MATH 635 HOMEWORK 2
DUE JANUARY 23, 2019.

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Reminder: homework is due at the *beginning* of class, handed to me or in my mailbox on the second floor of Fenton.

(1) Hatcher 2.2.22 (p. 157).
(2) Hatcher 2.2.23 (p. 157). (This is a special case of the Riemann-Hurwitz Formula (over $\mathbb{C}$), which includes branched covers. The proof of the Riemann-Hurwitz formula is not any harder.)
(3) Hatcher 3.1.11 (p. 205).
(4) Give an example of a space X so that:
  (a) For any field k, $\dim(H^*(X;k)) \leq 5$.
  (b) $H_*(X;\mathbb{Z})$ is not finitely generated.
(5) Hatcher 3.A.1 (p. 267).

Review / qualifying exam practice (not to turn in):

(1) Hatcher 2.2.20, 2.2.21, 2.2.25, 3.1.1.

More problems to think about but not turn in:

(1) Read the rest of the problems in Sections 2.2, 3.1, and 3.A and solve any that seem hard.
(2) If you're new to Ext and Tor, solve Hatcher 3.1.1, 3.1.2, 3.1.3, 3.A.2,
(3) Check that every time I have asked you to compute both homology and cohomology, the answers you got are consistent with the universal coefficient theorem.
(4) Here is a more abstract proof of the universal coefficient theorem; fill in the details.
  (a) Let C_* be a chain complex of free abelian groups, and H_* the homology of C_*, which we view as a chain complex with trivial homology. Assume that $C_i = 0$ for all sufficiently small i (i.e., C_* is *bounded below*). Show that there is a chain map $C_* \to H_*$ inducing an isomorphism on homology (i.e., a *quasi-isomorphism* from C_* to H_*).
  (b) Given a bounded-below chain complex C_* over a ring R, choose a chain complex P_* of free R-modules which is quasi-isomorphic to C_*. For M an R-module, define $\mathrm{Tor}^i(C_*, M)$ to be the i^{th} homology group of $P_* \otimes M$. Prove that:
    (i) $\mathrm{Tor}^i(C_*, M)$ is independent of the choice of P_*, up to (canonical) isomorphism.
    (ii) If C_* is quasi-isomorphic to D_* then $\mathrm{Tor}^i(C_*, M) \cong \mathrm{Tor}^i(D_*, M)$.
  (c) If the differential on C_* vanishes, then

$$\mathrm{Tor}^i(C_*, M) \cong (C_i \otimes M) \oplus \mathrm{Tor}^1(C_{i-1}, M) \oplus \mathrm{Tor}^2(C_{i-2}, M) \oplus \cdots.$$

(d) By definition, if C_* is a chain complex of free R-modules then $\mathrm{Tor}^i(C_*, M) \cong H_i(C_* \otimes M) =: H_i(C_*; M)$. So, deduce the universal coefficient theorem for homology.

(5) Prove the universal coefficient theorem with $\mathbb{Z}$ replaced by any PID.

(6) Look up and prove the Riemann-Hurwitz formula.

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