MATH 307 SPRING 2025
LIST OF USEFUL TAUTOLOGIES, APRIL 7 VERSION

INSTRUCTOR: ROBERT LIPSHITZ

Do not memorize this list, especially the names. I will give you a copy of this sheet on exams. (Exception: modus ponens. Remember it.)

Inference Rules

For each rule, we give the prototypical example of how it's used, and the tautology that leads to it. More general examples are obtained by substituting formulas for the propositional variables (P, Q, etc.).

Modus Ponens (MP): this is our only basic inference rule (not induced by MP and some tautologies).

(1)	P	
(2)	$P \Rightarrow Q$	
(3)	Q	MP (2), (1)

Conjunctive Inference (CI), from $P \Rightarrow (Q \Rightarrow (P \wedge Q))$:

(1)	P	
(2)	Q	
(3)	$P \wedge Q$	CI (1), (2)

Conjunctive Simplification (CS), from $(P \wedge Q) \Rightarrow P$ and $(P \wedge Q) \Rightarrow Q$:

(1)	$P \wedge Q$	
(2)	P	CS (1) (technically, RCS)
(3)	Q	CS (1) (technically, LCS)

Contrapositive Inference (CPI), from $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$:

(1)	$P \Rightarrow Q$	
(2)	$\neg Q \Rightarrow \neg P$	IC (1)

Modus Tollens (MT), from $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$:

(1)	$P \Rightarrow Q$	
(2)	$\neg Q$	
(3)	$\neg P$	MT (1), (2)

Syllogistic Inference (SI), from $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$:

(1)	$P \Rightarrow Q$	
(2)	$Q \Rightarrow R$	
(3)	$P \Rightarrow R$	SI (1), (2)

Inference by Cases (IC), from $((P \Rightarrow Q) \wedge (S \Rightarrow Q)) \Rightarrow ((P \vee S) \Rightarrow Q)$:

$$\begin{array}{lll} (1) & P \Rightarrow R & \\ (2) & Q \Rightarrow R & \\ (3) & (P \vee Q) \Rightarrow R & \text{IC (1), (2)} \end{array}$$

Other Tautologies

- Disjunctive inference: $((P \vee Q) \wedge \neg P) \Rightarrow Q$, $((P \vee Q) \wedge \neg Q) \Rightarrow P$.
- Rule of the Conditional: $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$. (Recall: $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.)
- Commutativity of and, or: $(P \wedge Q) \Rightarrow (Q \wedge P)$, $(P \vee Q) \Rightarrow (Q \vee P)$.
- Associativity of and, or: $((P \wedge Q) \wedge R) \Leftrightarrow (P \wedge (Q \wedge R))$ and similarly for $\vee$.
- De Morgan's Law: $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$. And $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$.
- De Morgan's Cousin (more classically, another Rule of the Conditional): $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$.
- Tollendo Ponens: $(\neg P \wedge (P \vee Q)) \Rightarrow Q$.
- Double negation: $P \Leftrightarrow \neg(\neg P)$.
- The Excluded Middle: $P \vee \neg P$.
- Absurdity: $(P \wedge (\neg P)) \Rightarrow Q$.

Tautology Principle

If $\mathcal{F}$ is a tautology and $\mathcal{F}'$ is the result of substituting some formula $\mathcal{G}$ for all instances of one of the variables in $\mathcal{F}$ then $\mathcal{F}'$ is also a tautology.

General Substitution Principle

If you know $\mathcal{F}$ and $\mathcal{R} \Leftrightarrow \mathcal{S}$ (i.e., these appear in earlier lines), you can conclude $\mathcal{F}'$ obtained by substituting $\mathcal{R}$ for $\mathcal{S}$ with justification GSP and the line numbers. That is:

$$\begin{array}{lll} (1) & \mathcal{F} & \\ (2) & \mathcal{R} \Leftrightarrow \mathcal{S} & \\ (3) & \mathcal{F}' & \text{GSP (2), (1)} \end{array}$$

Deduction Theorem

Easy application: If you want to deduce a conclusion of the form $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$ from premises $\mathcal{A}_1, \dots, \mathcal{A}_n$, it suffices to deduce $\mathcal{F}_2$ from $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{F}_1$

Harder application: You can do this in the middle of a proof: to obtain a formula of the form $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$, you can assume $\mathcal{F}_1$ as a hypothesis, deduce $\mathcal{F}_2$ somehow, and then conclude $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$. *However*, you can't use the intermediate steps (from where $\mathcal{F}_1$ was assumed to where you obtained $\mathcal{F}_2$) later in the argument. That is:

(n)	$\mathcal{F}_1$	Hyp for DT
⋮	⋮	⋮
(n+k)	$\mathcal{F}_2$	(some reason)
(n+k+1)	$\mathcal{F}_1 \Rightarrow \mathcal{F}_2$	DT for Hyp (n), (n)–(n+k) now unusuable.

Indirect Inference (Proof by Contradiction)

If you want to deduce a formula $\mathcal{F}$, you can assume $\neg\mathcal{F}$ as a hypothesis, obtain a contradiction, and then assert $\mathcal{F}$ with justification IC. *However*, you can't use the intermediate steps (from where $\mathcal{F}$ was assumed to the contradiction) laer in the argument. That is:

(n)	$\mathcal{F}_1$	Hyp for II
⋮	⋮	⋮
(n+k)	$P \wedge \neg P$	(some reason)
(n+k+1)	$\mathcal{F}_1$	II for Hyp (n) via (n+k"), (n)–(n+k) now unusuable.

Email address: `lipshitz@uoregon.edu`