MATH 635 HOMOLOGICAL ALGEBRA WORKSHEET

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The main goal of this worksheet is Problem (4), which we started to prove in class. To prove it, you need problem (3) and part (b) of problem (2). The remaining problems are for context. The last few problems are more challenging than earlier ones.

(0) An R-module P is called *projective* if for any R-modules M and N and homomorphisms $f\colon M \to N$, $g\colon P \to N$ with f surjective, there is a map $h\colon P \to M$ so that $g = f \circ h$. i.e.:

$$M@- >>[d]^f P[r]_g @-- >[ur]^h N.$$

- (a) Show that free modules are projective.
- (b) Show that a module P is projective if and only if the functor $\mathrm{Hom}(P, \cdot)$ is exact, i.e., for any short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \leftarrow \mathrm{Hom}(P, A) \leftarrow \mathrm{Hom}(P, B) \leftarrow \mathrm{Hom}(P, C) \leftarrow 0$ is exact.
- (c) Show that P is projective if and only if every short exact sequence of the form

$$0 \to M \to N \to P \to 0$$

splits.
- (d) Let $R = \mathbb{Q} \oplus \mathbb{Q}$. Let $P = \mathbb{Q}$, and give P the structure of an R-module by defining $(1, 0) \cdot p = p$ and $(0, 1) \cdot p = 0$ (where $(i, j) \in R$ and $p \in P$). Show that P is a projective R-module.
- (e) Generalizing the previous part, show that P is projective if and only there is a module Q so that $P \oplus Q$ is a free module. (Hint: "if" should be easy. For "only if", consider a free resolution of P and use the definition of projective.)

(0) An R-module I is called *injective* if for any R-modules M and N and homomorphisms $f\colon N \to M$, $g\colon N \to I$ with f injective, there is a map $h\colon M \to I$ so that $g = h \circ f$. i.e.:

$$M@-- >[dl]_h I N@^(- >[u]_f [l]^g.$$

- () Show that $\mathbb{Z}$ is *not* an injective $\mathbb{Z}$-module.
- () Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. (Hint: use Zorn's lemma to define the map h, by considering pairs (P, h_P) where $f(N) \subset P \subset M$ and $h_P\colon P \to I$ satisfies $h_P \circ f = g$. Order these pairs by $(P, h_P) < (Q, h_Q)$ if $P \subset Q$ and $h_Q|_P = h_P$.)
- () Similarly, show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. (Hint: same proof.)
- () Show that I is injective if and only if the functor $\mathrm{Hom}(\cdot, I)$ is exact.
- () Show that I is injective if and only if every short exact sequence of the form

$$0 \to I \to M \to N \to 0$$

splits.
- () An *injective resolution* of M is a cochain complex $0 \to I_0 \to I_1 \to I_2 \to \cdots$ and a chain map

$$0[r] M[r][d] 0[r][d] 0[r][d] \cdots 0[r] I_0[r] I_1[r] I_2[r] \cdots$$

inducing an isomorphism on homology. Show that every abelian group has an injective resolution. (Hint: use parts (b) and (c).)

(0) Let M be a $\mathbb{Z}$-module (abelian group). Consider the map $j\colon M \cong M\otimes_{\mathbb{Z}\mathbb{Z}\to M\otimes_{\mathbb{Z}\mathbb{Q}}}$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Show that $j(m \otimes 1) = 0$ if and only if m is torsion (i.e., there is a $c > 0$ so that $c \cdot m = 0$). (Hint: "if" is easy. For "only if", use part (b) of the previous problem to define a map $M \to \mathbb{Q}$ sending m to 1 and consider the induced map $M\otimes_{\mathbb{Z}\mathbb{Q}\to\mathbb{Q}}$.)

(0) Show that for any abelian group M, $\mathrm{Tor}^1_{\mathbb{Z}(M,\mathbb{Q})=0}$. (Hint: by considering a free resolution of M, it suffices to show that if $f\colon P \to Q$ is an injective map of free abelian groups then $f \otimes \mathbb{I}\colon P \otimes M \to Q \otimes M$ is injective. Suppose $(f \otimes \mathbb{I})(\sum p_i \otimes \frac{r_i}{s_i}) = 0$, clear denominators, and apply the previous problem.)

(0) Show that you can compute $\mathrm{Ext}^i_R(M,N)$ by taking a projective resolution of M, rather than a free resolution. (Hint: show that any two projective resolutions are chain homotopy equivalent.)

(0) Show that you can compute $\mathrm{Ext}^i_R(M,N)$ by taking an injective resolution of N instead of a projective resolution of M.

(0) Show that $\mathrm{Tor}^i_R(M,N) \cong \mathrm{Tor}^i_R(N,M)$.

(0) An R-module M is called *flat* if $M \otimes_R \cdot$ is an exact functor, i.e., if for any short exact sequence of R-modules $0 \to A \to B \to C \to 0$, the sequence

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is exact. A *flat resolution* of M is a chain complex $\cdots \to P_2 \to P_1 \to P_0 \to 0$ together with a chain map to $\cdots \to 0 \to 0 \to M \to 0$ so that the induced map on homology is an isomorphism. Show that $\mathrm{Tor}^i(M,N)$ can be computed by taking a flat resolution of M (or N), rather than a free resolution.

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