

MATH 635 WINTER 2024
HOMEWORK 1
DUE JANUARY 19, 2024.
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Required problems:

- (1) Let M be the cokernel of the map $\mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ given by the matrix

$$A = \begin{bmatrix} 2 & 4 & 10 \\ -2 & 2 & 2 \\ 8 & -2 & 4 \end{bmatrix}$$

- (So, A is a presentation matrix for M .) Decompose M as $\mathbb{Z}^k \oplus \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_\ell)$.
 (2) Consider the chain complex C_* given by

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\begin{bmatrix} 2 & -4 & 2 \end{bmatrix}} \mathbb{Z}^3 \xleftarrow{\begin{bmatrix} 10 & 36 \\ 4 & 15 \\ -2 & -6 \end{bmatrix}} \mathbb{Z}^2 \longleftarrow 0$$

where the left-most $\mathbb{Z}$ is in grading 0. Compute the homology of C_* .

- (3) Identify the following abelian groups ($\mathbb{Z}$ -modules):
 (a) $\text{Hom}_{\mathbb{Z}(\mathbb{Z}/(3), \mathbb{Z}/(4))}$
 (b) $\text{Hom}_{\mathbb{Z}(\mathbb{Z}/(3), \mathbb{Z}/(9))}$
 (c) $\text{Hom}_{\mathbb{Z}(\mathbb{Z}/(3), \mathbb{Z}/(15))}$
 (d) $\text{Hom}_{\mathbb{Z}(\mathbb{Z}/(3), \mathbb{Q})}$
 (e) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}\mathbb{Z}/(4)}$
 (f) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}\mathbb{Z}/(9)}$
 (g) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}\mathbb{Z}/(15)}$
 (h) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}\mathbb{Q}}$
 (4) Prove that the Hom and tensor product operations distribute with finite direct sums. That is

$$\begin{aligned} \text{Hom}_R(M, N_1 \oplus N_2) &\cong \text{Hom}_R(M, N_1) \oplus \text{Hom}_R(M, N_2) \\ \text{Hom}_R(M_1 \oplus M_2, N) &\cong \text{Hom}_R(M_1, N) \oplus \text{Hom}_R(M_2, N) \\ M \otimes_R (N_1 \oplus N_2) &\cong (M \otimes_R N_1) \oplus (M \otimes_R N_2). \end{aligned}$$

Which of these statements are true for infinite direct sums? (You don't have to prove your answer to this part, though you should be able to.)

- (5) Prove that if $f: R^n \rightarrow R^m$ is represented by the $m \times n$ matrix A then $f^T: R^m \cong \text{Hom}_R(R^m, R) \rightarrow \text{Hom}_R(R^n, R) \cong R^n$ is represented by A^T . (Here, the outer isomorphisms are the ones constructed in the previous problem, say.)
 (6) Let R be a ring and S an R -algebra. Suppose $f: R^n \rightarrow R^m$ is given by the $m \times n$ matrix A . Describe as explicitly as you can the map $(f \otimes \mathbb{I}): R^n \otimes S \rightarrow R^m \otimes S$.

(7) Let C_* be the chain complex

$$0 \longleftarrow \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} \mathbb{Z} \longleftarrow 0$$

where the left-most $\mathbb{Z}^2$ is in grading 0. Compute the (co)homology of:

- (a) C_* .
- (b) The dual cochain complex $\text{Hom}_{\mathbb{Z}(C_*, \mathbb{Z})}$.
- (c) The cochain complex $\text{Hom}_{\mathbb{Z}(C_*, \mathbb{Z}/(2))}$.
- (d) The chain complex $C_* \otimes_{\mathbb{Z}\mathbb{Z}/(2)}$.
- (e) The chain complex $C_* \otimes_{\mathbb{Z}\mathbb{Q}}$.
- (f) The chain complex $C_* \otimes_{\mathbb{Z}\mathbb{Z}/(3)}$.

Optional problems:

Solve these if you have *not* seen this material before this class. (That is, if you don't know how to do it, do it.) You can turn them in or not, as you prefer.

- (8) Given an R -module M and a submodule N , recall that we defined M/N to be the abelian group M/N with R -action given by $r \cdot [m] = [r \cdot m]$. Prove that this action is well defined and does, in fact, make M/N into an R -module.
- (9) Given R -modules M , N , and P and homomorphisms $f: M \rightarrow P$ and $g: N \rightarrow P$, prove that there is a unique homomorphism $M \oplus N \rightarrow P$ so that the following diagram commutes:

$$\begin{array}{ccc} & & \xrightarrow{f} \\ \uparrow & \nearrow j & \\ M \oplus N & & \end{array}$$

Here, the maps i and j are the inclusions $i(m) = (m, 0)$ and $j(n) = (0, n)$. (This is called the *universal property of coproducts*.)

Moreover, prove that this property characterizes $M \oplus N$ up to (unique) isomorphism, in the sense that given any other module Q and maps $i': M \rightarrow Q$ and $j': N \rightarrow Q$ with the same property as $M \oplus N$, there is a (unique) isomorphism $M \oplus N \cong Q$ so that the diagram

$$\begin{array}{ccc} & & \xrightarrow{i} \\ \uparrow & \nearrow \cong & \\ Q & & \end{array}$$

commutes.

Generalize to direct sums of arbitrarily many modules.

- (10) Given R -modules M and N and homomorphisms $f: P \rightarrow M$ and $g: P \rightarrow N$, prove that there is a unique homomorphism $P \rightarrow M \oplus N$ so that the following diagram commutes:

$$\begin{array}{ccc} & \searrow & \\ \uparrow & \downarrow & \\ M \oplus N & \xrightarrow{q} & \end{array}$$

Here, the maps p and q are given by $p(m, n) = m$ and $q(m, n) = n$. Again, prove that this characterizes $M \oplus N$ up to (unique) isomorphism.

(This is the *universal property of products*.) Give an example showing, however, that this does not hold for infinite direct sums: it is not true that given R -modules M_i , $i \in \mathbb{N}$, and maps $f_i: P \rightarrow M_i$ there is necessarily a homomorphism $P \rightarrow \bigoplus_{i \in \mathbb{N}} M_i$ so that for all i the diagram

$$\begin{array}{ccc} & \uparrow & \searrow \\ & P & \\ & \downarrow f_i & \\ \bigoplus_i M_i & & \end{array}$$

commutes. (Hint: you can find a very easy example illustrating this.)

- (11) Let $R\langle I \rangle$ denote the free module generated by a set I . Suppose M is any R -module, and $f: I \rightarrow M$ is a map of sets. Prove that there is a unique map of modules $R\langle I \rangle \rightarrow M$ so that the following diagram (of sets) commutes:

$$\begin{array}{ccc} & \xrightarrow{f} & \\ \downarrow j & \nearrow & \\ R\langle I \rangle & & \end{array}.$$

Here, j is the standard inclusion of I into $R\langle I \rangle$.

Moreover, prove that this property characterizes $R\langle I \rangle$ up to (unique) isomorphism.

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