# MATH 607 WINTER 2017
# HOMEWORK 2

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(1) Show that endowing a vector bundle $E$ with an almost complex structure $J$ is equivalent to identifying $E$ with a complex vector bundle (and formulate precisely what "equivalent" means).

(2) Fix a compact symplectic manifold $(M, \omega)$. Let $\mathrm{Symp}(M, \omega)$ denote the group of symplectomorphisms $(M, \omega) \to (M, \omega)$ and $\mathrm{Symp}_0(M, \omega)$ the path component of $\mathrm{Symp}(M, \omega)$ containing the identity map.

Let $H_t \colon M \to \mathbb{R}$, $t \in [0, 1]$, be a one-parameter family of smooth functions. Define a one-parameter family of vector fields $X_{H_t}$ by $\omega(X_{H_t}, \cdot) = dH_t$. Given $x \in M$ let $\gamma_x(t)$ satisfy $\gamma_x(0) = x$ and $\gamma_x'(t) = X_{H_t}(\gamma_x(t))$. Define $\phi \colon M \to M$ by $\phi(x) = \gamma_x(1)$. Any map $\phi$ that arises this way (for some choice of $H_t$) is a *Hamiltonian diffeomorphism*. We showed in class that $\phi$ is a symplectomorphism.

Let $Ham(M, \omega)$ be the set of Hamiltonian diffeomorphisms of $(M, \omega)$. Prove:

(a) $Ham(M, \omega)$ is a subgroup of $\mathrm{Symp}_0(M, \omega)$, i.e., is closed under composition and inverses. (Hint: this takes some thought.)

(b) Prove that if $H^1(M) = 0$ then $Ham(M, \omega) = \mathrm{Symp}_0(M, \omega)$. (Hint: let $\phi \in \mathrm{Symp}_0(M, \omega)$, choose a path $\phi_t$ from the identity to $\phi$, and consider the vector field $X_t = \phi_t'$.)

(3) Let $(S, j)$ be a Riemann surface, $(M, \omega)$ a symplectic manifold, and $J$ an almost complex structure on $M$ compatible with $\omega$. There is an induced Riemannian metric $g(v, w) = \omega(v, Jw)$ on $M$. The Riemannian metric $g$ on $TM$ induces a Riemannian metric on $T^*M$, and hence a norm on $T^*M$.

(a) Prove: if $u \colon S \to M$ is $J$-holomorphic then

$$\int_S u^*\omega = \int_S |du|^2,$$

where the right hand side uses the norm induced by the Riemannian metric. This quantity is called the *energy* of the map $u$.

(b) Deduce that if $S$ is closed then the energy depends only on the homology class of $u$ in $H_2(M)$.

(c) Suppose $S$ is a compact Riemann surface with boundary, $L$ is a Lagrangian submanifold of $M$, and $u \colon (S, \partial S) \to (M, L)$ is $J$-holomorphic. Deduce that the energy of $u$ depends only on the homology class $[u] \in H_2(M, L)$.

(d) Suppose that $L_0$ and $L_1$ are Lagrangian submanifolds of $M$ with $L_0 \quad L_1$, and that $u, v \colon ([0, 1] \times \mathbb{R}, \{1\} \times \mathbb{R}, \{0\} \times \mathbb{R}) \to (M, L_0, L_1)$ are $J$-holomorphic maps which converge to $x \in L_0 \cap L_1$ at $-\infty$ and $y \in L_0 \cap L_1$ at $+\infty$. Show that if $u$ is homotopic to $v$ then the energy of $u$ is equal to the energy of $v$.

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