

**MATH 635 HOMEWORK 8**  
**DUE MARCH 6, 2019.**  
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Problems to turn in:

- (1) Hatcher 3.3.21 (p. 259).
- (2) Hatcher 3.3.30 (p. 260).
- (3) Hatcher 3.3.31 (p. 260).
- (4) Hatcher 3.3.32 (p. 260).
- (5) (a) Let  $Y$  be a closed, connected, orientable 3-manifold with  $H_1(Y)$  finite and  $\phi: Y \rightarrow Y$  a map of degree  $-1$ . Show that  $\phi$  has a fixed point.  
 (b) Construct an example of a closed, connected, orientable 3-manifold  $Y$  with  $H_1(Y) \cong \mathbb{Z}$  and a map  $\phi: Y \rightarrow Y$  of degree  $-1$  with no fixed points.

Review / qualifying exam practice (not to turn in):

- (1) Hatcher 3.3.20, 3.3.25, 3.3.26.

More problems to think about but not turn in:

- (1) Hatcher 3.3.22, 3.3.23, 3.3.27–29.
- (2) Let  $X$  be a topological space. A *locally-finite singular  $n$ -chain* in  $X$  is a (possibly infinite) formal linear combination  $\sum k_\sigma \sigma \in \prod_{\sigma: \Delta^n \rightarrow X} \mathbb{Z}$  so that for each compact set  $C \subset X$ ,

$$\{\sigma \mid k_\sigma \neq 0 \text{ and } \sigma^{-1}(C) \neq \emptyset\}$$

is finite. Let  $C_n^{BM}(X)$  denote the abelian group of locally-finite singular  $n$ -chains in  $X$ .

- (a) Show that the “obvious” boundary operator  $\partial: C_n^{BM}(X) \rightarrow C_{n-1}^{BM}(X)$ ,

$$\partial \sum_{\sigma} k_{\sigma} \sigma = \sum_{\sigma} \sum_{i=0}^n (-1)^i k_{\sigma} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

is well-defined and makes  $C_n^{BM}(X)$  into a chain complex. (*Hint.* This would *not* be true without the “locally-finite” condition, so your proof had better use it!)

- (b) The homology of  $(C_*^{BM}, \partial)$  is called the *Borel-Moore homology of  $M$* , and denoted  $H_*^{BM}(X)$ . Observe that if  $X$  is compact then  $H_*^{BM}(X) \cong H_*(X)$ .  
 Compute  $H_*^{BM}(\mathbb{R})$ .

- (c) Show that if  $M$  is a connected, orientable  $n$ -manifold, not necessarily compact, then  $H_n^{BM}(M) \cong \mathbb{Z}$ . In particular, any oriented  $n$ -manifold  $M$  has a fundamental class  $[M] \in H_n^{BM}(M)$ .
- (d) Show that there is a well-defined cap product

$$H_{i+j}^{BM}(X) \otimes H^i(X) \rightarrow H_j^{BM}(X).$$

- (e) Imitate Hatcher’s proof of Poincaré duality to show that for any oriented  $n$ -manifold  $M$ ,

$$[M] \cap \cdot: H^i(X) \rightarrow H_{n-i}^{BM}(X)$$

is an isomorphism. In particular, this gives an alternative proof of Poincaré duality for compact manifolds.

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