MATH 607 WINTER 2017
HOMEWORK 7

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(1) Following Ozsváth-Szabó following Turaev, we defined a spin^c-structure on Y as a homology class of non-vanishing vector fields. Prove that this agrees with the usual definition of spin^c structures, as a lift of the classifying map

$$B\,\mathrm{Spin}^c(3)[d]Y[r]\_[TY]@-->[ur]B\,SO(3)$$

or, equivalently, a principal spin^c(3)-bundle P over Y and a bundle map

$$P[rr][dr]\mathrm{Fr}(TY)[dl]Y$$

intertwining the actions of $\mathrm{Spin}^c(3)$ and $SO(3)$ (via the projection $\mathrm{Spin}^c(3) \to SO(3)$). (If you use the principal bundle route, flesh out the definition first.) (Hint: this takes some thought. For one direction, given a non-vanishing vector field, the orthogonal complement is a 2-plane bundle, which gives a lift to $B\,SO(2) = BS^1$.)

(2) Prove that the set of spin^c-structures on Y is an affine copy of $H^2(Y)$. Also, given a spin^c-structure there is an induced complex line bundle, coming from the quotient map $\mathrm{Spin}^c(n) \to U(1)$. This line bundle has a first Chern class. How does this first Chern class relate to the $H^2(Y)$-action on the set of spin^c-structures?

(3) Prove directly that if $m(K)$ is the mirror of K then $\tau(m(K)) = -\tau(K)$.

(4) Compute $CFK^-(S^3, K)$ and $\tau(K)$ for some more knots represented by genus 1 Heegaard diagrams (e.g., the figure 8 knot).

(5) Use a genus 1 Heegaard diagram to compute $\tau(K)$ for all torus knots $T_{p,q}$. (You should get $(p-1)(q-1)/2$. *Warning.* I don't know how to do the computation this way.)

(6) Find an unknotting of $T_{p,q}$ using $(p-1)(q-1)/2$ crossing changes.

(7) Prove that the Euler characteristic of $\widehat{HFK}$ is the Alexander polynomial as follows. Fix a knot diagram for a knot K and let $\mathcal{H}$ be the associated Heegaard diagram. Show that the generators for $\widehat{CFK}(\mathcal{H})$ correspond to the Kauffman states (in the sense of Louis Kauffman, *Formal knot theory*). Study how the gradings behave under this identification. (This is involved; start with some examples. See Oszváth-Szabó, "Heegaard Floer homology and alternating knots", for details. There are other proofs.)

(8) Use a genus 1 Heegaard diagram to compute CFK^∞ of the figure 8 knot, and then use the large surgeries formula to compute $\widehat{HF}$ and HF^+ for large surgeries on the figure 8 knot.

(9) The simplest version of the *surgery exact triangle* is an exact sequence

$$\cdots \to HF^+(S^3) \to HF^+(S^3_n(K)) \to HF^+(S^3_{n+1}(K)) \to HF^+(S^3) \to \cdots.$$

Deduce that the surgery exact triangle holds for large n surgeries on a knot K in S^3 from the large surgery formula. As a corollary, if some large surgery on K is an L-space then all larger surgeries on K are L-spaces, as well.

(10) Prove that the differential on CFK^∞ preserves the property that $A(x) + i - j = 0$.

(11) Prove that the intersection sign of a point in $T_\alpha \cap T_\beta$ gives a $\mathbb{Z}/2$-grading on $\widehat{CF}(\Sigma, T_\alpha, T_\beta)$. (This grading depends on the orientations of T_α and T_β, and hence is not an invariant. It is an invariant as a relative $\mathbb{Z}/2$-grading, and can be lifted to a well-defined absolute $\mathbb{Z}/2$-grading. What I'm asking you to prove is just that the differential on $\widehat{CF}$ changes this grading by 1.)

(12) Deduce a large n surgery formula for $\widehat{HF}(S^3_n(K))$ from the surgery formula for HF^+ and the exact sequence relating HF^+ and $\widehat{HF}$.

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