# MATH 635 HOMEWORK 4
# DUE FEBRUARY 6, 2019.

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(1) Let $f, g\colon S^n \to S^n$. The goal of this exercise is to prove that $f \sim g$ if and only if $\deg(f) = \deg(g)$.

(a) Show that if $f \sim g$ then $\deg(f) = \deg(g)$. (So, the rest of the proof focuses on the converse.)

(b) Prove the result in the case $n = 1$. So, we may assume $n \geq 2$.

(c) Fix an identification $S^n \cong D^n/\partial D^n$. We introduce a special class of maps $S^n \to S^n$. By a *linear embedding of $D^n$ in $D^n$* I mean an injection $F\colon D^n \to \mathrm{int}(D^n)$ of the form $F(v) = Av + w$ for some invertible $n \times n$ matrix $A$ and vector $w$. By a *standard map* $S^n \to S^n$ I mean a map $G$ of the form

$$S^n = D^n/\partial D^n \to D^n/(D^n \setminus \mathrm{int}(F(D^n))) = F(D^n)/F(\partial D^n) \xrightarrow{F^{-1}} D^n/\partial D^n = S^n$$

where $F$ is a linear embedding. (This is not a standard term.) Show that any standard map is homotopic to either the identity map $S^n \to S^n$ or the reflection $S^n \to S^n$, $(x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$, depending on the sign of $\det(A)$. (You may assume that $GL_n(\mathbb{R})$ has two connected components, corresponding to the signs of $\det(A)$; you essentially proved this on a previous homework assignment.)

(d) More generally, let

$$F\colon D^n \amalg \cdots \amalg D^n \to D^n$$

be a union of $k$ linear embeddings with disjoint images, with matrices $A_1, \dots, A_k$. Consider the corresponding map

$$\begin{aligned} S^n = D^n/\partial D^n &\to D^n/(D^n \setminus \mathrm{int}(F(D^n \amalg \cdots \amalg D^n))) \\ &\xrightarrow{F^{-1}} (D^n/\partial D^n) \vee \cdots \vee (D^n/\partial D^n) = S^n \vee \cdots \vee S^n \to S^n, \end{aligned}$$

where the last map sends each summand by the identity map to $S^n$. Call such a map a *multi-standard map*, let $k(F) = k$, the number of balls involved, and let $p(F)$ be the number of $A_i$ with $\det(A_i) > 0$. Show that if $G_1$ and $G_2$ are multi-standard maps with $k(G_1) = k(G_2)$ and $p(G_1) = p(G_2)$ then $G_1$ and $G_2$ are homotopic.

(e) Prove that any map $f\colon S^n \to S^n$ is homotopic to a multi-standard map, as follows. Triangulate $S^n$ as $\partial\Delta^{n+1}$. Applying the simplicial approximation theorem, there is another triangulation $T$ of $S^n$ so that $f\colon T \to S^n$ is a simplicial map. We may choose the identification $S^n = D^n/\partial D^n$ so that the point $\partial D^n$ is inside one of the $n$-simplices of $S^n = \partial\Delta^{n+1}$ and every other simplex in $\partial\Delta^{n+1}$ is embedded linearly in $D^n$. (You do not have the prove this, though you could.) Let $g\colon S^n \to S^n$ be the standard map associated to a linear embedding with positive determinant into the interior of one of these other $n$-simplices in $D^n$. Observe that $g \circ f$ is a multi-standard map and $g \circ f \sim f$.

   (f) By a *special map* I mean the iterated suspension of a map $S^1 \to S^1$. Show that there is a special map homotopic to any multi-standard map.
   (g) By the 1-dimensional case, two special maps are homotopic if and only if they have the same degree. So, the result follows; explain why, briefly.
(2) Hatcher 2.B.8, p. 177.
(3) Hatcher 2.B.10, p. 177.
(4) Hatcher 2.C.4, p. 184. (Convince yourself that the fixed set is a subcomplex of the first barycentric subdivision, but you don't have to write down a proof. If you do write down a proof, you might want to use the following fact: there is one $n$-simplex in the first barycentric subdivision of $\Delta^n$ for each permutation $\sigma \in S_{n+1}$. The $n$-simplex in the barycentric subdivision of $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ x_0 + \cdots + x_n = 1\}$ corresponding to $\sigma$ is $\{(x_0, \dots, x_n) \mid x_{\sigma(0)} \geq x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}\}$. So, to specify a $k$-simplex in the barycentric subdivision one specifies a partial ordering of the coordinates *and* that $k$ pairs of consecutive elements in the partial ordering are actually equal.)

Review / qualifying exam practice (not to turn in):

(1) Hatcher 2.B.3, 2.B.4, 2.B.9, 2.B.11, 2.C.2, 2.C.3, 2.C.5.

More problems to think about but not turn in:

(1) Read the rest of the exercises in Sections 2.B and 2.C and solve any that seem hard.

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