MATH 607 WINTER 2017
HOMEWORK 6

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(1) Show that the manifolds which admit genus 1 Heegaard splittings are exactly S^3, $S^2 \times S^1$, and the lens spaces $L(p, q)$.

(2) Every closed, orientable 3-manifold can be presented by a Heegaard diagram, and also as surgery on a framed link. Given a 3-manifold Y described as surgery on a framed link L (i.e., $Y = S^3(L)$), find a Heegaard diagram for Y. (Hint: find a collection of arcs A in S^3 so that $L \cup A$ is isotopic to a handlebody embedded in the standard way.)

Remark. Going the other direction is harder, and is due to Lickorish; see his book for a nice explanation.

(3) Use the previous problem to draw a Heegaard diagram for $+1$ surgery on the right-handed trefoil. (This is the Poincaré homology sphere.)

(4) Draw a Heegaard diagram for T^3.

(5) Given a Heegaard diagram for Y, explain how to compute $H_1(Y)$ and $H_2(Y)$.

(6) Prove that every closed, orientable 3-manifold Y is the boundary of a compact, orientable 4-manifold. (Hint: describe Y as integer surgery on a link.)

(7) Prove that the tangent bundle to any orientable 3-manifold is trivializable.

(8) Homotopy classes of vector fields...
   (a) Show that the set of homotopy classes of non-vanishing vector fields on S^3 is in canonical bijection with $\mathbb{Z}$.
   (b) Classify homotopy classes of non-vanishing vector fields on $S^2 \times S^1$. (Hint: this is a bit tricky.)

(9) Flesh out the details in the sketch that given a Riemann surface Σ (i.e., a smooth complex curve), $\mathrm{Sym}^n(\Sigma)$ is naturally a complex manifold.

(10) Given a Heegaard diagram (Σ, α, β), prove that the intersection number of T_α and T_β is $|H_1(Y)|$ if $H_1(Y)$ is finite (i.e., Y is a rational homology sphere) and 0 if $H_1(Y)$ is infinite.

(11) Compute $\widehat{HF}(L(p, q))$, the Heegaard Floer homology of the lens space $L(p, q)$.

(12) Let $-Y$ denote the orientation reverse of Y. Prove that $\widehat{CF}(-Y)$ is the dual complex to $\widehat{CF}(Y)$. (Hint: take the same Heegaard diagram but with the opposite orientation and the conjugate almost complex structure.)

(13) Let ϕ be a domain in a Heegaard diagram which is an embedded polygon with concave corners, and so that the edges are segments in α_1, β_1, α_2, β_s, ..., α_g, β_g (in that order). Prove that ϕ has a unique holomorphic representative with respect to the standard complex structure on $\mathrm{Sym}^g(\Sigma)$. Harder: prove that this representative is transversally cut out (i.e., that the standard complex structure on $\mathrm{Sym}^g(\Sigma)$ is generic enough). (See also the appendix to [Ras02].)

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