MATH 432/532 SPRING 2017
HOMEWORK 9
DUE MARCH 10, 2017.

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Problems:

(1) Let $M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$.
   (a) Prove that M is a smooth submanifold of $\mathbb{R}^3$.
   (b) For each $p \in M$, describe T_pM as a subspace of $\mathbb{R}^3$.
   (c) Prove that TM is trivial, i.e., find a smooth map $F\colon TM \to M \times \mathbb{R}^2$ so that for all $p \in M$, $F(T_pM) = \{p\} \times \mathbb{R}^2$ and $F|_{T_pM}$ is a vector space isomorphism.

(2) Lee, Problem 5-11.

(3) Lee, Problem 5-17.

(4) Let P_2 denote the vector space of polynomials of degree ≤ 2. Define $F\colon P_2 \to P_2$ by

$$F(p(t)) = p'(t)^2 - 2p(t)p''(t).$$

   (a) Compute the total derivative of F.
   (b) Observe that $F(1 + 2t + t^2) = 0$. Is 0 a regular value of F?

(5) Lee, Problem 15-3. (Ignore the book's hint. Identify T_pS^n with a subspace of $\mathbb{R}^{n+1}$. If $\{v_1, \dots, v_n\}$ is a basis for T_pS^n then $\{p, v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^{n+1}$. Define $[v_1, \dots, v_n]$ to be positively oriented if and only if $[p, v_1, \dots, v_n]$ is a positively oriented basis for $\mathbb{R}^{n+1}$; this is the "outward-normal first" convention. Check that this defines an orientation for S^n, and then think about what the antipodal map does.)

Challenge problems (required for Math 532, optional for 432):

(6) A *Lie group* is a smooth manifold G with a smooth map $m\colon G \times G \to G$ so that which makes G into a group, i.e., which is associative and has an identity element and inverses.
   (a) Prove that matrix multiplication makes $O(n)$ into a Lie group. (Hint: you did most of the work already.)
   (b) Prove that $O(n)$ is compact. (Hint: use the Heine-Borel theorem.)
   (c) Describe $T_IO(n)$, the tangent space at the identity to $O(n)$, as explicitly as you can. (This is the *Lie algebra* of $O(n)$.)

(7) Prove that if G is an n-dimensional Lie group then the tangent bundle to G is trivial, i.e., there is a smooth map $\phi\colon G \times \mathbb{R}^n \to TG$ with the properties that:
   (a) The map ϕ commutes with the projections $\pi\colon TG \to G$ and $\pi'\colon G \times \mathbb{R}^n \to G$.
   (b) The map ϕ respects vector addition and scalar multiplication, in the sense that for $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $\phi(p, v + \lambda w) = \phi(p, v) + \lambda\phi(p, w) \in T_pG$.

(Hint: this is easier than it sounds. Use the differential of the map $L_g\colon G \to G$, $L_g(h) = gh$ to identify T_gG with T_eG (where e is the identity element).)

(8) Prove that $U(n)$, the group of $n \times n$ unitary matrices, and $SU(n)$, the group of $n \times n$ unitary matrices with determinant 1, are Lie groups.

(9) Prove that $SU(2)$ is diffeomorphic to S^3, and conclude that TS^3 is trivial. (Hint: $SU(2)$ acts on the unit sphere in $\mathbb{C}^2$. Choose a point $v \in S^3$ and define a map $SU(2) \to S^3$ by $A \mapsto Av$. Prove that this map is bijective. Check that the differential gives an isomorphism $T_ISU(2) \to T_vS^3$ and use the Lie group structure to deduce that the differential is an isomorphism at all points in $SU(2)$.)

Optional practice (not required for anyone):

(10) Lee, Problem 5-1.
(11) Lee, Problem 5-7.
(12) Lee, Problem 5-10.
(13) Lee, Problem 15-12.

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