MATH 607 WINTER 2017
HOMEWORK 1

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`V2: correction to problem 3 (thanks, Bronson).`

(1) Prove that all $2n$-dimensional symplectic vector spaces are isomorphic.

(2) The symplectic form on T^*M...

(a) The *canonical* 1-*form* on T^*M is defined as follows. Fix $\alpha \in T_p^*M$. Then $T_\alpha T^*M \cong T_pM \oplus T_p^*M$ non-canonically; but there is a canonical linear map $d\pi\colon T_\alpha T^*M \to T_pM$, the total derivative of the projection $\pi\colon T^*M \to M$. The *canonical* 1-*form* on T^*M is defined by $\lambda_{can}(\alpha, v) = \alpha(d\pi(v))$, where $\alpha \in T^*M$ and $v \in T_\alpha T^*M$. Prove that $\omega_{can} = d\lambda_{can}$ is a symplectic form on T^*M.

(b) Given local coordinates $q_1, \dots, q_n$ on some $U \subset M$, there are induced local coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ on T^*U. (So, $(q_1, \dots, q_n)$ is a map $U \to V$ for some $V \subset \mathbb{R}^n$ open; and $(p_1, \dots, p_n; q_1, \dots, q_n)\colon T^*U \to \mathbb{R}^n \times V$.) Explain how.

(c) Show that, with respect to these local coordinates,

$$\lambda_{can} = p_1 dq_1 + \cdots + p_n dq_n$$

so

$$\omega_{can} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

(d) Suppose that α is a 1-form on M, i.e., $\alpha\colon M \to T^*M$ is a smooth section. Show that the image of α is a Lagrangian submanifold of T^*M if and only if α is closed.

(3) Let (M, ω) be a symplectic manifold and consider $(M \times M, \omega \oplus (-\omega))$. Prove that a diffeomorphism $\phi\colon M \to M$ is a symplectomorphism if and only if $\Gamma_\phi = \{(x, \phi(x)) \mid x \in M\} \subset M \times M$ is Lagrangian.

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