D^2u, we have

$$\arctan\lambda_1 + \arctan\lambda_2 > \frac{\delta}{2}.$$

So

$$D^2u(x) > \tan\left(\frac{\delta}{2}-\frac{\pi}{2}\right) > -\infty$$

when $x \in \Omega$. Now we may apply the Lewy-Yuan rotation argument in [Yua06, Step 1 in section 2], (cf. [CW16, Proposition 4.1]). In particular, there is a domain $\bar{\Omega} \subset \mathbb{R}^n$, and a function $\bar{u}$ on $\bar{\Omega}$ such that the surface $\bar{\Gamma} \subset \mathbb{R}^2 + \sqrt{-1}\mathbb{R}^2$ defined by

$$\bar{\Gamma} = \left\{(\bar{x}, D\bar{u}(\bar{x})) : \bar{x} \in \bar{\Omega}\right\}$$

is isometric to Γ via a unitary rotation of $\mathbb{C}^2$. Expressly, letting

$$\bar{x}(x)=\cos\left(\frac{\delta}{4}\right)x+\sin\left(\frac{\delta}{4}\right)Du(x) \tag{3.1}$$

we have

$$\bar{\Omega}=\left\{\bar{x}(x):x\in\mathbb{R}^2\backslash\mathbb{B}^2_{R_0}(0)\right\}.$$

Also, when $\bar{x}=\bar{x}(x)$,

$$D\bar{u}\left(\bar{x}\right)=-\sin\left(\frac{\delta}{4}\right)x+\cos\left(\frac{\delta}{4}\right)Du(x). \tag{3.2}$$

When u is radial, the derivative satisfies

$$Du(x)=u'(|x|)\frac{x}{|x|}$$

so (3.1) becomes

$$\bar{x}(x)=\cos\left(\frac{\delta}{4}\right)x+\sin\left(\frac{\delta}{4}\right)u'(|x|)\frac{x}{|x|} \tag{3.3}$$

$$=\left(\cos\left(\frac{\delta}{4}\right)+\sin\left(\frac{\delta}{4}\right)\frac{u'(|x|)}{|x|}\right)x \tag{3.4}$$

and (3.2) becomes

$$D\bar{u}\left(\bar{x}\right)=\left(-\sin\left(\frac{\delta}{4}\right)+\cos\left(\frac{\delta}{4}\right)\frac{u'(|x|)}{|x|}\right)x \tag{3.5}$$

$$=\left(-\sin\left(\frac{\delta}{4}\right)+\cos\left(\frac{\delta}{4}\right)\frac{u'(|x|)}{|x|}\right)\left(\cos\left(\frac{\delta}{4}\right)+\sin\left(\frac{\delta}{4}\right)\frac{u'(|x|)}{|x|}\right)^{-1}\bar{x}. \tag{3.6}$$

We conclude that $D\bar{u}(\bar{x})$ is a multiple of $\bar{x}$, so the function $\bar{u}$ is radial in the $\bar{x}$-coordinates.

Under this representation, we have (cf. [CW16, Proposition 4.1]) that the inverse map $\bar{x}^{-1}$ exists and is Lipschitz, that is

$$D\bar{x}\geq c_\delta I_2>0.$$

It follows that the complement of $\bar{\Omega}$ must be contained in a compact set:

$$\mathbb{R}^2\backslash\bar{\Omega}\subset\mathbb{B}^2_{\bar{R}_0}(0)$$

for some $\bar{R}_0$. Further, we have that

$$\tan\left(\frac{\delta}{2}-\frac{\pi}{2}-\frac{\delta}{4}\right)<D^2\bar{u}<\tan\left(\frac{\pi}{2}-\frac{\delta}{4}\right).$$

The induced metric on the gradient graph of $\bar{u}$ is still g because the two gradient graphs are isometric. On the gradient graph of $\bar{u}$, the metric g is given in terms of the $\bar{x}$-coordinates by

$$g=g_{\bar{\imath}\bar{j}}d\bar{x}^id\bar{x}^j$$

with

$$g_{\bar{\imath}\bar{j}}=\delta_{\bar{\imath}\bar{j}}+\bar{u}_{\bar{\imath}\bar{k}}\delta^{\bar{k}\bar{l}}\bar{u}_{\bar{l}\bar{j}}.$$

Thus, in these coordinates we have

$$d\bar{x}^2\leq g\leq C_1d\bar{x}^2.$$

The volume form has the expression

$$dV_g=\sqrt{g}\,d\bar{x}_1\wedge d\bar{x}_2,$$

so for any $R > \bar{R}_0$, if we define

$$\Gamma_R := \left\{(\bar{x}, D\bar{u}(\bar{x})) : \bar{x} \in \bar{\Omega}, |\bar{x}| < R\right\} \subset \bar{\Gamma},$$

then we have

$$\mathrm{Vol}(\Gamma_R) = \int_{\Gamma_R} dV_g \leq C_2 R^2.$$

Now because $\bar{u}$ is radial, so is the function $\bar{\theta}$, where

$$\bar{\theta} = \arctan\left(\bar{\lambda}_1\right) + \arctan\left(\bar{\lambda}_2\right)$$

for $\bar{\lambda}_1, \bar{\lambda}_2$ eigenvalues of $D^2\bar{u}$. One can check (cf. [CW16, pg. 20]) that

$$\bar{\theta} = \theta - \frac{\delta}{2}$$

so $\bar{\theta}$ is still harmonic with respect to the metric g on $\bar{\Gamma}$ and must satisfy an ordinary differential equation of the form (2.2).

Now choose any interval $[a, b]$ with $a > \bar{R}_0$, and some larger R. Taking $\bar{x}$ as coordinates for $\bar{\Gamma}$, we can define the function

$$\rho(\bar{x}) = |\bar{x}|.$$

We then define a Lipschitz test function $\eta(\rho(\bar{x}))$ on $\bar{\Gamma}$ as follows:

$$\begin{aligned}
\eta &= 0 && \text{if } \rho < a\\
\eta' &= \frac{1}{b-a} && \text{if } a \leq \rho \leq b\\
\eta &= 1 && \text{if } b \leq \rho \leq b + R\\
\eta' &= -\frac{1}{R} && \text{if } b + R \leq \rho \leq b + 2R\\
\eta &= 0 && \text{if } \rho > b + 2R.
\end{aligned}$$

We integrate (1.1) by parts and replace $\bar{\theta}$ by $\theta - \frac{\delta}{2}$:

$$0 = \int_{\bar{\Gamma}} \eta \Delta_g \theta \, dV_g = -\int_{\bar{\Gamma}} \langle \nabla_g \theta, \nabla_g \eta \rangle dV_g.$$

Thus

$$\begin{aligned}
\int_{\Gamma_b \backslash \Gamma_a} \langle \nabla_g \theta, \nabla_g \eta \rangle dV_g &= -\int_{\Gamma_{b+2R} \backslash \Gamma_{b+R}} \langle \nabla_g \theta, \nabla_g \eta \rangle dV_g\\
&\leq \left(\int_{\Gamma_{b+2R} \backslash \Gamma_{b+R}} \left|\nabla_g \theta\right|^2 dV_g\right)^{1/2} \left(\int_{\Gamma_{b+2R} \backslash \Gamma_{b+R}} \left|\nabla_g \eta\right|^2 dV_g\right)^{1/2}\\
&\leq \left(\int_{\Gamma \backslash \Gamma_{b+R}} \left|\nabla_g \theta\right|^2 dV_g\right)^{1/2} \frac{1}{R} \left(\mathrm{Vol}(\Gamma_{b+2R} \backslash \Gamma_{b+R})\right)^{1/2}\\
&\leq \left(\int_{\Gamma \backslash \Gamma_{b+R}} \left|\nabla_g \theta\right|^2 dV_g\right)^{1/2} C_2.
\end{aligned}$$

Now

$$\lim_{R \to \infty} \left(\int_{\Gamma \backslash \Gamma_{b+R}} \left|\nabla_g \theta\right|^2 dV_g\right)^{1/2} = 0$$

as the energy of θ is finite by Theorem 2.1. We conclude that

$$\int_{\Gamma_b\setminus\Gamma_a} \langle \nabla_g \theta, \nabla_g \eta\rangle dV_g = 0.$$

But

$$\int_{\Gamma_b\setminus\Gamma_a} \langle \nabla_g \theta, \nabla_g \eta\rangle dV_g = \frac{1}{b-a}\int_{\Gamma_b\setminus\Gamma_a} g^{\bar r\bar r}\theta_{\bar r} dV_g.$$

Because θ is radial in $\bar{x}$, and the above two lines are true for every $b > a > R_0$, we conclude that

$$\theta_{\bar r} \equiv 0.$$

It follows that θ and $\bar\theta$ are both constant, and both Γ and $\bar\Gamma$ are special Lagrangian. The potential function u must be a solution to the special Lagrangian equation. □

In light of Proposition 3.1, to show any radial solution to (2.3) over (a, ∞) is special Lagrangian, it suffices to consider the remaining case $\lim_{r\to\infty} \theta(r) = 0$.

Proposition 3.2. *When $n = 2$, any radial solution on (a, ∞) to (2.3) must be special Lagrangian.*

Proof. It suffices to show $C = 0$. For $n = 2$ the equation (2.2) is simply

$$\theta_r = C\frac{1}{r}\sqrt{\frac{1 + (u'')^2}{1 + \left(\frac{u'}{r}\right)^2}}. \tag{3.7}$$

Because θ stays finite, we have that

$$-\pi < \int_a^\infty C\frac{1}{r}\sqrt{\frac{1 + (u'')^2}{1 + \left(\frac{u'}{r}\right)^2}}dr < \pi. \tag{3.8}$$

We proceed by contradiction to show $C = 0$. Without loss of generality, we may assume that $C > 0$, hence θ is increasing. If $C < 0$, note that the left hand side of (2.2) is odd in u, while the right hand side is even. Thus we may replace u with $-u$ to obtain $C > 0$.

In view of Proposition 3.1, we only need to consider the case that $\lim_{r\to\infty} \theta = 0$. Note that $\theta < 0$ because θ is increasing in r (as $C > 0$) and θ vanishes at infinity.

It follows from

$$\theta = \arctan(u'') + \arctan\left(\frac{u'}{r}\right) < 0$$

that

$$u'' + \frac{u'}{r} < 0.$$

Now

$$(ru')' < 0$$

from which it follows that for any t_0

$$ru' \le t_0 u'(t_0) \text{ for } r > t_0. \tag{3.9}$$

First note that u'/r must be unbounded. If not, the integral expression in (3.8) would be unbounded. It follows that there is a sequence of r_k for which $|u'(r_k)/r_k| \to \infty$ as $k \to \infty$. Noting (3.9) we must have $u'(r_k)/r_k \to -\infty$. We conclude from (3.9) that there is r_0 so that

$$u' \le 0, \quad \text{for } r > r_0. \tag{3.10}$$

Now letting

$$q(r) = \operatorname{arcsinh}\left(\frac{u'(r)}{r}\right)$$

we conclude

$$q(r_k) \to -\infty, \quad \text{as } k \to \infty.$$

Computing the derivative,

$$q' = \frac{1}{\sqrt{1+\left(\frac{u'}{r}\right)^2}}\left(\frac{u''}{r} - \frac{u'}{r^2}\right).$$

Now because q is unbounded below, for any given large $M_k > 0$, we can find an interval $[a+1, s_k] \subset (a, \infty)$ such that

$$\int_{a+1}^{s_k} q'(r)dr = q(s_k) - q(a+1) = -M_k.$$

It follows that

$$\int_{a+1}^{s_k} \frac{1}{\sqrt{1+\left(\frac{u'}{r}\right)^2}} \frac{u''}{r} dr = -M_k + \int_{a+1}^{s_k} \frac{1}{\sqrt{1+\left(\frac{u'}{r}\right)^2}} \frac{u'}{r^2} dr. \tag{3.11}$$

Now

$$\begin{aligned} \left|\int_{a+1}^{s_k} \frac{1}{\sqrt{1+\left(\frac{u'}{r}\right)^2}} \frac{u''}{r} dr\right| &\le \left|\int_{a+1}^{s_k} \frac{1}{r}\sqrt{\frac{1+(u'')^2}{1+\left(\frac{u'}{r}\right)^2}} dr\right| \\ &= \frac{1}{C}\left|\int_{a+1}^{s_k} \theta_r \, dr\right| \\ &= \frac{1}{C}\left|\theta(s_k) - \theta(a+1)\right| \\ &< +\infty, \end{aligned} \tag{3.12}$$

using (3.7). Taking $M_k \to \infty$ we conclude from (3.11) and (3.12) that

$$\lim_{s_k\to\infty} \int_{a+1}^{s_k} \frac{1}{\sqrt{1+\left(\frac{u'}{r}\right)^2}} \frac{u'}{r^2} \, dr = +\infty.$$

Clearly, for this to happen we must have $u' > 0$ for at least a sequence of $r \to \infty$. But this contradicts (3.10).

We conclude that $C = 0$ and $\theta_r = 0$. Thus u solves the special Lagrangian equation. □

Remark 3.3. For $n = 2$, the special Lagrangian equation can be written as

$$\left(1 - u_{11}u_{22} + u_{12}^2\right)\sin\theta + (u_{11} + u_{22})\cos\theta = 0$$

where u_{ij} stands for the second order derivative of u in x_i, x_j. When u is radial, the above equation takes the form

$$\left(1 - \frac{u'u''}{r}\right)\sin\theta + \left(u'' + \frac{u'}{r}\right)\cos\theta = 0.$$

This can also be written as

$$\left(r^2 - (u')^2\right)'\sin\theta + 2(ru')'\cos\theta = 0.$$

Completing the square and integrating

$$\int_a^r \left[\left(u^{'}(t) - t\cot\theta\right)^2 - t^2 \csc^2\theta\right]^{\prime} dt = 0$$

we get

$$\left(u^{'}(r) - r\cot\theta\right)^2 - r^2\csc^2\theta = \left(u^{'}(a) - a\cot\theta\right)^2 - a^2\csc^2\theta$$

or

$$\left(u^{'}(r) - r\cot\theta\right)^2 = \csc^2\theta\left[r^2 + \beta_{a,u,\theta}\right]$$

for

$$\beta_{a,u,\theta} = \sin^2\theta\left[u^{'}(a)\right]^2 - a^2\sin^2\theta - 2a\cos\theta\sin\theta.$$

Thus

$$u'(r) = \frac{1}{\sin\theta}\left(r\cos\theta \pm \sqrt{r^2 + \beta_{a,u,\theta}}\right).$$

Integrating

$$\begin{aligned}\int_a^r u'(t)dt &= \frac{\cot\theta}{2}\left(r^2 - a^2\right) \pm \csc\theta\int_a^r \sqrt{t^2 + \beta_{a,u,\theta}}dt \\ &= \frac{\cot\theta}{2}\left(r^2 - a^2\right) + \csc\theta\frac{1}{2}\left(r\sqrt{r^2+\beta} - a\sqrt{a^2+\beta}\right) + \frac{\beta}{2}\csc\theta\ln\left(\frac{r + \sqrt{r^2+\beta}}{a + \sqrt{a^2+\beta}}\right).\end{aligned}$$

Thus

$$u(r) = u(a) + \frac{\cot\theta}{2}(r^2 - a^2) \pm \frac{\csc\theta}{2}\left(r\sqrt{r^2+\beta} - a\sqrt{a^2+\beta} + \beta\ln\left(\frac{r + \sqrt{r^2+\beta}}{a + \sqrt{a^2+\beta}}\right)\right). \tag{3.13}$$

Therefore, according to Proposition 3.2, all radial solutions to (1.1) defined on an unbounded domain $\left\{x \in \mathbb{R}^2 : |x| \geq a\right\}$ are given by the above formula (3.13) explicitly.

4. Existence of non special Lagrangian solutions on unbounded domain for $n > 2$

Observe that one instance of the equation (2.4), i.e. $C = 1 - n$, can be written as

$$F_\lambda(v) = 0 \tag{4.1}$$

where

$$\lambda = n - 1$$

and

$$F_\lambda(v) = \frac{1}{1 + (v')^2}v'' + \lambda\frac{v'r - v}{r^2 + v^2} + \lambda\frac{\left(1 + (v')^2\right)^{1/2}}{\left(r^2 + v^2\right)^{\lambda/2}}. \tag{4.2}$$

Note that solutions to (4.1) cannot be special Lagrangian since $C \neq 0$.

Proposition 4.1. *For $\lambda \geq 2$, the equation (4.1) admits nontrivial solutions on* $(0, \infty)$.

Proof. Let

$$v(0) = 1$$
$$v'(0) = 1.$$

Equivalent to (4.1), we are trying to solve

$$v'' = G_\lambda(r, v, v') = -\lambda \left[\frac{\left(1+(v')^2\right)^{1/2}}{(r^2+v^2)^{\lambda/2}} + \frac{v'r - v}{r^2+v^2} \right] \left(1+(v')^2\right). \tag{4.3}$$

Notice that when $v > 0$ or $r > 0$, the function $G_\lambda(r, v, v')$ is smooth in terms of its arguments. Thus by the standard ODE theory (see for example [BR69] Theorem 8, Chapter 6), by choosing $v(0) = 1 > 0$ we guarantee the existence of a solution on some interval $[0, T)$.

Suppose that $T < \infty$ is the maximum of such T. First we claim that given these initial conditions we have that on $[0, T)$,

$$\begin{aligned} v &\geq 1 \\ v' &> 0. \end{aligned}$$

To see this, we argue by contradiction. Assume it is not the case. At the first time r_1 where $v' = 0$, it must hold that

$$v'' \leq 0$$

and (4.3) reads

$$\begin{aligned} v'' &= -\lambda \left[\frac{1}{(r^2+v^2)^{\lambda/2}} - \frac{v}{r^2+v^2} \right] \\ &= -\lambda \frac{1}{(r^2+v^2)} \left[\frac{1}{(r^2+v^2)^{\lambda/2-1}} - v \right]. \end{aligned}$$

Now $v > 1$ and $\lambda \geq 2$, so the right hand side is clearly positive, and we have contradiction, hence proving the claim that $v' > 0$.

Next, we observe the following. Rewriting (4.3) as

$$v'' = -\lambda \left[\frac{\left(1+(v')^2\right)^{3/2}}{(r^2+v^2)^{\lambda/2}} + \frac{v'r}{r^2+v^2}\left(1+(v')^2\right) - \frac{v}{r^2+v^2}\left(1+(v')^2\right) \right]$$

when $v' > 0$ we have

$$\begin{aligned} v'' &\leq -\lambda \frac{v'r}{r^2+v^2}\left(1+(v')^2\right) + \lambda \frac{v}{r^2+v^2}\left(1+(v')^2\right) \\ &\leq \lambda \frac{v}{r^2+v^2} + \lambda \frac{(v')^2\, v - r\,(v')^3}{r^2+v^2}. \end{aligned}$$

In particular, after some small $\delta > 0$ where the solution exists, (choose $\delta < 1/\sqrt{\lambda}$) we see that on $[\delta, T)$

$$v'' \leq \lambda + \lambda \frac{(v')^2}{r^2+v^2}(v - \delta v').$$

Certainly $v - \delta v' > 0$ for sufficiently small δ since $v \geq 1$ for all r. We claim that $v - \delta v' > 0$ for all r. We argue by contradiction. Suppose this is not the case. At the first time r_0 for which $v - \delta v' = 0$, we must have

$$\begin{aligned} v' - \delta v'' &\leq 0 \\ v &= \delta v' \end{aligned}$$

so at r_0

$$v' \le \delta v'' \le \delta\lambda + \delta\lambda \frac{(v')^2}{r^2 + v^2}(v - \delta v') = \delta\lambda.$$

But $v(r_0) > 1$ since v is increasing from 1, so at r_0

$$v' = \frac{v}{\delta} > \frac{1}{\delta} > \delta\lambda$$

for δ chosen small enough, which is a contradiction. Thus $v - \delta v' > 0$ for all r and we can integrate

$$\frac{v'}{v} < \frac{1}{\delta}$$

that is

$$v(r) \le v(\delta) e^{\frac{1}{\delta} r}.$$

Thus on the interval $[\delta, T)$ we have

$$v' \le \frac{1}{\delta} v \le \frac{1}{\delta} v(\delta) e^{\frac{T}{\delta}} \le C_1$$

This implies that v' is bounded.

It is easy to see that as long as $v > 1$ and $0 < v' < C_1$ the right hand side $G_\lambda(r, v, v')$ of the ODE (4.3) is smooth in its arguments on any fixed bounded interval $[0, T)$, so in particular, the solution must be well-defined at T. Again (c.f. [BR69]) we may extend this to $[T, T + \tau)$ for some $\tau > 0$. It follows that for $T < \infty$, the interval $[0, T)$ cannot be the maximal domain of existence. □

Some explicit solutions and geometric examples. For dimensions 3 and 4 we can write down the following explicit solutions to (1.1), respectively

$$u(r) = r$$
$$u(r) = \ln r.$$

For the solution $u = r$, and $n = 3$, consider the gradient graph

$$\Gamma = (x, Du(x)) = \left\{ \left(x, \frac{x}{|x|} \right) : x \in \mathbb{B}^3 \backslash \{0\} \right\} \subset \mathbb{R}^3 \oplus \mathbb{R}^3$$

over the unit ball (or ball of any size) in $\mathbb{R}^3$. While the function $Du(x) = \frac{x}{|x|}$ has an isolated non-continuous singularity at $0 \in \mathbb{B}^3$, the submanifold is a smooth embedded submanifold with a smooth boundary consisting of disjoint copies of $\mathbb{S}^2$. A smooth embedding

$$\mathbb{S}^2 \times [0, 1] \to \bar{\Gamma}$$

becomes obvious when writing

$$\bar{\Gamma} = \left\{ (sV, V) : V \in \mathbb{S}^2, 0 \le s \le 1 \right\}.$$

This topological cylinder cannot be isotopic (Hamiltonian or otherwise) to the graph of a smooth function over the unit ball.

Inverting $\bar{\Gamma}$ through the origin in $\mathbb{R}^3$ and gluing, we can obtain a nongraphical smooth Hamiltonian stationary manifold

$$\Gamma_0 = \left\{ (sV, V) : V \in \mathbb{S}^2, -1 < s < 1 \right\}.$$

5. Point singularities for radial solutions to Hamiltonian stationary equations

Motivated by the examples at the end of last section, we consider the alternative possibility, that $\lim_{r\to 0} u'(r) = 0$. In this case the gradient graph is homeomorphic to the domain of the function.

To begin we establish an analytic result for later application:

Lemma 5.1. *Suppose that v is a positive differentiable function on $(0, 2R)$ and continuous at 0. Suppose that there is a sequence of decreasing positive numbers $r_k \to 0$ as $k \to \infty$ and a constant $c > 0$ such that both*

$$v(r_k) \geq c r_k, \tag{5.1}$$

$$v(r_k) = \max_{r\in[0,r_k]} v(r). \tag{5.2}$$

If

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^R \sqrt{\frac{1+(v')^2}{r^2+v^2}}\, dr < \infty \tag{5.3}$$

then

$$\lim_{r\to 0} v(r) = \delta > 0.$$

Proof. For the sequence of decreasing positive numbers $r_k \to 0$ given by the assumption,

$$\begin{aligned} \int_{r_k}^{r_{k-1}} \sqrt{\frac{1+(v')^2}{r^2+v^2}} dr &\geq \int_{r_k}^{r_{k-1}} \frac{v'}{\sqrt{r^2+v^2}} dr \\ &\geq \frac{1}{\max \sqrt{r^2+v^2}} \int_{r_k}^{r_{k-1}} v' dr \\ &= \frac{1}{\sqrt{r_{k-1}^2 + v^2(r_{k-1})}} \left[v(r_{k-1}) - v(r_k)\right] \end{aligned} \tag{5.4}$$

using that

$$v(r_{k-1}) = \max_{r\in[0,r_k]} v(r) \geq \max_{r\in[r_k,r_{k-1}]} v(r).$$

Now using

$$v(r_k) \geq c r_k,$$

we have

$$\begin{aligned} \sqrt{r_{k-1}^2 + v^2(r_{k-1})} &\leq \sqrt{\frac{v^2(r_{k-1})}{c^2} + v^2(r_{k-1})} \\ &= \sqrt{\frac{1}{c^2}+1}\, v(r_{k-1}) \end{aligned}$$

thus

$$\frac{1}{\sqrt{r_{k-1}^2 + v^2(r_{k-1})}} \geq \frac{c}{\sqrt{1+c^2}} \frac{1}{v(r_{k-1})}.$$

It then follows from (5.4)

$$\int_{r_k}^{r_{k-1}} \sqrt{\frac{1+(v')^2}{r^2+v^2}}dr \geq \frac{c}{\sqrt{1+c^2}}\frac{1}{v(r_{k-1})}\left[v(r_{k-1})-v(r_k)\right] \geq \frac{c}{\sqrt{1+c^2}}\left[1-q_k\right] \tag{5.5}$$

where

$$q_k = \frac{v(r_k)}{v(r_{k-1})} \leq 1$$

as $r_k < r_{k-1}$ from (5.2). Now let

$$a_k = 1 - q_k$$

so that $0 \leq a_k < 1$, and then sum (5.5) over k

$$\lim_{k\to\infty}\int_{r_k}^{r_0} \sqrt{\frac{1+(v')^2}{r^2+v^2}}dr \geq \frac{c}{\sqrt{1+c^2}}\sum_{k=1}^{\infty} a_k.$$

Thus

$$\sum_{k=1}^{\infty} a_k \leq \frac{\sqrt{1+c^2}}{c}\int_0^{r_0} \sqrt{\frac{1+(v')^2}{r^2+v^2}}dr < \infty$$

by (5.3). In particular,

$$\lim_{k\to\infty} a_k = 0.$$

Recall that for a sequence q_k of positive numbers, $\prod_{k=1}^{\infty} q_k$ converges if and only if $\sum_{k=1}^{\infty} \ln q_k$ converges. Now

$$\sum_{k=1}^{\infty} \ln q_k = \sum_{k=1}^{\infty} \ln(1-a_k).$$

For large enough K_0 there exists a constant C so that

$$\left|\sum_{k>K_0} \ln(1-a_k)\right| \leq C\left|\sum_{k>K_0} a_k\right| < \infty.$$

Thus, we may assume

$$\sum_{k=1}^{\infty} \ln q_k = q > -\infty.$$

Then

$$\prod_{k=1}^{\infty} q_k = e^q \neq 0.$$

But

$$q_k q_{k-1}\cdots q_1 = \frac{v(r_k)}{v(r_{k-1})}\frac{v(r_{k-1})}{v(r_{k-2})}\cdots\frac{v(r_2)}{v(r_1)} = \frac{v(r_k)}{v(r_1)}.$$

Hence

$$\lim_{k\to\infty} v(r_k) = e^q v(r_1) > 0.$$

This concludes the proof of the lemma. □

Corollary 5.2. *Suppose that v is a solution of (2.4) on $(0,R)$. If $C \neq 0$, then $\lim_{r\to 0} v(r) \neq 0$.*

Proof. Suppose not. Assume that v is a solution and $\lim_{r\to 0} v(r) = 0$. For $0 < \varepsilon < R$, integrating (2.2) leads to

$$\begin{aligned}
|\theta(R) - \theta(\varepsilon)| &= |C| \int_\varepsilon^R \sqrt{\frac{1+(v')^2}{(r^2+v^2)^{n-1}}}dr \\
&= |C| \int_\varepsilon^R \frac{1}{(r^2+v^2)^{\frac{n-2}{2}}} \sqrt{\frac{1+(v')^2}{r^2+v^2}}dr \\
&\geq \min_{r\in(\varepsilon,R)} \frac{1}{(r^2+v^2)^{\frac{n-2}{2}}} |C| \int_\varepsilon^R \sqrt{\frac{1+(v')^2}{r^2+v^2}}dr.
\end{aligned}$$

Now assuming $\lim_{r\to 0} v(r) = 0$, v is bounded, and we have

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^R \sqrt{\frac{1+(v')^2}{r^2+v^2}}dr \leq C_R < \infty. \tag{5.6}$$

Next, if there is constant $t_0 > 0$ so that $|v(r)| \leq cr$ for all $r < t_0$, (5.6) cannot hold. Thus, there is a sequence of decreasing positive numbers $r_k \to 0$ such that

$$|v(r_k)| \geq cr_k. \tag{5.7}$$

Now we show that v does not frequently change sign. Using the odd and even properties of the equation (2.2), we may assume that $C > 0$. In this case, if v has a nonnegative local maximum, then at the point where the maximum is attained, we have

$$\begin{aligned}
0 \geq \frac{1}{1+(v')^2}v'' &= -(n-1)\,\frac{v'r - v}{r^2+v^2} + C\,\frac{(1+(v')^2)^{\frac{1}{2}}}{(r^2+v^2)^{\frac{n-1}{2}}} \\
&= (n-1)\,\frac{v}{r^2+v^2} + C\,\frac{1}{(r^2+v^2)^{\frac{n-1}{2}}} > 0
\end{aligned}$$

a contradiction. In particular, if $v'(a) > 0$ and $v(a) > 0$, then we must have $v'(r) > 0$ for all $r > a$. It follows that $v(r) = 0$ can occur no more than once (not counting at $r = 0$). We conclude that there is an interval $(0, t_1)$ where v is either positive or negative.

If $v > 0$ on $(0, t_1)$, we can apply Lemma 5.1 to conclude that $\lim_{r\to 0} v(r) \neq 0$, recalling (5.6) and (5.7).

If $v < 0$ on $(0, t_1)$, we note that if there exists a positive number $t_2 < t_1$ such that $v' \leq 0$ on $(0, t_2)$, then Lemma 5.1 applies to $-v$ and yields the contradiction. So we assume that there is a sequence of decreasing positive numbers $s_k^{(1)} \to 0$ with $v'(s_k^{(1)}) > 0$. Note that there also exists another decreasing sequence $s_k^{(2)} \to 0$ where $v'(s_k^{(2)}) \leq 0$, otherwise v would be positive near 0 as we are assuming $\lim_{r\to 0} v(r) = 0$, this would contradict $v < 0$ on $(0, t_1)$. Then, there would exist a decreasing sequence $t_k \to 0$ where v attains local maximum. We now treat the the two cases $n \geq 3$ and $n = 2$ separately. When $n \geq 3$

$$0 \geq v''(t_k) = (n-1)\frac{v(t_k)}{t_k^2 + v^2(t_k)} + \frac{C}{(t_k^2 + v^2(t_k))^{\frac{n-1}{2}}}$$

which gives a contradiction since $v(t_k) \to 0$ when $t_k \to 0$, recalling again that $C > 0$.

We consider next the case $n = 2$. In this case, we may find some r_0 with $v'(r_0) > 0$. Choose an interval (s_0, t_0) so that $r_0 \in (s_0, t_0)$ and

$$\begin{aligned} v'(s_0) &= 0 \\ v'(t_0) &= 0 \\ v'(r) &> 0 \text{ on } (s_0, t_0) \end{aligned}$$

At t_0,

$$0 \geq \left(t_0^2 + v^2(t_0)\right) v''(t_0) = v(t_0) + C\left(t_0^2 + v^2(t_0)\right)^{\frac{1}{2}}.$$

This yields that

$$v^2(t_0) \geq C^2\left(t_0^2 + v^2(t_0)\right)$$

so clearly $C < 1$ and

$$|v(t_0)| \geq \frac{C}{\sqrt{1 - C^2}} t_0.$$

Now at $s_0 < r_0$ we will have

$$|v(t_0)| = -v(t_0) < -v(s_0) = |v(s_0)| \tag{5.8}$$

and

$$0 \leq \left(s_0^2 + v^2(s_0)\right) v''(s_0) = v(s_0) + C\left(s_0^2 + v^2(s_0)\right)^{\frac{1}{2}}.$$

The latter inequality gives us

$$|v(s_0)| \leq \frac{C}{\sqrt{1 - C^2}} s_0 < \frac{C}{\sqrt{1 - C^2}} t_0 \leq |v(t_0)|$$

which is clearly a contradiction of (5.8). □

We now prove Theorem 1.2.

Proof. Suppose that u is a radial solution to (1.1) on a neighborhood $\mathbb{B}_R(0)$ with

$$\lim_{r \to 0} u'(r) = 0.$$

Since $\lim_{r \to 0} u'(r) = 0$, the contrapositive of Corollary 5.2 asserts that θ is constant, so the gradient graph of u is a calibrated submanifold which is smooth away from the origin and continuous everywhere. For any $\rho \in (0, R]$, define

$$u^{(\rho)}(x) = \frac{u'(\rho)}{2\rho}|x|^2.$$

Notice that on $\partial B_\rho(0)$ we have

$$Du^{(\rho)}(x) = u'(\rho)\frac{x}{\rho} = Du(x)$$

and that because the latter is a continuous graph over the whole domain, these gradient graphs will be isotopic. Now we may use the special Lagrangian calibration argument ([HL82, Section III]) to compare the volumes of the graphs

$$\Gamma = \left\{(x, Du(x)) : \ x \in B_\rho(0)\right\}$$

and

$$\Gamma^{(\rho)} = \left\{(x, Du^{(\rho)}(x)) : \ x \in B_\rho(0)\right\}.$$

First, let
$$\phi = \operatorname{Re}\left(e^{-\sqrt{-1}\theta} dz_1 \wedge \ldots \wedge dz_n\right).$$
Note that the closed n-form ϕ calibrates Γ. By Stokes' Theorem, as $d\phi = 0$, we have
$$\mathrm{Vol}(\Gamma) = \int_\Gamma \phi = \int_{\Gamma^{(\rho)}} \phi \leq \mathrm{Vol}(\Gamma^{(\rho)}).$$
On the other hand, letting
$$\phi^{(\rho)} = \operatorname{Re}\left(e^{-\sqrt{-1}\theta^{(\rho)}} dz_1 \wedge \ldots \wedge dz_n\right)$$
where θ_ρ is the constant given by
$$\theta^{(\rho)} = n \arctan\left(\frac{u'(\rho)}{\rho}\right)$$
and noting that the closed n-form $\phi^{(\rho)}$ calibrates $\Gamma^{(\rho)}$. We conclude that
$$\mathrm{Vol}(\Gamma^{(\rho)}) = \int_{\Gamma^{(\rho)}} \phi^{(\rho)} = \int_\Gamma \phi^{(\rho)} \leq \mathrm{Vol}(\Gamma).$$
It then follows that
$$\mathrm{Vol}(\Gamma^{(\rho)}) = \mathrm{Vol}(\Gamma) = \int_\Gamma \phi$$
and
$$\theta = \theta^{(\rho)}.$$
This implies that
$$u'(\rho) = \rho \tan\left(\frac{\theta}{n}\right)$$
for any $\rho \in (0, R]$, therefore u is quadratic since θ is a constant.

Next, we assume that
$$\lim_{r\to 0} u'(r) = c \neq 0.$$
Letting $u(0) = c$, we have a map (extending (2.1)),
$$F : [0,\infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n \times \mathbb{R}^n$$
defined by
$$F(r,\xi) = (r\xi, u'(r)\,\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{5.9}$$
It is not hard to see that this immersion is proper and injective, so must be an embedding. □

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