

MATH 635 HOMEWORK 5
DUE FEBRUARY 12, 2021.
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- (1) Hatcher 3.1.9 (p. 205).
- (2) Hatcher 3.1.10 (p. 205), part about the universal coefficient theorem (you did the rest last week).
- (3) Hatcher 3.1.11 (p. 205).
- (4) Hatcher 3.A.1 (p. 267).

Suggested review / qualifying exam practice (not to turn in):

- (1) Hatcher 3.1.1, 3.1.2, 3.1.3, 3.1.4.
- (2) Hatcher 3.A.2, 3.A.3, 3.A.4, 3.A.5, 3.A.6.
- (3) Here is a more abstract proof of the universal coefficient theorem; fill in the details.
 - (a) Let C_* be a chain complex of free abelian groups, and H_* the homology of C_* , which we view as a chain complex with trivial homology. Assume that $C_i = 0$ for all sufficiently small i (i.e., C_* is *bounded below*). Show that there is a chain map $C_* \rightarrow H_*$ inducing an isomorphism on homology (i.e., a *quasi-isomorphism* from C_* to H_*).
 - (b) Given a bounded-below chain complex C_* over a ring R , choose a chain complex P_* of free R -modules which is quasi-isomorphic to C_* . For M an R -module, define $\text{Tor}^i(C_*, M)$ to be the i^{th} homology group of $P_* \otimes M$. Prove that:
 - (i) $\text{Tor}^i(C_*, M)$ is independent of the choice of P_* , up to (canonical) isomorphism.
 - (ii) If C_* is quasi-isomorphic to D_* then $\text{Tor}^i(C_*, M) \cong \text{Tor}^i(D_*, M)$.
 - (c) If the differential on C_* vanishes, then
$$\text{Tor}^i(C_*, M) \cong (C_i \otimes M) \oplus \text{Tor}^1(C_{i-1}, M) \oplus \text{Tor}^2(C_{i-2}, M) \oplus \cdots$$
 - (d) By definition, if C_* is a chain complex of free R -modules then $\text{Tor}^i(C_*, M) \cong H_i(C_* \otimes M) =: H_i(C_*; M)$. So, deduce the universal coefficient theorem for homology.
- (4) Prove the universal coefficient theorem with $\mathbb{Z}$ replaced by any PID.

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