MATH 307 SPRING 2025
TAUTOLOGIES AND DEDUCTION RULES, APRIL 16 VERSION

INSTRUCTOR: ROBERT LIPSHITZ

Inference Rules

For each rule, we give the prototypical example of how it's used, and the tautology that leads to it. More general examples are obtained by substituting formulas for the propositional variables (P, Q, etc.).

Modus Ponens (MP): this is our only basic inference rule (not induced by MP and some tautologies).

(1)	P	
(2)	$P \Rightarrow Q$	
(3)	Q	MP (2), (1)

Conjunctive Inference (CI), from $P \Rightarrow (Q \Rightarrow (P \wedge Q))$:

(1)	P	
(2)	Q	
(3)	$P \wedge Q$	CI (1), (2)

Conjunctive Simplification (CS), from $(P \wedge Q) \Rightarrow P$ and $(P \wedge Q) \Rightarrow Q$:

(1)	$P \wedge Q$	
(2)	P	CS (1) (technically, RCS)
(3)	Q	CS (1) (technically, LCS)

Contrapositive Inference (CPI), from $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$:

(1)	$P \Rightarrow Q$	
(2)	$\neg Q \Rightarrow \neg P$	IC (1)

Modus Tollens (MT), from $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$:

(1)	$P \Rightarrow Q$	
(2)	$\neg Q$	
(3)	$\neg P$	MT (1), (2)

Syllogistic Inference (SI), from $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$:

(1)	$P \Rightarrow Q$	
(2)	$Q \Rightarrow R$	
(3)	$P \Rightarrow R$	SI (1), (2)

Inference by Cases (IC), from $((P \Rightarrow Q) \wedge (S \Rightarrow Q)) \Rightarrow ((P \vee S) \Rightarrow Q)$:

(1)	$P \Rightarrow R$	
(2)	$Q \Rightarrow R$	
(3)	$(P \vee Q) \Rightarrow R$	IC (1), (2)

Other Tautologies

- Disjunctive inference: $((P \vee Q) \wedge \neg P) \Rightarrow Q$, $((P \vee Q) \wedge \neg Q) \Rightarrow P$.
- Rule of the Conditional: $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.
  (Recall: $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.)
- Commutativity of and, or: $(P \wedge Q) \Rightarrow (Q \wedge P)$, $(P \vee Q) \Rightarrow (Q \vee P)$.
- Associativity of and, or: $((P \wedge Q) \wedge R) \Leftrightarrow (P \wedge (Q \wedge R))$ and similarly for $\vee$.
- De Morgan's Law: $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$. And $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$.
- De Morgan's Cousin (a.k.a. another Rule of the Conditional): $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$.

- Tollendo Ponens: $(\neg P \wedge (P \vee Q)) \Rightarrow Q$.
- Double negation: $P \Leftrightarrow \neg(\neg P)$.
- The Excluded Middle: $P \vee \neg P$.
- Absurdity: $(P \wedge (\neg P)) \Rightarrow Q$.

Tautology Principle

If $\mathcal{F}$ is a tautology and $\mathcal{F}'$ is the result of substituting some formula $\mathcal{G}$ for all instances of one of the variables in $\mathcal{F}$ then $\mathcal{F}'$ is also a tautology.

General Substitution Principle

If you know $\mathcal{F}$ and $\mathcal{R} \Leftrightarrow \mathcal{S}$ (i.e., these appear in earlier lines), you can conclude $\mathcal{F}'$ obtained by substituting $\mathcal{R}$ for $\mathcal{S}$ with justification GSP and the line numbers. That is:

(1)	$\mathcal{F}$	
(2)	$\mathcal{R} \Leftrightarrow \mathcal{S}$	
(3)	$\mathcal{F}'$	GSP (2), (1)

Deduction Theorem

Easy application: If you want to deduce a conclusion of the form $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$ from premises $\mathcal{A}_1, \dots, \mathcal{A}_n$, it suffices to deduce $\mathcal{F}_2$ from $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{F}_1$

Harder application: You can do this in the middle of a proof: to obtain a formula of the form $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$, you can assume $\mathcal{F}_1$ as a hypothesis, deduce $\mathcal{F}_2$ somehow, and then conclude $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$. *However*, you can't use the intermediate steps (from where $\mathcal{F}_1$ was assumed to where you obtained $\mathcal{F}_2$) later in the argument. That is:

(n)	$\mathcal{F}_1$	Hyp for DT
$\vdots$	$\vdots$	$\vdots$
(n+k)	$\mathcal{F}_2$	(some reason)
(n+k+1)	$\mathcal{F}_1 \Rightarrow \mathcal{F}_2$	DT for Hyp (n), (n)–(n+k) now unusuable.

Indirect Inference (Proof by Contradiction)

If you want to deduce a formula $\mathcal{F}$, you can assume $\neg\mathcal{F}$ as a hypothesis, obtain a contradiction, and then assert $\mathcal{F}$ with justification IC. *However*, you can't use the intermediate steps (from where $\mathcal{F}$ was assumed to the contradiction) laer in the argument. That is:

(n)	$\mathcal{F}_1$	Hyp for II
$\vdots$	$\vdots$	$\vdots$
(n+k)	$P \wedge \neg P$	(some reason)
(n+k+1)	$\mathcal{F}_1$	II for Hyp (n) via (n+k”), (n)–(n+k) now unusuable.

Negation and Quantifiers

$$\neg((\forall x)P) \Leftrightarrow (\exists x)\neg P \qquad \neg((\exists x)P) \Leftrightarrow (\forall x)\neg P$$

Inference Rules for Quantifiers

A *term* is a formula built just from constants, variables, and functions. No predicate symbols (assertions that are true or false), logical symbols ($\wedge$, $\vee$, $\neg$, $\Rightarrow$), or quantifiers ($\exists$, $\forall$). Substituting a term t for x in a formula (open sentence) $P(x)$ means replacing all instances of x by t. The substitution is *valid* if no variable in t becomes bound in $P(t)$, i.e., gets hit by a quantifier.

Universal Generalization

If a formula (open sentence) $P(x)$ appears in your proof, and no (active) hypotheses involve the variable x, then you can write $(\forall x)(P(x))$. Example:

(1)	Assume $x \in \mathbb{Z}$	Hyp for DT
(2)	$x = x \cdot 1$	Axiom (using (1))
(3)	$1 \vert x$	Definition of $\vert$
(4)	$(x \in \mathbb{Z}) \Rightarrow (1 \vert x)$	DT for Hyp (1)
(5)	$(\forall x)((x \in \mathbb{Z}) \Rightarrow (1 \vert x))$	UG (4)

Existential Generalization

If $P(x)$ is a formula, t is a term which is valid to substitute for x in $P(x)$, and $P(t)$ appears in your proof, then you can write $(\exists x)(P(x))$. Example:

(1)	$5^2 > 0$	(some reason)
(2)	$(\exists x)(x^2 > 0)$	EG (1)

Instance of an Existential

If $(\exists x)(P(x))$ appears in your proof, and no (active) hypothesis involves the variable y, then you can write "$P(y)$ [is true] for some value of y", and use $P(y)$ in later steps in the proof. *However*, this counts as a hypothesis in the sense that y is no longer available to reuse in UG or IE. Example:

(1)	$(\exists x)(x \in \mathbb{Z} \wedge x \text{ is prime})$	(some reason)
(2)	$(y \in \mathbb{Z}) \wedge (y \text{ is prime})$, for some y	IE (1)
(3)	y is prime	CS (2)
(4)	$y \neq 4$	(some reason, i.e., that 4 isn't prime)

It's okay (and more common) to write this in the following less stilted form:

(1)	$(\exists x)(x \in \mathbb{Z} \wedge x \text{ is prime})$	(some reason)
(2)	y is prime, for some $y \in \mathbb{Z}$	IE (1)
(3)	$y \neq 4$	(some reason, i.e., that 4 isn't prime)

If x is not in a hypothesis (available to use), you also don't have to change variables.

Universal Specialization

If $(\forall x)(P(x))$ appears in a proof, and $P(t)$ is a valid substitution, then you can deduce $P(t)$. Example:

(1)	$(\forall x)((x \in \mathbb{R}) \Rightarrow (x^2 \geq 0))$	(some reason)
(2)	$(5 \in \mathbb{R}) \Rightarrow (5^2 \geq 0)$	US (1)

Email address: `lipshitz@uoregon.edu`