

MATH 635 WINTER 2024
HOMEWORK 3
DUE FEBRUARY 5, 2024
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Required problems:

- (1) Let C_* and D_* be chain complexes of R -modules, and M any R -module. Prove that if $f, g: C_* \rightarrow D_*$ are chain homotopic chain maps then the induced maps $f^T, g^T: \text{Hom}(D_*, M) \rightarrow \text{Hom}(C_*, M)$ are chain homotopic.
- (2) (Homotopy invariance of cohomology) Prove that if $f, g: X \rightarrow Y$ are homotopic maps then $f^* = g^*: H^n(Y) \rightarrow H^n(X)$. Conclude that if $X \simeq Y$ then $H^n(X) \cong H^n(Y)$ for each n .
- (3) (Excision for cohomology) Recall that for $A \subset X$,

$$C^n(X, A) = \{c \in C^n(X) \mid c(\sigma) = 0 \text{ for all } \sigma: \Delta^n \rightarrow A\}.$$

- (a) Prove that $C^*(X, A)$ is a subcomplex of $C^*(X)$. So, we can define $H^n(X, A) = \ker(\delta|_{C^n(X, A)}) / \text{Im}(\delta|_{C^{n-1}(X, A)})$.
- (b) Suppose $Z \subset A \subset X$ with the closure of Z contained in the interior of A . Prove that $H^n(X, A) \cong H^n(X \setminus Z, A \setminus Z)$. (This isomorphism is induced by the inclusion map $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$, though you don't have to prove that if you don't want to.)
- (4) Hatcher 2.1.17 (p. 132).
- (5) Hatcher 2.1.20 (p. 132). (The first half of this is an important observation.)
- (6) Hatcher 2.1.27 (p. 133).
- (7) Imitate our computation of $H_*(S^n)$ to compute the singular cohomology groups $H^*(S^n)$.

Optional problems:

Some homological algebra. This is optional for now. Next week I will do some of this in class, and the rest will probably be non-optional homework.

- (1) Given a chain map $f: C_* \rightarrow D_*$, the *mapping cone* of f is the chain complex $\text{Cone}(f)$ with $\text{Cone}(f)_n = C_{n-1} \oplus D_n$ and differential $\partial(c, d) = (-\partial(c), f(c) + \partial(d))$.
 - (a) Suppose X and Y are simplicial complexes and $f: X \rightarrow Y$ is a simplicial map. Explain how the mapping cone of f , $\text{Cone}(f)$, is a simplicial complex and show that the simplicial chain complex of $\text{Cone}(f)$ is isomorphic to $\text{Cone}(f_\# : C_*^{\text{simp}}(X) \rightarrow C_*^{\text{simp}}(Y))$.
 - (b) Let C_* and D_* be chain complexes, and f a collection of homomorphisms $f_n: C_n \rightarrow D_n$. Even if f_n is not a chain map, we can still form $\text{Cone}(f)$ as above. Show that the differential on $\text{Cone}(f)$ satisfies $\partial^2 = 0$ if and only if f is a chain map.

- (c) Show that the chain map $f: C_* \rightarrow D_*$ induces an isomorphism on homology if and only if $\text{Cone}(f)$ is acyclic (i.e., has trivial homology). (Hint: there is an obvious short exact sequence involving C_* , D_* , and $\text{Cone}(f)$.)
- (2) Let C_* and D_* be chain complexes with $C_n = D_n = 0$ for $n < 0$, C_n free for all n , and $H_*(D) = 0$. Let $f: C_* \rightarrow D_*$ be a chain map. Prove that f is nullhomotopic. (Hint: construct a nullhomotopy h of f inductively, starting with $h_0: C_0 \rightarrow D_1$.)
- (3) Let C_* and D_* be chain complexes with $C_n = D_n = 0$ for $n < 0$. Assume further that each D_n is a free R -module. Let $f: C_* \rightarrow D_*$ be a chain map so that the induced map $f_*: H_*(C) \rightarrow H_*(D)$ on homology is an isomorphism. Prove that there is a map $g: D_* \rightarrow P_*$ so that $g \circ f$ is chain homotopic to the identity map. (Hint: consider $\text{Cone}(f)$. By the previous problem, the canonical map $D_* \hookrightarrow \text{Cone}(f)$ is nullhomotopic. A nullhomotopy is a pair of maps $\alpha: D_* \rightarrow C_*$, $\beta: D_* \rightarrow D_*$. Unpack the fact that (α, β) is a chain map to obtain the result.)
- (4) Let C_* and D_* be chain complexes over R so that $C_n = D_n = 0$ for $n < 0$ and C_n and D_n are free R -modules for each n . Suppose $f: C_* \rightarrow D_*$ is a chain map which induces an isomorphism on homology. Prove that f is a chain homotopy equivalence. (Hint: this should be easy from the previous problem.)
- (5) Let C_* and D_* be chain complexes over R so that $C_n = D_n = 0$ for $n < 0$, each C_n is free, and $H_n(D) = 0$ for $n > 0$. Prove that if $f, g: C_* \rightarrow D_*$ induce the same map $H_0(C) \rightarrow H_0(D)$ then they are chain homotopic. (Hint: let $M = H_0(D)$. Consider the complex $0 \leftarrow M \leftarrow D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots$. Observe that $f - g$ induces a map from C_* to this extended version of D and use Problem 2.)

Some other optional problems; think about these, but you don't have to turn them in.

(6) Hatcher 2.1.26 (p. 133).

(7) Hatcher 2.1.29 (p. 133).

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