Representations of G and $L^1(G)$

INTRODUCTION: This note was motivated by lectures by Gabriel Nagy at Kansas State. We refer frequently to D&E, which refers to *Principles of Harmonic Analysis* by Anton Deitmar and Siegfried Echterhoff (Springer 2009). We also refer to H&R, this being volumes 1 and 2 of Hewitt & Ross's *Abstract Harmonic Analysis*. Observe that a version of Theorem 1 below is in H&R, 22.10 and 22.7, but that its proof is much too complicated. It uses the fact (21.13) that every $*$-representation on a Hilbert space is a direct sum of cyclic subrepresentations.

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This is about "duality" between norm-continuous representations π of the algebra $L^1(G)$ by bounded linear operators $\mathcal{B}(V)$, V a Banach space, and bounded continuous group representations U_x of G on V. Here G is an LC-group with left Haar measure μ. One direction is, of course, easy: From U_x one builds π by

$$\pi(f)v := \int_G f(x)U_x(v)d\mu(x) \quad \text{for} \quad f \in L^1(G), v \in V, \tag{0}$$

a V-valued integral. Let (e_j) be a nice approximate identity which D&E call a Dirac net. As noted in D&E 6.2.2, $\lim_j \pi(e_j)v = v$ for all $v \in V$. In other words, the net of operators $\pi(e_j)$ on V converges to the identity operator on V in the strong operator topology.

Now we begin with an algebra representation π of $A = L^1(G)$ satisfying for some finite $C \geq 1$,

$$\| \pi(f) \|_{op} \leq C \, \| f \|_1 \quad \text{for} \quad f \in L^1(G). \tag{1}$$

Following D&E (and Nagy), we say that π is **non-degenerate** if

$$\pi(A)V = \{\pi(f)v : f \in L^1(G), v \in V\} \quad \text{spans a dense subspace of} \quad V, \tag{2}$$

where $\pi(A)V = \{\pi(f)v : f \in L^1(G), v \in V\}$. (If V is a Hilbert space and π is a $*$-representation, then (2) is plainly the same as saying $\pi(f)(v) = 0$ for all $f \in L^1(G)$ implies $v = 0$.)

Lemma 1. If π is non-degenerate as above, then $\pi(A)V = V$.

Proof. We define $(f, v) \to f \bullet v$ from $A = L^1(G) \times V \to V$, where $f \bullet v = \pi(f)v$. Then V is a Banach $L^1(G)$-module (H&R 32.14). In particular, $(f * g) \bullet v = \pi(f * g)v = \pi(f)(\pi(g)v) = f \bullet (\pi(g)v) = f \bullet (g \bullet v)$ and

$$\| f \bullet v \| = \| \pi(f)v \| \leq \| \pi(f) \|_{op} \cdot \| v \| \leq C \, \| f \|_1 \cdot \| v \| .$$

The Dirac net (e_j) is a bounded approximate unit for $L^1(G)$, so the Cohen Factorization Theorem (H&R 32.22) tells us that $\pi(A)V = A \bullet V$ is a closed linear subspace of V, and thus $\pi(A)V = V$. $\square$

For Lemma 2 and Theorem 1 below, we will give two proofs, one using Lemma 1 and the other not.

Lemma 2. If π is non-degenerate, then $\lim_j \pi(e_j)v = v$ for all $v \in V$.

Proof using Lemma 1. Given $v \in V$, we have $v = \pi(f)w = f \bullet w$ for some $f \in L^1(G)$ and $w \in V$. Therefore

$$\lim_j \pi(e_j)v = \lim_j e_j \bullet w = \lim_j e_j \bullet (f \bullet w) = \lim_j (e_j * f) \bullet w = f \bullet w = v. \qquad \square$$

Proof avoiding Lemma 1. Given $v \in V$ and $\epsilon > 0$, use (2) to select $\sum_{i=1}^n \pi(f_i)v_i$ satisfying

$$\| \sum_{i=1}^n \pi(f_i)v_i - v \| < \frac{\epsilon}{3C}.$$

Then $\| \pi(e_j)v - v \|$ is bounded by

$$\| \pi(e_j)v - \sum_{i=1}^n \pi(e_j)\pi(f_i)v_i \| + \| \sum_{i=1}^n \pi(e_j)\pi(f_i)v_i - \sum_{i=1}^n \pi(f_i)v_i \| + \| \sum_{i=1}^n \pi(f_i)v_i - v \| .$$

The third and first terms in the sum are bounded by $\epsilon/3$, and the middle term is bounded by

$$\sum_{i=1}^n \| \pi(e_j * f_i - f_i)v_i \| \leq C \sum_{i=1}^n \| e_j * f_i - f_i \|_1 \cdot \| v_i \|,$$

which is also less than $\epsilon/3$ for sufficiently "large" j in the Dirac net. $\square$

NOTE. Thus, for non-degenerate π: if $\pi(f)v = 0$ for all $f \in L^1(G)$, then $v = 0$. This is the non-degeneracy-like hypothesis in H&R 22.10 and 22.7.

Theorem 1. Given a norm-continuous representation π of $A = L^1(G)$, as described above, there exists a representation U_x of G, where each U_x is a bounded linear operator on V with norm not greater than C, and such that equation (0) holds.

Proof using Lemma 1. For $x \in G$, we propose to define $U_x : V \to V$ by

$$U_x(\pi(f)v) := \pi(L_x(f))v \quad \text{for all} \quad x \in G \quad \text{and} \quad f \in L^1(G), \tag{3}$$

where L_x is left translation by x^{-1} in $L^1(G)$. Since $V = \pi(A)V$, U_x is defined for all $v \in V$, provided that it is well-defined. But note that

$$\pi(L_x(e_j))(\pi(f)v) = \pi(L_x(e_j) * f)v = \pi(L_x(e_j * f))v \to \pi(L_x f)v,$$

since π is continuous and $e_j * f \to f$ in $L^1(G)$. So the right side of (3) is unambiguously determined by the vector $\pi(f)v$. This calculation also shows

$$\| U_x(\pi(f)v) \| \leq \liminf_j \| \pi(L_x(e_j)) \|_{op} \cdot \| \pi(f)v \| \leq C \liminf_j \| L_x(e_j) \|_1 \cdot \| \pi(f)v \| = C \| \pi(f)v \| .$$

It remains to verify equation (0): $\pi(f)v = \int_G f(x)U_x(v)d\mu(x)$. For fixed f, both sides are continuous linear functions of v, so it suffices to verify (0) for elements $\pi(g)v$ in $\pi(A)V = V$. So we want

$$\pi(f)(\pi(g)v) = \int_G f(x)U_x(\pi(g)v)d\mu(x).$$

For fixed v, both sides are continuous as functions of f, so it suffices to consider f in $C_c(G)$. Now we're done:

$$\int_G f(x)U_x(\pi(g)v)d\mu(x) = \int_G f(x)\pi(L_x(g))v\, d\mu(x) = \int_G \pi(f(x)L_x(g))v\, d\mu(x),$$

and, as elaborated on in the next paragraph, by D&E B.6.1(c) this is equal to

$$\pi\left(\int_G f(x)L_x(g)d\mu(x)\right)v = \pi(f * g)v = \pi(f)(\pi(g)v),$$

where the first equality follows from D&E B.6.5. Note that $f*g(y) = \int_G f(x)g(x^{-1}y)d\mu(x) = \int_G f(x)L_x(g)(y)d\mu(x)$.

To apply D&E B.6.1(c) above, let $\varphi(x) := f(x)L_x(g)$, so that $\varphi : G \to L^1(G)$ and we want

$$\int_G \pi(\varphi(x))v\, d\mu(x) = \pi\left(\int_G \varphi(x)\, d\mu(x)\right)(v). \tag{4}$$

As noted in the first paragraph of the proof of D&E B.6.5, φ is continuous with compact support. So is the function $x \to \pi(\varphi(x))v$ from G into V, so both integrals in (5) exist. Now, fix v. Then $T(f) = \pi(f)v$ clearly defines a continuous linear operator $T : L^1(G) \to V$, and we can apply D&E to φ to obtain $\pi\left(\int_G \varphi(x)\, d\mu(x)\right)v =$

$$T\left(\int_G \varphi(x)d\mu(x)\right) = \int_G (T(\varphi))(x)\, d\mu(x) = \int_G T(\varphi(x))\, d\mu(x) = \int_G \pi(\varphi(x))v\, d\mu(x),$$

which is equation (4). $\square$

Proof avoiding Lemma 1. For $x \in G$, we want U_x to satisfy $U_x(\pi(f)v) = \pi(L_x(f))v$ for $f \in A = L^1(G)$ and $v \in V$, where L_x is left translation by x^{-1} in $L^1(G)$. So we first define U_x on the span of $\pi(A)V$:

$$U_x\left(\sum_{i=1}^n \pi(f_i)v_i\right) := \sum_{i=1}^n \pi(L_x(f_i))v_i. \tag{5}$$

Since left translation satisfies $L_x(e) * f = L_x(e * f)$ (H&R 20.11), elements of our Dirac net (e_j) satisfy

$$\pi(L_x(e_j))\left(\sum_{i=1}^n \pi(f_i)v_i\right) = \sum_{i=1}^n \pi(L_x(e_j) * f_i)v_i = \sum_{i=1}^n \pi(L_x(e_j * e_i))v_i.$$

Since (e_j) is an approximate unit in $L^1(G)$, and L_x and π are continuous, we see that

$$\lim_j \pi(L_x(e_j))\left(\sum_{i=1}^n \pi(f_i)v_i\right) = \lim_j \sum_{i=1}^n \pi(L_x(e_j * e_i))v_i = \sum_{i=1}^n \pi(L_x(f_i))v_i. \tag{6}$$

Since the left-hand side of (6) depends only on $\sum_{i=1}^n \pi(f_i)v_i$, this shows that definition (5) is well defined.

Equation (6) implies that

$$\| \sum_{i=1}^{n} \pi(L_x(f_i))v_i \| \leq \liminf_j \| \pi(L_x(e_j)) \|_{op} \cdot \| \sum_{i=1}^{n} \pi(f_i)v_i \| \,.$$

Since $\| L_x \|_{op} = 1$ and $\| \pi \| \leq C$, we conclude that

$$\| U_x \left(\sum_{i=1}^{n} \pi(f_i)v_i \right) \| \leq C \, \| \sum_{i=1}^{n} \pi(f_i)v_i \| \,.$$

Thus U_x is a bounded linear operator on the span of $\pi(A)V$, with norm no greater than C. So each U_x can be extended to an operator on V, with norm no greater than C.

The rest of the proof, verifying equation (0), is the same as the Proof using Lemma 1. $\square$

Here is D&E's 6.2.3.

Theorem 2. If π in Theorem 1 is a $*$-representation on $L^1(G)$, acting on a Hilbert space V, then the representation U on G is a unitary representation.

Proof. This follows from the computation at the bottom of D&E's page 133 together with Lemma 1. $\square$

References:

[D&E] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, Springer Science-Business Media, LLC 2009.

[H&R] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis*, Springer-Verlag, volumes I (2nd edition, 1979) and II (1970).