MATH 607 WINTER 2017
HOMEWORK 5

INSTRUCTOR: ROBERT LIPSHITZ

(1) Prove that the Hofer norm defines a bi-invariant semi-metric on the Hamiltonian diffeomorphism group. (That is, prove it is symmetric and satisfies the triangle inequality, and is bi-invariant.)

(2) (a) Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold of a compact symplectic manifold. Fix an almost complex structure J on M compatible with ω. Show that there is a constant $C > 0$ so that if $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$ is J-holomorphic then $\int u^*\omega > C$.

(b) Give an example where

$$\left\{\int u^*\omega \mid u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)\right\}$$

is dense in $\mathbb{R}$. (Here, we do not require u to be J-holomorphic.) So, the previous part is rather striking.

(3) Let α be a closed 1-form on M. Compute the Novikov-Morse homology of α in classical algebraic topology terms.

(4) Let L be a compact Lagrangian submanifold of an affine algebraic variety V, and let J be the standard complex structure on V. Show that there is an open neighborhood U of L with compact closure so that if $u\colon \mathbb{D}^2 \to M$ is holomorphic and $u(S^1) \subset U$ then $u(\mathbb{D}^2) \subset U$. (Hint: use the maximum modulus theorem from 1-variable complex analysis.)

(5) Compute the displacement energy of the unit circle $S^1 \subset \mathbb{R}^2$, by:
(a) Using results from class to get a lower bound.
(b) Finding a Hamiltonian isotopy with displacement energy as small as possible.

(6) Compute the displacement energy of any circle inside $\mathbb{R}^2$. (Hint: use the fact that the Hofer metric is bi-invariant.)

(7) Fix a Lagrangian $L \subset M$. Let

$$\mathcal{L}(L) = \{\phi(L) \mid \phi \in Ham(M, \omega)\}.$$

Given $L_0, L_1 \in \mathcal{L}(L)$ let

$$\delta(L_0, L_1) = \inf\{\|\phi\| \mid \phi(L_0) = L_1\}$$

where $\|\phi\|$ denotes the Hofer norm of ϕ. Prove that δ defines a non-degenerate metric on $\mathcal{L}(L)$. (These definitions are due to Chekanov [Che00]; see also Usher [Ush13], Usher-Zhang [UZ16].)

Email address: `lipshitz@uoregon.edu`